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A completion for distributive nearlattices

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Abstract. The aim of this article is to propose an adequate completion for distributive nearlattices. We give a proof of the existence of such a completion through a representation theorem, which allows us to prove that this completion is a completely distributive algebraic lattice. We show several properties about this completion, and we present a connection with the free distributive lattice extension of a distributive nearlattice. Finally, we consider how can be extended n-ary operations on distributive nearlattices, and we study the basic properties of these extensions.

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1. Introduction

In the literature there exist different completions for several ordered algebraic structures. For instance, we can mention the completion for Boolean algebras with operators given by Jónsson and Tarski in [22] and [21], which was called *canonical extension*. This completion was extended and generalized to bounded distributive lattices with operators by Gehrke and Jónsson in [15,16] and [17], and they proved that the canonical extension for bounded distributive lattices has as nice properties as the canonical extension for Boolean algebras with operators. Moreover, the concept of a canonical extension was generalized to bounded lattices not necessarily distributive in [13], and to partially ordered sets in [12]. The theory of completions for ordered algebraic structures have different proposes, for instance, the canonical extension for posets introduced

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by Dunn et al. in [12] was proved to be important to obtain a complete relational semantic for implication and fusion fragments of several substructural logics.

Gehrke et al. [14] studied in a general and uniform way the completions for posets for which each element of the completion is reachable by joins of meets and by meets of joins from the original poset. This kind of completions are called Δ_1-*completions*. The canonical extension for posets given in [12] is a particular case of Δ_1-completion.

Nearlattices are join-semilattices with the greatest element in which every principal filter is a bounded lattice. They are a natural generalization of implication algebras, in the sense of [1], and also of bounded distributive lattices. Nearlattices were studied mainly by Cornish and Hickman in [10,20], and by Chajda, Halaš, Kühr and Kolařík in [19,7,6,8,5,9]. Nearlattices can be regarded as total algebras via an everywhere defined ternary operation satisfying some identities. An important class of nearlattices is the class of *distributive nearlattices*. In [3] and [4], a full duality is developed for distributive nearlattices, and some applications are shown. Recently, in [18] the first author proposes a sentential logic associated with the variety of distributive nearlattices.

Since a distributive nearlattice has a natural order relation associated, we can apply the theory of Δ_1-completions given in [14] to obtain different completions. But some of these completions may not be fully adequate. For instance, we can consider the canonical extension of a nearlattice, as a poset, and extend the ternary operation as in [12]. If the nearlattice is distributive, its canonical extension need not be a distributive lattice, and so we think that the canonical extension is not an adequate completion for nearlattices. In Figure 1, we show a distributive nearlattice A and its canonical extension, which is a non-distributive lattice.

The purpose of this paper is to introduce an adequate notion of completion for distributive nearlattices and to study the extensions of n-ary operations defined on distributive nearlattices. The paper is organized as follows. In Section 2, we recall the necessary concepts and results on Δ_1-completions and distributive nearlattices. In Section 3, we provide an alternative proof of the existence of certain Δ_1-completions [14] for distributive nearlattices; we call these Δ_1-completions as *DN-completions*. Section 4 is devoted to study the connection between the free distributive lattice extension [10,4] and the DN-completion of a distributive nearlattice. In Section 5, we study how to extend the operations between distributive nearlattices to operations between their DN-completions. We show that the extensions of the join and the ternary operation of a distributive nearlattice correspond respectively to the join and the natural ternary operation on its DN-completion.

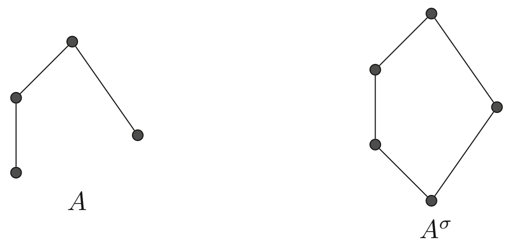

FIGURE 1. An example of a distributive nearlattice A and its canonical extension, as a poset

2. Preliminaries

In this section, we present the main notions and results about the theories of Δ_1-completions and distributive nearlattices that we shall need for our purposes in this paper. For more details about Δ_1-completions for posets see [14]. Our main references for the theory of distributive nearlattices are [10, 20, 5]. Moreover, our reference on order theoretical notions is [11].

2.1. Δ_1-completions

Definition 2.1. A *polarity* is a triple $\langle X, Y, R\rangle$ where X and Y are nonempty sets and $R \subseteq X \times Y$ is a binary relation.

Every polarity $\langle X, Y, R\rangle$ gives rise to the following Galois connection (Φ_R, Ψ_R):

$$\begin{array}{llll} \Phi_R\colon & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & A & \mapsto & \Phi_R(A) = \{y \in Y : (\forall x \in X)(x \in A \implies xRy)\} \\ \Psi_R\colon & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & B & \mapsto & \Psi_R(B) = \{x \in X : (\forall y \in Y)(y \in B \implies xRy)\} \end{array}$$

We thus have the lattice of *Galois closed subsets* of X

$$\mathcal{G}(X) = \{A \in \mathcal{P}(X) : (\Psi_R \circ \Phi_R)(A) = A\}.$$

For further details and background on polarities see [11, 14].

Let P be a poset. A *completion* of P is a pair $\langle L, e\rangle$ where L is a complete lattice and $e\colon P \to L$ is an order embedding. For each $u \in L$, we consider the following sets:

$$[u)_P = \{a \in P : u \leq e(a)\} \quad \text{and} \quad (u]_P = \{a \in P : e(a) \leq u\}.$$

A collection $\mathcal{F}$ of upsets of P is called *standard* provided that $\{[a) : a \in P\} \subseteq \mathcal{F}$, where $[a) = \{b \in P : a \leq b\}$ (dually $(a]$). Dually, a collection $\mathcal{I}$ of downsets of P is called *standard* if $\{(a] : a \in P\} \subseteq \mathcal{I}$. For each standard collection of upsets $\mathcal{F}$ and each standard collection of downsets $\mathcal{I}$, we consider the polarity $\langle \mathcal{F}, \mathcal{I}, R\rangle$ where $R \subseteq \mathcal{F} \times \mathcal{I}$ is defined as follows:

$$FRI \iff F \cap I \neq \emptyset,$$

for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$. We will say that a pair $\langle \mathcal{F}, \mathcal{I}\rangle$ is a *standard Δ_1-polarity* of P if $\mathcal{F}$ is a standard collection of upsets of P and $\mathcal{I}$ is a standard collection of downsets of P, and we consider the binary relation $R \subseteq \mathcal{F} \times \mathcal{I}$ as just defined.

Let P be a poset and let $\langle \mathcal{F}, \mathcal{I} \rangle$ be a standard Δ_1-polarity of P. We know that the polarity $\langle \mathcal{F}, \mathcal{I}, R \rangle$ gives rise to the Galois connection (Φ_R, Ψ_R) and to the lattice of Galois closed subsets $\mathcal{G}(\mathcal{F}) = \{X \in \mathcal{P}(\mathcal{F}) : (\Psi_R \circ \Phi_R)(X) = X\}$ of $\mathcal{F}$. Then, the map $\alpha \colon P \to \mathcal{G}(\mathcal{F})$ defined by $\alpha(a) = \{F \in \mathcal{F} : a \in F\}$ is an order embedding, and thus the pair $\langle \mathcal{G}(\mathcal{F}), \alpha \rangle$ is a completion of P. If $\langle L, e \rangle$ is an arbitrary completion of P, then an element $x \in L$ is called $\mathcal{F}$-*closed* if there is $F \in \mathcal{F}$ such that $x = \bigwedge e[F]$, and an element $y \in L$ is called $\mathcal{I}$-*open* if there is $I \in \mathcal{I}$ such that $y = \bigvee e[I]$. Let us denote the collection of all $\mathcal{F}$-closed elements of L by $\mathsf{K}_{\mathcal{F}}(L)$ and the collection of all $\mathcal{I}$-open elements of L by $\mathsf{O}_{\mathcal{I}}(L)$. We will drop the subscript when confusion is unlikely.

Definition 2.2. Let P be a poset and let $\langle \mathcal{F}, \mathcal{I} \rangle$ be a standard Δ_1-polarity of P. We will say that a completion $\langle L, e \rangle$ of P is:

(C) $\langle \mathcal{F}, \mathcal{I} \rangle$-*compact* when for each $F \in \mathcal{F}$ and each $I \in \mathcal{I}$, if $\bigwedge e[F] \leq \bigvee e[I]$, then $F \cap I \neq \emptyset$,

(D) $\langle \mathcal{F}, \mathcal{I} \rangle$-*dense* if $u = \bigwedge \{y \in \mathsf{O}(L) : u \leq y\}$ and $u = \bigvee \{x \in \mathsf{K}(L) : x \leq u\}$, for every $u \in L$.

Definition 2.3 [14]. Let P be a poset and let $\langle \mathcal{F}, \mathcal{I} \rangle$ be a standard Δ_1-polarity of P. We say that a completion $\langle L, e \rangle$ of P is an $\langle \mathcal{F}, \mathcal{I} \rangle$-*completion* if it is $\langle \mathcal{F}, \mathcal{I} \rangle$-compact and $\langle \mathcal{F}, \mathcal{I} \rangle$-dense.

Theorem 2.4 [14]. *Let P be a poset and let $\langle \mathcal{F}, \mathcal{I} \rangle$ be a standard Δ_1-polarity of P. Then the completion $\langle \mathcal{G}(\mathcal{F}), \alpha \rangle$ is, up to isomorphism, the unique $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P.*

2.2. Nearlattices

Let $\langle A, \vee, 1 \rangle$ be a join-semilattice with a greatest element. We use "*semilattice*" as an abbreviation of "join-semilattice with a greatest element". A *filter* is a nonempty subset F of A such that (1) if $x \in F$ and $x \leq y$, then $y \in F$ and (2) if $x, y \in F$ then $x \wedge y \in F$, whenever $x \wedge y$ exists. A proper filter F of A is called *prime* if for all $x, y \in A$, if $x \vee y \in F$, then $x \in F$ or $y \in F$. We denote by $\mathsf{Fi}(A)$ and $\mathsf{Fi}_{\mathsf{pr}}(A)$ the collections of all filters and all prime filters of A, respectively. Notice that the collection $\mathsf{Fi}(A)$ is a closure system on A, and thus $\langle \mathsf{Fi}(A), \subseteq \rangle$ is a complete lattice. We denote by $\mathrm{Fig}_A(.)$ the closure operator associated with $\mathsf{Fi}(A)$.

A nonempty subset I of A is called an *ideal* when (1) if $y \in I$ and $x \leq y$, then $x \in I$ and (2) if $x, y \in I$, then $x \vee y \in I$. A proper ideal I of A is called *prime* if for all $x, y \in A$, $x \wedge y \in I$ implies $x \in I$ and $y \in I$, whenever $x \wedge y$ exists. We denote by $\mathsf{Id}(A)$ and $\mathsf{Id}_{\mathsf{pr}}(A)$ the collections of all ideals and all prime ideals of A, respectively. It is easy to check that the intersection of any collection of ideals is either an ideal or an empty set. Then, for every nonempty set X of A, there exists the least ideal containing X and it is denoted by $\mathrm{Idg}_A(X)$.

Definition 2.5. A *nearlattice* is a semilattice $\langle A, \vee, 1 \rangle$ such that for each $a \in A$, the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded lattice with respect to the induced order.

Let A be a nearlattice. For every element $a \in A$, we denote the meet in $[a)$ by $\wedge_a$. It should be noted that for all $x, y \in A$, the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A. Thus, for all $x, y \in [a)$, the meet of x and y in $[a)$ coincides with their meet in A, i.e., $x \wedge_a y = x \wedge y$. This should be kept in mind since we will use it without mention.

As we mentioned before, nearlattices can be considered as algebras with one ternary operation satisfying some identities, and therefore they form a variety. This fact was proved by Hickman in [20] and by Chajda and Kolařík in [9]. Later, in [2] Araújo and Kinyon found a smaller equational base.

Theorem 2.6 [2]. *Let A be a nearlattice. Let $m\colon A^3 \to A$ be a ternary operation given by $m(x, y, z) := (x \vee z) \wedge_z (y \vee z)$. Then the following identities are satisfied:*

(1) $m(x, y, x) = x$,
(2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,
(3) $m(x, x, 1) = 1$.

Conversely, let $\langle A, m, 1\rangle$ be an algebra of type (3,0) satisfying the identities (1)–(3). *If we define $x \vee y := m(x, x, y)$, then $\langle A, \vee, 1\rangle$ is a nearlattice. Moreover, for each $a \in A$ and for all $x, y \in [a)$, we have $x \wedge_a y = m(x, y, a)$.*

Definition 2.7. A nearlattice A is said to be *distributive* if each principal filter is a bounded distributive lattice.

Theorem 2.8 [9]. *Let A be a nearlattice. Then, A is distributive if and only if satisfies either of the following equivalent identities:*

(1) $m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w))$,
(2) $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

Let A be a distributive nearlattice and let X be a nonempty subset of A. Then, there is a nice characterization of the generated filter $\mathrm{Fig}_A(X)$:

$$\mathrm{Fig}_A(X) = \{a \in A : \exists a_1, \ldots, a_n \in [X)(a = a_1 \wedge \cdots \wedge a_n)\}, \tag{2.1}$$

where $[X) = \{b \in A : x \leq b \text{ for some } x \in X\}$.

Theorem 2.9 [10]. *Let A be a nearlattice. Then, A is distributive if and only if the lattice $\langle \mathsf{Fi}(A), \subseteq\rangle$ is distributive.*

Theorem 2.10 [19]. *Let A be a distributive nearlattice. Let $I \in \mathsf{Id}(A)$ and $F \in \mathsf{Fi}(A)$ such that $I \cap F = \emptyset$. Then there exists $P \in \mathsf{Id}_{\mathsf{pr}}(A)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Let us consider the poset $\langle \mathsf{Id}_{\mathsf{pr}}(A), \subseteq\rangle$, and we denote by $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ the collection of all downsets of $\langle \mathsf{Id}_{\mathsf{pr}}(A), \subseteq\rangle$. Then $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \cup, \cap, \mathsf{Id}_{\mathsf{pr}}(A), \emptyset\rangle$ is a completely distributive algebraic lattice and $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), m, \mathsf{Id}_{\mathsf{pr}}(A)\rangle$ is a distributive nearlattice where $m(U, V, W) = (U \cup W) \cap (V \cup W)$, for every $U, V, W \in \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$. Recall that if A and B are two distributive nearlattices, a map $f\colon A \to B$ is a *homomorphism* if $f(1) = 1$, $f(a \vee b) = f(a) \vee f(b)$, for every $a, b \in A$, and $f(a \wedge b) = f(a) \wedge f(b)$ whenever $a \wedge b$ exists. We have the following representation theorem given by Halaš in [19].

Theorem 2.11 [19]. *Let A be a distributive nearlattice. The map $\varphi_A\colon A \to \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ defined by*

$$\varphi_A(a) = \{P \in \mathsf{Id}_{\mathsf{pr}}(A) : a \notin P\}$$

is an injective homomorphism.

As we mentioned in the Introduction, distributive lattices and implication algebras [1] are distributive nearlattices. Moreover, it is easy to construct finite distributive nearlattices. Next, we present an infinity distributive nearlattice, which shall be used throughout the article.

Example 2.12. Consider the set $A = \mathcal{P}_{\aleph_0}(\mathbb{N}) = \{X \subseteq \mathbb{N} : |X| = \aleph_0\}$. Then, it is easy to check that $\langle A, \cup, \mathbb{N}\rangle$ is a distributive nearlattice. For what follows (Examples 3.12 and 4.5), we need to characterize the prime ideals of A:

- The only prime principal ideals of A are $(X_n]$, where $X_n = \mathbb{N} - \{n\}$ and $n \in \mathbb{N}$.
- A non-principal ideal I is prime if and only if the following conditions are satisfied:
  (1) $\bigcup I = \mathbb{N}$,
  (2) for all $Y \subseteq \mathbb{N}$ such that $|Y| = |Y^c| = \aleph_0$, $Y \in I$ or $Y^c \in I$, and
  (3) if $Y \in I$, then $|Y^c| = \aleph_0$.

3. DN-completion

If A is a distributive nearlattice, then $\langle \mathsf{Fi}(A), \mathsf{Id}(A)\rangle$ is a standard Δ_1-polarity of A, as a poset, and by Theorem 2.4 we have the $\langle \mathsf{Fi}(A), \mathsf{Id}(A)\rangle$-completion of A.

Definition 3.1. Let A be a distributive nearlattice. The *DN-completion* of A is the $\langle \mathsf{Fi}(A), \mathsf{Id}(A)\rangle$-completion of A. We denote the DN-completion of A by $\langle A^*, \alpha\rangle$.

Without loss of generality we can consider that A is a sub-poset of A^* and the order embedding α is the identity map. So, if $\langle A^*, \vee^*, \wedge^*, 1^*, 0^*\rangle$ is the DN-completion of A, then $A \subseteq A^*$, $1^* = 1$ and $a \leq_A b$ if and only if $a \leq_{A^*} b$, for every $a, b \in A$. Moreover, the collections of closed and open elements of A^* are given by

- $\mathsf{K}(A^*) = \{x \in A^* : x = \bigwedge F \text{ for some } F \in \mathsf{Fi}(A)\}$,
- $\mathsf{O}(A^*) = \{y \in A^* : y = \bigvee I \text{ for some } I \in \mathsf{Id}(A)\}$,

and conditions (C) and (D) of Definition 2.2 are

(C) for each $F \in \mathsf{Fi}(A)$ and each $I \in \mathsf{Id}(A)$, if $\bigwedge F \leq \bigvee I$, then $F \cap I \neq \emptyset$,
(D) $u = \bigwedge\{y \in \mathsf{O}(A^*) : u \leq y\}$ and $u = \bigvee\{x \in \mathsf{K}(A^*) : x \leq u\}$, for every $u \in A^*$.

Remark 3.2. By property (C), notice that $A = \mathsf{K}(A^*) \cap \mathsf{O}(A^*)$. Moreover, $0^* \in \mathsf{K}(A^*)$ and not necessarily $0^* \in \mathsf{O}(A^*)$, since this should imply that A has a least element, which is not necessarily the case. We show that $0^* \in \mathsf{K}(A^*)$. Let $y \in \mathsf{O}(A^*)$. So, there is $I \in \mathsf{Id}(A)$ such that $y = \bigvee I$. As I is nonempty, there is $a \in I$. Then $\bigwedge A \leq a \leq y$, i.e., $\bigwedge A \leq y$, for every $y \in \mathsf{O}(A^*)$. Thus, by property (D), we have that $\bigwedge A = 0^*$ and since $A \in \mathsf{Fi}(A)$, it follows that $0^* \in \mathsf{K}(A^*)$.

Now we give some basic results about the DN-completion of a distributive nearlattice, which will be useful for what follows.

Lemma 3.3. *Let A be a distributive nearlattice and let A^* be the DN-completion of A. Let $F \in \mathsf{Fi}(A)$ and $I \in \mathsf{Id}(A)$. Then the following properties are satisfied:*

(1) $\bigwedge F \leq a$ *iff* $a \in F$, *for every* $a \in A$,
(2) $a \leq \bigvee I$ *iff* $a \in I$, *for every* $a \in A$.

Lemma 3.4 [14]. *Let A be a distributive nearlattice and let A^* be the DN-completion of A. Then $a \vee^* b = a \vee b$ and $a \wedge^* b = a \wedge b$, whenever $a \wedge b$ exists, for every $a, b \in A$.*

Since A^* is a distributive lattice, it follows that the structure $\langle A^*, m^*, 1^* \rangle$ is a distributive nearlattice, where the ternary operation m^* is naturally defined by $m^*(u, v, w) := (u \vee^* w) \wedge^* (v \vee^* w)$.

Proposition 3.5. *Let A be a distributive nearlattice and A^* its DN-completion. Then $\langle A, m, 1 \rangle$ is a subalgebra of $\langle A^*, m^*, 1^* \rangle$.*

Let $a, b, c \in A$. Then, by Lemma 3.4, we have

$$m(a, b, c) = (a \vee c) \wedge_c (b \vee c) = (a \vee^* c) \wedge^* (b \vee^* c) = m^*(a, b, c). \quad \square$$

Lemma 3.6. *Let A be a distributive nearlattice and let A^* be its DN-completion. Let B be a nonempty subset of A. Then $\bigwedge B = \bigwedge \mathrm{Fig}_A(B)$ and $\bigvee B = \bigvee \mathrm{Idg}_A(B)$.*

Proof. Using the characterization of generated filter (2.1), it can be proved that $\bigwedge B$ is the greatest lower bound of the set $\mathrm{Fig}_A(B)$. By a similar argumentation, it can be proved that $\bigvee B = \bigvee \mathrm{Idg}_A(B)$. $\square$

Lemma 3.7. *Let A be a distributive nearlattice and A^* its DN-completion. Let D and E be nonempty subsets of A. If $\bigwedge D \leq \bigvee E$, then there exist nonempty finite subsets $D_0 \subseteq [D)$ and $E_0 \subseteq E$ such that $\bigwedge D_0 \leq \bigvee E_0$ in A.*

Proof. It follows by property (C) and from the characterization of generated filter (2.1). $\square$

Lemma 3.8. *Let A be a distributive nearlattice and A^* its DN-completion. Let X and Y be nonempty subsets of $\mathsf{K}(A^*)$ and $\mathsf{O}(A^*)$, respectively. Then $\bigwedge X \in \mathsf{K}(A^*)$ and $\bigvee Y \in \mathsf{O}(A^*)$.*

Proof. For each $x \in X$, there is $F_x \in \mathsf{Fi}(A)$ such that $x = \bigwedge F_x$. Then, by Lemma 3.6, we have

$$\bigwedge X = \bigwedge_{x \in X} \left(\bigwedge F_x \right) = \bigwedge \left(\bigcup_{x \in X} F_x \right) = \bigwedge \mathrm{Fig}_A \left(\bigcup_{x \in X} F_x \right) \in \mathsf{K}(A^*).$$

An analogous argument shows that $\bigvee Y \in \mathsf{O}(A^*)$. $\square$

Now we focus on the DN-completion of the direct product of distributive nearlattices. Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. It is clear that $A^* \times B^*$ is a completion of the distributive nearlattice $A \times B$. The following result is straightforward, and we thus leave the details to the reader.

Lemma 3.9. *Let A and B be distributive nearlattices. Then:*

(1) $\mathsf{Fi}(A \times B) = \{F_1 \times F_2 : F_1 \in \mathsf{Fi}(A)$ *and* $F_2 \in \mathsf{Fi}(B)\}$.
(2) $\mathsf{Id}(A \times B) = \{I_1 \times I_2 : I_1 \in \mathsf{Id}(A)$ *and* $I_1 \in \mathsf{Id}(B)\}$.

Proposition 3.10. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Then:*

(1) $\mathsf{K}(A^* \times B^*) = \mathsf{K}(A^*) \times \mathsf{K}(B^*)$,
(2) $\mathsf{O}(A^* \times B^*) = \mathsf{O}(A^*) \times \mathsf{O}(B^*)$,
(3) $(A \times B)^* \cong A^* \times B^*$.

Proof. (1) By Lemma 3.9, we have

$$\begin{aligned}
\mathsf{K}(A^* \times B^*) &= \left\{\bigwedge F : F \in \mathsf{Fi}(A \times B)\right\} \\
&= \left\{\bigwedge (F_1 \times F_2) : F_1 \in \mathsf{Fi}(A),\ F_2 \in \mathsf{Fi}(B)\right\} \\
&= \left\{\left(\bigwedge F_1, \bigwedge F_2\right) : F_1 \in \mathsf{Fi}(A),\ F_2 \in \mathsf{Fi}(B)\right\} \\
&= \mathsf{K}(A^*) \times \mathsf{K}(B^*).
\end{aligned}$$

(2) It follows similarly to (1).

(3) By Theorem 2.4, it is enough to show that the completion $A^* \times B^*$ satisfies the properties (C) and (D). Let $F \in \mathsf{Fi}(A \times B)$ and $I \in \mathsf{Id}(A \times B)$ be such that $\bigwedge F \leq \bigvee I$. Thus, there are $F_1 \in \mathsf{Fi}(A)$, $F_2 \in \mathsf{Fi}(B)$, $I_1 \in \mathsf{Id}(A)$ and $I_2 \in \mathsf{Id}(B)$ such that $F = F_1 \times F_2$ and $I = I_1 \times I_2$. It follows that $\bigwedge F = \bigwedge (F_1 \times F_2) = (\bigwedge F_1, \bigwedge F_2)$ and $\bigvee I = \bigvee (I_1 \times I_2) = (\bigvee I_1, \bigvee I_2)$. Since $\bigwedge F \leq \bigvee I$, we have $\bigwedge F_1 \leq \bigvee I_1$ and $\bigwedge F_2 \leq \bigvee I_2$. Then, by property (C) for A^* and B^*, $F_1 \cap I_1 \neq \emptyset$ and $F_2 \cap I_2 \neq \emptyset$. This implies that $F \cap I \neq \emptyset$. Now, we prove property (D). Let $(u, v) \in A^* \times B^*$. Using property (D) for A^* and B^* and from (1), we obtain that

$$\begin{aligned}
(u, v) &= \left(\bigvee \{x_1 \in \mathsf{K}(A^*) : x_1 \leq u\}, \bigvee \{x_2 \in \mathsf{K}(B^*) : x_2 \leq v\}\right) \\
&= \bigvee \left(\{x_1 \in \mathsf{K}(A^*) : x_1 \leq u\} \times \{x_2 \in \mathsf{K}(B^*) : x_2 \leq v\}\right) \\
&= \bigvee \{(x_1, x_2) \in \mathsf{K}(A^*) \times \mathsf{K}(B^*) : (x_1, x_2) \leq (u, v)\} \\
&= \bigvee \{(x_1, x_2) \in \mathsf{K}(A^* \times B^*) : (x_1, x_2) \leq (u, v)\}.
\end{aligned}$$

Similarly, we can prove that

$$(u, v) = \bigwedge \{(y_1, y_2) \in \mathsf{O}(A^* \times B^*) : (u, v) \leq (y_1, y_2)\}.$$

Then $A^* \times B^*$ is the DN-completion of $A \times B$, i.e., $(A \times B)^* \cong A^* \times B^*$. □

Next, we provide an alternative proof of the existence of the DN-completion of a distributive nearlattice A to that given in [14]. We show that in fact $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ is, up to isomorphism, the DN-completion of A.

Theorem 3.11. *Let A be a distributive nearlattice. Then $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ is the DN-completion of A.*

Proof. By Definition 2.3 and Theorem 2.4, we need to prove that the completion $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ satisfies conditions (C) and (D).

Let $F \in \mathsf{Fi}(A)$ and $I \in \mathsf{Id}(A)$. Suppose that $\bigcap \varphi_A[F] \subseteq \bigcup \varphi_A[I]$. If $F \cap I = \emptyset$, by Theorem 2.10, there is $P \in \mathsf{Id}_{\mathsf{pr}}(A)$ such that $F \cap P = \emptyset$ and $I \subseteq P$. Thus, $P \in \bigcap \varphi_A[F]$ and $P \notin \bigcup \varphi_A[I]$, which is a contradiction. So $F \cap I \neq \emptyset$, and the completion $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ satisfies condition (C).

We show condition (D). Let $u \in \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$. We need to prove that

(1) $u = \bigcup\{\bigcap \varphi_A[F] : F \in \mathsf{Fi}(A),\ \bigcap \varphi_A[F] \subseteq u\}$,
(2) $u = \bigcap\{\bigcup \varphi_A[I] : I \in \mathsf{Id}(A),\ u \subseteq \bigcup \varphi_A[I]\}$.

Let $P \in u$. Then $P \in \mathsf{Id}_{\mathsf{pr}}(A)$ and thus $P^c \in \mathsf{Fi}(A)$. Since $P \in u$ and u is a downset of $\mathsf{Id}_{\mathsf{pr}}(A)$, we have $\bigcap \varphi_A[P^c] \subseteq u$. Given that $P \cap P^c = \emptyset$, $P \in \bigcap \varphi_A[P^c]$. Then $P \in \bigcup\{\bigcap \varphi_A[F] : F \in \mathsf{Fi}(A),\ \bigcap \varphi_A[F] \subseteq u\}$, and hence $u \subseteq \bigcup\{\bigcap \varphi_A[F] : F \in \mathsf{Fi}(A),\ \bigcap \varphi_A[F] \subseteq u\}$. The inverse inclusion is straightforward and therefore (1) holds.

To prove (2), we note first that if $u = \emptyset$, then $\bigcap\{\bigcup \varphi_A[I] : I \in \mathsf{Id}(A),\ u \subseteq \bigcup \varphi_A[I]\} = \emptyset$. Otherwise, there is $P \in \bigcap\{\bigcup \varphi_A[I] : I \in \mathsf{Id}(A),\ u \subseteq \bigcup \varphi_A[I]\}$. So, in particular, $P \in \bigcup \varphi_A[P]$. Then there is $a \in P$ such that $P \in \varphi_A(a)$, i.e., $a \in P$ and $a \notin P$, which is a contradiction. Now, we assume that $u \neq \emptyset$. Let $P \in \bigcap\{\bigcup \varphi_A[I] : I \in \mathsf{Id}(A),\ u \subseteq \bigcup \varphi_A[I]\}$. Suppose that $P \notin u$. Since u is a downset of $\mathsf{Id}_{\mathsf{pr}}(A)$, it follows that for each $Q \in u$, $P \nsubseteq Q$. So, there is $a_Q \in P \setminus Q$. Let $B = \{a_Q : Q \in u\}$ and we consider the ideal $I = \mathrm{Idg}_A(B)$. It is easy to check that $u \subseteq \bigcup \varphi_A[I]$. As $B \subseteq P$, we have $I \subseteq P$ and $P \notin \bigcup \varphi_A[I]$, which is a contradiction. Thus, $\bigcap\{\bigcup \varphi_A[I] : I \in \mathsf{Id}(A),\ u \subseteq \bigcup \varphi_A[I]\} \subseteq u$. The other inclusion is trivial, and thus (2) holds. Therefore, by Theorem 2.4, the pair $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ is, up to isomorphism, the DN-completion of A. □

Example 3.12. Consider the distributive nearlattice $A = \mathcal{P}_{\aleph_0}(\mathbb{N})$ of Example 2.12. From Theorem 3.11, we have that the DN-completion of A is $A^* = \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$. Let us prove that the prime ideals of A are incomparable. Let $I_1, I_2 \in \mathsf{Id}_{\mathsf{pr}}(A)$ be such that $I_1 \neq I_2$. We analysis the following cases:

- If I_1 and I_2 are principal, then there exist $n, m \in \mathbb{N}$ such that $I_1 = (X_n]$ and $I_2 = (X_m]$, where $X_n = \mathbb{N} - \{n\}$ and $X_m = \mathbb{N} - \{m\}$. Thus, it is clear that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$.
- If I_1 is principal and I_2 is non-principal, then there is $n \in \mathbb{N}$ such that $I_1 = (X_n]$, where $X_n = \mathbb{N} - \{n\}$. So, by (3) of Example 2.12, $X_n \notin I_2$. Then, $(X_n] \nsubseteq I_2$. Now suppose that $I_2 \subset (X_n]$. Then $n \notin Y$, for all $Y \in I_2$, i.e., $n \notin \bigcup I_2 = \mathbb{N}$, which is a contradiction. Thus, $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$.
- Finally, suppose that I_1 and I_2 are non-principal. If $I_1 \subset I_2$, then there is $Y \in I_2$ such $Y \notin I_1$. It follows that $|Y^c| = \aleph_0$, and since $Y \notin I_1$, we have

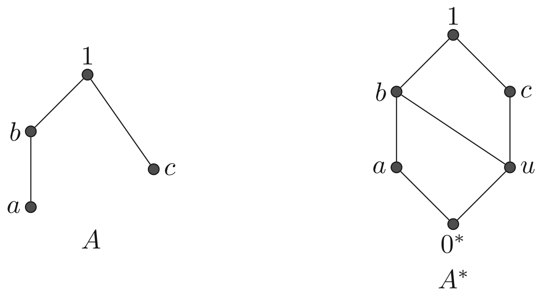

FIGURE 2. A distributive nearlattice and its DN-completion

$Y^c \in I_1$. So, $Y^c \in I_2$. Then $\mathbb{N} = Y \cup Y^c \in I_2$, which is a contradiction. Thus, $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$.

Hence, we have proved that $\langle \mathsf{Id}_{\mathsf{pr}}(A), \subseteq\rangle$ is an anti-chain. Therefore, $A^* = \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)) = \mathcal{P}(\mathsf{Id}_{\mathsf{pr}}(A))$.

In Lemma 3.8 we have proved that the set of closed elements $\mathsf{K}(A^*)$ of the DN-completion A^* of a distributive nearlattice A is closed under arbitrary meets. Now, we can prove that $\mathsf{K}(A^*)$ is, in fact, a sublattice of A^*. A different situation happens with the open elements. We saw that $\mathsf{O}(A^*)$ is closed under arbitrary joins, but $\mathsf{O}(A^*)$ need not be closed under meets. Consider the distributive nearlattice A and its DN-completion A^* in Figure 2. Then we have $\mathsf{O}(A^*) = \{a, b, c, 1\}$ and $b \wedge^* c = u \notin \mathsf{O}(A^*)$.

Proposition 3.13. *Let A be a distributive nearlattice and A^* its DN-completion. Then $\mathsf{K}(A^*)$ is a sublattice of A^*.*

Proof. By Lemma 3.8 and Remark 3.2, we only need to show that $\mathsf{K}(A^*)$ is closed under finite joins. Let $x_1, x_2 \in \mathsf{K}(A^*)$ and let $F_1, F_2 \in \mathsf{Fi}(A)$ be such that $x_1 = \bigwedge F_1$ and $x_2 = \bigwedge F_2$. By Theorem 3.11, A^* is completely distributive and we thus have

$$x_1 \vee x_2 = \bigwedge F_1 \vee \bigwedge F_2 = \bigwedge\{a \vee b : a \in F_1, b \in F_2\} = \bigwedge(F_1 \cap F_2).$$

Since $F_1 \cap F_2 \in \mathsf{Fi}(A)$, it follows that $x_1 \vee x_2 \in \mathsf{K}(A^*)$. □

Let A be a distributive nearlattice and let A^* be the DN-completion of A. Let X be a nonempty subset of A^*. An element $j \in A^*$ is called *completely join irreducible* when $j = \bigvee X$ implies $j \in X$. Dually, an element $m \in A^*$ is called *completely meet irreducible* when $m = \bigwedge X$ implies $m \in X$. Let us denote by $\mathsf{J}_\infty(A^*)$ and $\mathsf{M}_\infty(A^*)$ the collections of all completely join irreducible elements and all completely meet irreducible elements, respectively. By [14], we know that $\mathsf{K}(A^*)$ is dually isomorphic to $\mathsf{Fi}(A)$ and $\mathsf{O}(A^*)$ is isomorphic to $\mathsf{Id}(A)$ by the following maps:

$$\begin{array}{ccc} \bigwedge : \mathsf{Fi}(A) & \rightleftarrows & \mathsf{K}(A^*) : [.)_A \\ F & \mapsto & \bigwedge F \\ [x)_A & \leftarrow & x \end{array} \qquad \begin{array}{ccc} \bigvee : \mathsf{Id}(A) & \rightleftarrows & \mathsf{O}(A^*) : (.]_A \\ I & \mapsto & \bigvee I \\ (y]_A & \leftarrow & y \end{array}$$

Then, by [14], $\mathsf{J}_\infty(A^*) \subseteq \mathsf{K}(A^*)$ and $\mathsf{M}_\infty(A^*) \subseteq \mathsf{O}(A^*)$. In the following proposition we establish the correspondences between $\mathsf{J}_\infty(A^*)$ and some class of filters and between $\mathsf{M}_\infty(A^*)$ and some ideals.

Proposition 3.14. *Let A be a distributive nearlattice and A^* its DN-completion. Then:*

1. $F \in \mathsf{Fi}_{\mathsf{pr}}(A)$ *iff* $\bigwedge F \in \mathsf{J}_\infty(A^*)$, *for every* $F \in \mathsf{Fi}(A)$,
2. $I \in \mathsf{Id}_{\mathsf{pr}}(A)$ *iff* $\bigvee I \in \mathsf{M}_\infty(A^*)$, *for every* $I \in \mathsf{Id}(A)$.

Proof. (1) Let $F \in \mathsf{Fi}(A)$. Suppose that $F \in \mathsf{Fi}_{\mathsf{pr}}(A)$ and let $x_0 = \bigwedge F$. Let X be a nonempty subset of A^* such that $x_0 = \bigvee X$. Since $\mathsf{K}(A^*)$ is join-dense in A^*, we can assume without loss of generality that $X \subseteq \mathsf{K}(A^*)$. Suppose that $x_0 \notin X$. Thus $x < x_0$, i.e., $x < \bigwedge F$ for every $x \in X$. This implies that $F^c \cap [x)_A \neq \emptyset$ for every $x \in X$. Then, for each $x \in X$, there is $a_x \in F^c \cap [x)_A$. We obtain that

$$\bigwedge F = x_0 = \bigvee X \leq \bigvee_{x \in X} a_x \leq \bigvee F^c.$$

As $F \in \mathsf{Fi}_{\mathsf{pr}}(A)$, we have $F^c \in \mathsf{Id}(A)$, and by condition (C) we obtain that $F \cap F^c \neq \emptyset$. This is a contradiction. Then $\bigwedge F = x_0 \in \mathsf{J}_\infty(A^*)$.

Conversely, assume that $x_0 = \bigwedge F \in \mathsf{J}_\infty(A^*)$. Let $a, b \in A$ be such that $a \vee b \in F$. So, $x_0 \leq a \vee b$, and then $x_0 = x_0 \wedge (a \vee b) = (x_0 \wedge a) \vee (x_0 \wedge b)$. Since $x_0 \in \mathsf{J}_\infty(A^*)$, it follows that $x_0 \leq a$ or $x_0 \leq b$. Thus $a \in F$ or $b \in F$, and therefore $F \in \mathsf{Fi}_{\mathsf{pr}}(A)$.

(2) It can be proved by a dual argumentation. □

4. Connection between the free distributive lattice extension and the DN-completion

In this section, we are going to study the connection between the free distributive lattice extension and the DN-completion of a distributive nearlattice. In [10], the authors show that the free distributive lattice extension of a distributive nearlattice A is the distributive lattice of all finitely generated filters of A. Recently in [4], following the duality developed in [3], a new topological approach of the existence of the free distributive lattice extension of A is shown.

Definition 4.1. Let A be a distributive nearlattice. A pair $\langle L(A), e\rangle$, where $L(A)$ is a bounded distributive lattice and $e\colon A \to L(A)$ an embedding, is said to be a *free distributive lattice extension of* A if $e[A]$ is finitely meet-dense in $L(A)$ and the following universal property holds: for every bounded distributive lattice M and every homomorphism $h\colon A \to M$, there exists a unique homomorphism $\widehat{h}\colon L(A) \to M$ such that $h = \widehat{h} \circ e$.

Remark 4.2. The finitely meet-dense condition in Definition 4.1 implies that the free distributive lattice extension is unique up to isomorphism.

Let A be a distributive nearlattice. Recall that in [3] the dual space of A was defined as a topological space $\langle X, \mathcal{K}\rangle$ where $\mathcal{K}$ is a base satisfying certain additional conditions. By Theorem 2.11, A is isomorphic to the subalgebra $\varphi_A[A] = \{\varphi_A(a) : a \in A\}$ of $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$. By the results in [3], the pair $\langle \mathsf{Id}_{\mathsf{pr}}(A), \mathcal{K}_A\rangle$ is the dual space of A, where the topology $\mathcal{T}_A$ is generated by taking as base the family $\mathcal{K}_A = \{\varphi_A(a)^c : a \in A\}$. We will denote by $\mathcal{KO}(\mathsf{Id}_{\mathsf{pr}}(A))$ the family of all open and compact subsets of $\langle \mathsf{Id}_{\mathsf{pr}}(A), \mathcal{K}_A\rangle$, and we consider the family $D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)] = \{U : U^c \in \mathcal{KO}(\mathsf{Id}_{\mathsf{pr}}(A))\}$. If follows that $U \in D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)]$ if and only if there exist $a_1, \ldots, a_n \in A$ such that $U = \varphi_A(a_1) \cap \cdots \cap \varphi_A(a_n)$. Then, by the results presented in [4], we have that $\langle D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)], \varphi_A\rangle$ is the free distributive lattice extension of A. On the other hand, by Theorem 3.11, $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ is the DN-completion of A. We consider the lattice generated by $\varphi_A[A]$ in $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$, denoted by $L(\varphi_A[A])$. Since $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ is distributive, $L(\varphi_A[A])$ can be constructed as follows: at first, take the meets of all nonempty finite subsets of $\varphi_A[A]$. Then $L(\varphi_A[A])$ is the set of joins of all nonempty finite subsets of these meets. It follows that $L(\varphi_A[A]) = D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)]$. Indeed, if $V \in L(\varphi_A[A])$ then there exist $a_1^1, \ldots, a_{n_1}^1, \ldots, a_1^k, \ldots, a_{n_k}^k \in A$ such that

$$\begin{aligned} V &= [\varphi_A(a_1^1) \cup \cdots \cup \varphi_A(a_{n_1}^1)] \cap \cdots \cap [\varphi_A(a_1^k) \cup \cdots \cup \varphi_A(a_{n_k}^k)] \\ &= \varphi_A(\bar{a}_1) \cap \cdots \cap \varphi_A(\bar{a}_k), \end{aligned}$$

where $\bar{a}_i = a_1^i \vee \cdots \vee a_{n_i}^i \in A$, for every $i \in \{1, \ldots, k\}$. So, $V \in D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)]$. The other inclusion is immediate. Then the bounded distributive lattice $L(\varphi_A[A])$ embedded in the DN-completion $\mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ is the free distributive lattice extension of A. In summary, $L(A) \cong L(\varphi_A[A]) = D_{\mathcal{KO}}[\mathsf{Id}_{\mathsf{pr}}(A)]$.

Proposition 4.3. *Let A be a distributive nearlattice and let $\langle L(A), e\rangle$ be the free distributive lattice extension of A. Then the map*

$$\beta \colon \mathsf{Id}_{\mathsf{pr}}(L(A)) \to \mathsf{Id}_{\mathsf{pr}}(A)$$

defined by $\beta(P) = \{a \in A : e(a) \in P\}$ is an order embedding.

Proof. Since e is a homomorphism, it is easy to show that $\beta(P) \in \mathsf{Id}_{\mathsf{pr}}(A)$ for every $P \in \mathsf{Id}_{\mathsf{pr}}(L(A))$. Let $P_1, P_2 \in \mathsf{Id}_{\mathsf{pr}}(L(A))$. It is straightforward that $P_1 \subseteq P_2$ implies $\beta(P_1) \subseteq \beta(P_2)$. Now, assume that $\beta(P_1) \subseteq \beta(P_2)$. Suppose that $P_1 \nsubseteq P_2$, i.e., there is $x \in P_1$ such that $x \notin P_2$. Since $e[A]$ is meet-dense in $L(A)$, there exist $a_1, \ldots, a_n \in A$ such that $x = e(a_1) \wedge \cdots \wedge (a_n) \in P_1$ and as P_1 is prime, there is $i \in \{1, \ldots, n\}$ such that $e(a_i) \in P_1$. Then $a_i \in \beta(P_1) \subseteq \beta(P_2)$ and thus $e(a_i) \in P_2$. It follows that $x \in P_2$, which is a contradiction. Hence $P_1 \subseteq P_2$. This completes the proof. □

The notion of *canonical extension* of a distributive lattice was defined in [15]. It is showed that the canonical extension of a distributive lattice L is, up to isomorphism, $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(L)), \alpha\rangle$ where $\alpha \colon L \to \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(L))$ is given by $\alpha(x) = \{Q \in \mathsf{Id}_{\mathsf{pr}}(L) : x \notin Q\}$. Let us denote by L^σ the canonical extension of a distributive lattice L. Thus, we will consider $L^\sigma = \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(L))$. It is not hard to show that the canonical extension of L is in fact the $\langle \mathsf{Fi}(L), \mathsf{Id}(L)\rangle$-completion of L.

For a distributive nearlattice A, we show that the canonical extension $L(A)^\sigma$ of $L(A)$ is a homomorphic image of the DN-completion A^*.

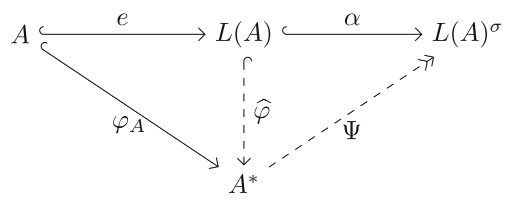

FIGURE 3. $L(A)^\sigma$ is a homomorphic image of A^*

Proposition 4.4. *Let A be a distributive nearlattice and let $\langle L(A), e\rangle$ be the free distributive lattice extension of A. Let $\langle A^*, \varphi_A\rangle$ be the DN-completion of A and let $\langle L(A)^\sigma, \alpha\rangle$ be the canonical extension of $L(A)$. Then there exist a lattice embedding $\widehat{\varphi}\colon L(A) \to A^*$ and an onto lattice homomorphism $\Psi\colon A^* \to L(A)^\sigma$ such that $\varphi_A = \widehat{\varphi} \circ e$ and $\alpha = \Psi \circ \widehat{\varphi}$, see Figure 3.*

Proof. We know that $L(A)^\sigma = \mathcal{P}_d\,(\mathsf{Id}_{\mathsf{pr}}(L(A))$ and $\alpha\colon L(A) \to L(A)^\sigma$ is given by $\alpha(x) = \{Q \in \mathsf{Id}_{\mathsf{pr}}(L(A)) : x \notin Q\}$. Recall also that $A^* = \mathcal{P}_d\,(\mathsf{Id}_{\mathsf{pr}}(A))$ and $\varphi_A\colon A \to A^*$ is defined by $\varphi_A(a) = \{P \in \mathsf{Id}_{\mathsf{pr}}(A) : a \notin P\}$.

The existence of the lattice embedding $\widehat{\varphi}\colon L(A) \to A^*$ such that $\varphi_A = \widehat{\varphi} \circ e$ is a direct consequence of Definition 4.1.

Now, since by Proposition 4.3 the map $\beta\colon \mathsf{Id}_{\mathsf{pr}}(L(A)) \to \mathsf{Id}_{\mathsf{pr}}(A)$ is an order embedding, it follows that the map $\Psi\colon A^* \to L(A)^\sigma$ defined by

$$\Psi(u) = \beta^{-1}[u] = \{P \in \mathsf{Id}_{\mathsf{pr}}(L(A)) : \beta(P) \in u\}$$

for every $u \in A^* = \mathcal{P}_d\,(\mathsf{Id}_{\mathsf{pr}}(A))$ is an onto lattice homomorphism. In order to show that $\Psi \circ \widehat{\varphi} = \alpha$, let $x \in L(A)$. By Definition 4.1, there are $a_1, \ldots, a_n \in A$ such that $x = e(a_1) \wedge \cdots \wedge e(a_n)$. Thus $\widehat{\varphi}(x) = \varphi_A(a_1) \cap \cdots \cap \varphi_A(a_n)$, and then $\Psi(\widehat{\varphi}(x)) = \Psi(\varphi_A(a_1)) \cap \cdots \cap \Psi(\varphi_A(a_n))$. Hence, for $P \in \mathsf{Id}_{\mathsf{pr}}(L(A))$, we have

$$\begin{aligned}
P \in \alpha(x) &\iff x \notin P \\
&\iff \forall i \in \{1, \ldots, n\}(e(a_i) \notin P) \\
&\iff \forall i \in \{1, \ldots, n\}(a_i \notin \beta(P)) \\
&\iff \forall i \in \{1, \ldots, n\}(\beta(P) \in \varphi_A(a_i)) \\
&\iff \forall i \in \{1, \ldots, n\}(P \in \Psi(\varphi_A(a_i))) \\
&\iff P \in \Psi(\varphi_A(a_1)) \cap \cdots \cap \Psi(\varphi_A(a_n)) = \Psi(\widehat{\varphi}(x)).
\end{aligned}$$

Hence $\alpha(x) = \Psi(\widehat{\varphi}(x))$. This completes the proof. □

Example 4.5. Let us show that the free distributive lattice extension and the DN-completion of a distributive nearlattice not necessarily coincide. Consider again the distributive nearlattice $A = \mathcal{P}_{\aleph_0}(\mathbb{N})$ of Example 2.12. Thus, by Example 3.12, we know that the DN-completion of A is $A^* = \mathcal{P}(\mathsf{Id}_{\mathsf{pr}}(A))$. Now, we shall see that the free distributive lattice extension $L(\varphi_A[A])$ of A is properly contained in $\mathcal{P}(\mathsf{Id}_{\mathsf{pr}}(A))$. Recall that

$$L(\varphi_A[A]) = \{\varphi_A(B_1) \cap \cdots \cap \varphi_A(B_n) : n \in \mathbb{N} \text{ and } B_1, \ldots, B_n \in A\},$$

where $\varphi_A\colon A \to \mathcal{P}(\mathsf{Id}_{\mathsf{pr}}(A))$ defined by $\varphi_A(B) = \{I \in \mathsf{Id}_{\mathsf{pr}}(A) : B \notin I\}$. Recall also that for every $n \in \mathbb{N}$, $X_n = \mathbb{N} - \{n\}$. Let $\Lambda = \{(X_n] : n \in \mathbb{N}\} \in \mathcal{P}(\mathsf{Id}_{\mathsf{pr}}(A))$. Suppose that there exists $\{B_1, \ldots, B_n\} \subseteq A$ such that $\Lambda = \varphi_A(B_1) \cap \cdots \cap \varphi_A(B_n)$. Since $\Lambda \neq \mathsf{Id}_{\mathsf{pr}}(A)$, it follows that $B_i \neq \mathbb{N}$, for some $i \in \{1, \ldots, n\}$. Thus, there is $m \in \mathbb{N}$ such that $m \notin B_i$. Then $B_i \subseteq X_m$, i.e., $B_i \in (X_m]$. Thus $(X_m] \notin \varphi_A(B_i)$, which is a contradiction. Hence, $\Lambda \notin L(\varphi_A[A])$.

5. Extensions of n-ary operations

In this section, we study two extensions of n-ary operations defined on distributive nearlattices. We shall describe how to extend the additional operations from a distributive nearlattice to its DN-completion. We focus on those operations that are order preserving in each argument.

Moreover, in order to study the extensions of n-ary operations, it is enough to consider just the extensions of unary operations, because DN-completions of distributive nearlattices commute with direct products (Proposition 3.10). That is, since $(A^*)^n \cong (A^n)^*$, it is equivalent to consider operations $f\colon (A^*)^n \to B^*$ and $f\colon (A^n)^* \to B^*$.

The following definition is similar to that given in [15] (see also [12]) in the framework of canonical extension for distributive lattices.

Definition 5.1. Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f\colon A \to B$ be an order preserving map. For each $u \in A^*$, we define $f^\sigma\colon A^* \to B^*$ and $f^\pi\colon A^* \to B^*$ as follows:

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{f(a) : x \leq a \in A\} : u \geq x \in \mathsf{K}(A^*) \right\}, \tag{5.1}$$

$$f^\pi(u) = \bigwedge \left\{ \bigvee \{f(a) : y \geq a \in A\} : u \leq y \in \mathsf{O}(A^*) \right\}. \tag{5.2}$$

In the following proposition, we show some basic but useful properties of f^σ and f^π. Its proof is straightforward, and thus we leave the details to the reader.

Proposition 5.2. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f\colon A \to B$ be an order preserving map. Then:*

(P1) *for each $x \in \mathsf{K}(A^*)$ and each $y \in \mathsf{O}(A^*)$,*
- $f^\sigma(x) = \bigwedge\{f(a) : x \leq a \in A\}$,
- $f^\sigma(x) \in \mathsf{K}(A^*)$,
- $f^\pi(y) = \bigvee\{f(a) : y \geq a \in A\}$,
- $f^\pi(y) \in \mathsf{O}(A^*)$,

(P2) *for each $u \in A^*$,*
- $f^\sigma(u) = \bigvee\{f^\sigma(x) : u \geq x \in \mathsf{K}(A^*)\}$,
- $f^\pi(u) = \bigwedge\{f^\pi(y) : u \leq y \in \mathsf{O}(A^*)\}$,

(P3) *f^σ and f^π are order preserving and extend f,*

(P4) *$f^\sigma \leq f^\pi$ and they coincide on $\mathsf{K}(A^*) \cup \mathsf{O}(A^*)$.*

By Theorem 3.11, the pair $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A \rangle$ is the DN-completion of a distributive nearlattice A. With this consideration, if $f\colon A \to B$ is an order preserving map between distributive nearlattices, then we have from (4.1) and (4.2) that for each $u \in A^* = \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$:

- $f^\sigma(u) = \bigcup \{\bigcap\{\varphi_B(f(a)) : x \subseteq \varphi_A(a), a \in A\} : u \supseteq x \in \mathsf{K}(A^*)\}$,
- $f^\pi(u) = \bigcap \{\bigcup\{\varphi_B(f(a)) : y \supseteq \varphi_A(a), a \in A\} : u \subseteq y \in \mathsf{O}(A^*)\}$.

Lemma 5.3. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f\colon A \to B$ be an order preserving map such that $f(1) = 1$. Then:*

1. *if f preserves finite joins, then f^π preserves arbitrary joins of open elements,*
2. *if f preserves existing finite meets, then f^σ preserves arbitrary meets of closed elements.*

Proof. (1) Let $Y \subseteq \mathsf{O}(A^*)$ and $y_0 = \bigcup Y$. Since f^π is order preserving, it follows that $\bigcup_{y\in Y} f^\pi(y) \subseteq f^\pi(y_0)$. Let $Q \in f^\pi(y_0) = \bigcup\{\varphi_B(f(a)) : a \in A, \varphi_A(a) \subseteq y_0\}$. Thus, there is $a_0 \in A$ such that $\varphi_A(a_0) \subseteq y_0$ and $Q \in \varphi_B(f(a_0))$. Then $a_0 \notin f^{-1}[Q]$. If $f^{-1}[Q] = \emptyset$, then $a \notin f^{-1}[Q]$ and $Q \in \varphi_B(f(a))$ for every $a \in A$. Hence $Q \in \bigcup\{\varphi_B(f(a)) : \varphi_A(a) \subseteq y\} = f^\pi(y)$ for every $y \in Y$ and we thus obtain that $Q \in \bigcup_{y\in Y} f^\pi(y)$. Assume that $f^{-1}[Q] \neq \emptyset$. Since f preserves finite joins, $f^{-1}[Q]$ is an ideal of A. As $a_0 \notin f^{-1}[Q]$, by Theorem 2.10, there is $P \in \mathsf{Id}_{\mathsf{pr}}(A)$ such that $a_0 \notin P$ and $f^{-1}[Q] \subseteq P$. Then $P \in \varphi_A(a_0) \subseteq y_0 = \bigcup Y$ and so, there is $y' \in Y$ such that $P \in y' = \bigcup\{\varphi_A(a) : a \in A, \varphi_A(a) \subseteq y'\}$. Thus, there is $a_1 \in A$ such that $\varphi_A(a_1) \subseteq y'$ and $P \in \varphi_A(a_1)$. Hence $a_1 \notin P$ and this implies that $a_1 \notin f^{-1}[Q]$. Then, $Q \in \varphi_B(f(a_1))$ and $\varphi_A(a_1) \subseteq y'$. It follows that $Q \in f^\pi(y')$ and $Q \in \bigcup_{y\in Y} f^\pi(y)$. We have proved that $f^\pi(y_0) \subseteq \bigcup_{y\in Y} f^\pi(y)$.

(2) Let $X \subseteq \mathsf{K}(A^*)$ and $x_0 = \bigcap X$. Since f^σ is order preserving, it follows that $f^\sigma(x_0) \subseteq \bigcap_{x\in X} f^\sigma(x)$. By (P1), for each $x \in \mathsf{K}(A^*)$ we have $f^\sigma(x) = \bigcap\{\varphi_B(f(a)) : a \in A, x \subseteq \varphi_A(a)\}$. Let $Q \in \bigcap_{x\in X} f^\sigma(x)$. Then, for each $x \in X$, $Q \in \varphi_B(f(a))$ for every $a \in A$ such that $x \subseteq \varphi_A(a)$. That is, for each $x \in X$, $a \notin f^{-1}[Q]$ for every $a \in A$ such that $x \subseteq \varphi_A(a)$. As Q is a prime ideal of B, Q^c is a filter. Since f preserves existing finite meets and $f(1) = 1$, it follows that $f^{-1}[Q^c] \in \mathsf{Fi}(A)$. Suppose that $Q \notin f^\sigma(x_0) = \bigcap\{\varphi_B(f(a)) : a \in A, x_0 \subseteq \varphi_A(a)\}$. Thus, there is $a_0 \in A$ such that $x_0 \subseteq \varphi_A(a_0)$ and $Q \notin \varphi_B(f(a_0))$. Then $a_0 \notin f^{-1}[Q^c]$. By Theorem 2.10, there is $P \in \mathsf{Id}_{\mathsf{pr}}(A)$ such that $a_0 \in P$ and $P \cap f^{-1}[Q^c] = \emptyset$. So, $P \subseteq f^{-1}[Q]$. Then $P \notin \varphi_A(a_0)$ and $P \notin x_0$. This implies that there is $x \in X$ such that $P \notin x$. Since x is a closed element of A^*, $x = \bigcap\{\varphi_A(a) : a \in A, x \subseteq \varphi_A(a)\}$. It follows that there is $a_1 \in A$ such that $x \subseteq \varphi_A(a_1)$ and $P \notin \varphi_A(a_1)$, i.e., $a_1 \in P$. Hence, for the closed element $x \in X$, $a_1 \in f^{-1}[Q]$ and $x \subseteq \varphi_A(a_1)$, which is a contradiction. Therefore, $Q \in f^\sigma(x_0)$ and $\bigcap_{x\in X} f^\sigma(x) \subseteq f^\sigma(x_0)$. □

Now, we can prove a stronger result that the previous one.

Proposition 5.4. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f\colon A \to B$ be an order preserving map such that $f(1) = 1$. Then:*

1. *if f preserves finite joins, then f^π preserves arbitrary joins,*
2. *if f preserves existing finite meets, then f^σ preserves arbitrary meets.*

Proof. (1) Let $U \subseteq A^*$ and $u_0 = \bigvee U$. Since f^π is order preserving, it follows that $\bigvee_{u\in U} f^\pi(u) \leq f^\pi(u_0)$. Since B^* is a completely distributive lattice, it follows by Lemma 5.3 that

$$\begin{aligned}\bigvee_{u\in U} f^\pi(u) &= \bigvee_{u\in U} \bigwedge\{f^\pi(y) : u \leq y \in \mathsf{O}(A^*)\}\\ &= \bigwedge_{\substack{\alpha\colon U\to\mathsf{O}(A^*)\\ u\leq\alpha(u)}} \bigvee_{u\in U} f^\pi(\alpha(u))\\ &= \bigwedge_{\substack{\alpha\colon U\to\mathsf{O}(A^*)\\ u\leq\alpha(u)}} f^\pi\left(\bigvee_{u\in U}\alpha(u)\right).\end{aligned}$$

As $u_0 = \bigvee U \leq \bigvee_{u\in U}\alpha(u)$ for every $\alpha\colon U \to \mathsf{O}(A^*)$ such that $u \leq \alpha(u)$ and since f^π is order preserving, it follows that $f^\pi(u_0) \leq f^\pi\left(\bigvee_{u\in U}\alpha(u)\right)$ for every α. Then

$$f^\pi(u_0) \leq \bigwedge_{\substack{\alpha\colon U\to\mathsf{O}(A^*)\\ u\leq\alpha(u)}} f^\pi\left(\bigvee_{u\in U}\alpha(u)\right) = \bigvee_{u\in U} f^\pi(u).$$

This completes the proof of (1).

(2) It follows by a dual argumentation. □

Let us see now that the extensions of the operations $\vee$ and m of a distributive nearlattice coincide respectively with the operations $\vee^*$ and m^* (see on page 7) of its DN-completion. First, we introduce the following definition as a generalization of a definition given in [16].

Definition 5.5. Let A and B be distributive nearlattices and let $f\colon A \to B$ be an order preserving map. We say that f is *smooth* if $f^\sigma = f^\pi$.

The proof of the following result is similar to that given in [16].

Lemma 5.6. *Let A and B be distributive nearlattices and let $f\colon A \to B$ be an order preserving map. If f^π preserves joins of up-directed subsets, then f is smooth.*

Theorem 5.7. *Let A be a distributive nearlattice and A^* its DN-completion. Then $\vee^\sigma = \vee^*$ and $\vee^\pi = \vee^*$. Hence, $\vee$ is smooth.*

Proof. First, we prove that $\vee^\pi$ and $\vee^*$ coincide on the open elements of A^*. Let $y_1, y_2 \in \mathsf{O}(A^*)$. By (P1), we have

$$y_1 \vee^\pi y_2 = \bigvee\{a \vee^* b : a \leq y_1,\ b \leq y_2,\ a, b \in A\}.$$

Since $y_1, y_2 \in \mathsf{O}(A^*)$, there are $I_1, I_2 \in \mathsf{Id}(A)$ such that $y_1 = \bigvee I_1$ and $y_2 = \bigvee I_2$. Then, by Lemma 3.3, we have

$$y_1 \vee^\pi y_2 = \bigvee\{a \vee^* b : a \in I_1,\ b \in I_2\}.$$

So, we obtain that $y_1 \vee^\pi y_2 = \bigvee I_1 \vee^* \bigvee I_2 = y_1 \vee^* y_2$.

Let now $u_1, u_2 \in A^*$. By (P2) and Lemma 3.8, we have

$$\begin{aligned} u_1 \vee^\pi u_2 &= \bigwedge\{y_1 \vee^\pi y_2 : u_1 \leq y_1,\ u_2 \leq y_2,\ y_1, y_2 \in \mathsf{O}(A^*)\} \\ &= \bigwedge\{y_1 \vee^* y_2 : u_1 \leq y_1,\ u_2 \leq y_2,\ y_1, y_2 \in \mathsf{O}(A^*)\} \\ &= \bigwedge\{y \in \mathsf{O}(A^*) : u_1 \vee^* u_2 \leq y\} \\ &= u_1 \vee^* u_2. \end{aligned}$$

Now, since $\vee$ preserves joins, it follows by Proposition 5.4 that $\vee^\pi$ preserves arbitrary joins and, in particular, $\vee^\pi$ preserves joins of up-directed subsets. Therefore, by Lemma 5.6, $\vee$ is smooth and hence $\vee^\sigma = \vee^\pi = \vee^*$. □

Let A be a distributive nearlattice and A^* its DN-completion. For $\overline{a} := (a_1, a_2, a_3) \in A^3$, we write

$$\varphi_A(\overline{a}) := (\varphi_A(a_1), \varphi_A(a_2), \varphi_A(a_3)).$$

By Theorem 3.11, the pair $\langle \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A)), \varphi_A\rangle$ is the DN-completion of A and $m^*(u_1, u_2, u_3) = (u_1 \cup u_3) \cap (u_2 \cup u_3) = (u_1 \cap u_2) \cup u_3$, for every $u_1, u_2, u_3 \in \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$. Moreover, since $\varphi_A \colon A \to \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))$ is a homomorphism, we have $\varphi_A(m(\overline{a})) = m^*(\varphi_A(\overline{a}))$, for each $\overline{a} \in A^3$. As the operation m is order preserving, by (4.1) and (4.2) we have that for each $\overline{u} = (u_1, u_2, u_3) \in \mathcal{P}_d(\mathsf{Id}_{\mathsf{pr}}(A))^3$:

- $m^\sigma(\overline{u}) = \bigcup\big\{\bigcap\{m^*(\varphi_A(\overline{a})) : \overline{x} \leq \varphi_A(\overline{a}), \overline{a} \in A^3\} : \overline{u} \geq \overline{x} \in \mathsf{K}(A^*)^3\big\}$,
- $m^\pi(\overline{u}) = \bigcap\big\{\bigcup\{m^*(\varphi_A(\overline{a})) : \varphi_A(\overline{a}) \leq \overline{y}, \overline{a} \in A^3\} : \overline{u} \leq \overline{y} \in \mathsf{O}(A^*)^3\big\}$.

Theorem 5.8. *Let A be a distributive nearlattice and let A^* be the DN-completion of A. Then $m^\sigma = m^*$ and $m^\pi = m^*$. Hence, m is smooth.*

Proof. We first prove that m^* and m^σ coincide on the closed elements of A^*. Let $\overline{x} = (x_1, x_2, x_3) \in \mathsf{K}(A^*)^3$. By (P1), we have

$$m^\sigma(\overline{x}) = \bigcap\{m^*(\varphi_A(\overline{a})) : \overline{x} \leq \varphi_A(\overline{a}),\ \overline{a} \in A^3\}.$$

Let $P \in m^\sigma(\overline{x})$. So, $P \in (\varphi_A(a_1) \cap \varphi_A(a_2)) \cup \varphi_A(a_3)$ for every $\overline{a} \in A^3$ such that $\overline{x} \leq \varphi_A(\overline{a})$. Since $x_1, x_2, x_3 \in \mathsf{K}(A^*)$, there are $F_1, F_2, F_3 \in \mathsf{Fi}(A)$ such that $x_i = \bigcap \varphi_A[F_i]$ for $i = 1, 2, 3$. Then, we obtain that

$$P \in (\varphi_A(a_1) \cap \varphi_A(a_2)) \cup \varphi_A(a_3) \tag{5.3}$$

for every $\overline{a} \in F_1 \times F_2 \times F_3$. Suppose that $P \notin m^*(\overline{x})$. Thus, $P \notin (x_1 \cap x_2) \cup x_3$ and this implies that $P \notin x_1$ or $P \notin x_2$, and $P \notin x_3$. So, $F_1 \cap P \neq \emptyset$ or $F_2 \cap P \neq \emptyset$, and $F_3 \cap P \neq \emptyset$. If $F_1 \cap P \neq \emptyset$, then there are $b_1 \in F_1 \cap P$ and $b_3 \in F_3 \cap P$. It follows that $P \notin \varphi_A(b_1) \cup \varphi_A(b_3)$, which is a contradiction by (5.3). Analogously if $F_2 \cap P \neq \emptyset$. Hence, $m^\sigma(\overline{x}) \subseteq m^*(\overline{x})$.

Let $P \in m^*(\overline{x})$. Let $\overline{a} \in A^3$ be such that $\overline{x} \leq \varphi_A(\overline{a})$. Since m^* is order preserving, we have $m^*(\overline{x}) \subseteq m^*(\varphi_A(\overline{a})) = \varphi_A(m(\overline{a}))$. Then $P \in m^*(\varphi_A(\overline{a}))$ for every $\overline{a} \in A^3$ such that $\overline{x} \leq \varphi_A(\overline{a})$, i.e., $P \in m^\sigma(\overline{x})$. So, $m^*(\overline{x}) \subseteq m^\sigma(\overline{x})$. Therefore, $m^*(\overline{x}) = m^\sigma(\overline{x})$ for every $\overline{x} \in \mathsf{K}(A^*)^3$.

Now, we prove the general case. Let $\overline{u} = (u_1, u_2, u_3) \in {A^*}^3$. By (P2), we have

$$m^\sigma(\overline{u}) = \bigcup\{m^\sigma(\overline{x}) : \overline{u} \geq \overline{x} \in \mathsf{K}(A^*)^3\} = \bigcup\{m^*(\overline{x}) : \overline{u} \geq \overline{x} \in \mathsf{K}(A^*)^3\}.$$

Since m^* is order preserving, it follows that $m^\sigma(\overline{u}) \subseteq m^*(\overline{u})$. In order to prove the other inclusion, let $P \in m^*(\overline{u})$. Then $P \in u_1 \cap u_2$ or $P \in u_3$. Let $F = P^c \in \mathsf{Fi}(A)$ and $x = \bigcap \varphi_A[F]$. We consider the two possible cases:
Case 1: Suppose that $P \in u_1 \cap u_2$. Let us see that $x \subseteq u_1 \cap u_2$. Let $Q \in x$. So, $Q \cap P^c = \emptyset$ and thus $Q \subseteq P$. Since u_1 and u_2 are downsets of $\mathsf{Id}_{\mathsf{pr}}(A)$, $Q \in u_1 \cap u_2$. As $0^* \in \mathsf{K}(A^*)$, it follows that $\overline{x} = (x, x, 0^*) \in \mathsf{K}(A^*)^3$ and $\overline{x} \leq \overline{u}$. Moreover, $m^*(\overline{x}) = (x \cup 0^*) \cap (x \cup 0^*) = x$ and $P \in x = m^*(\overline{x})$. Thus,

$$P \in \bigcup\{m^*(\overline{x}) : \overline{u} \geq \overline{x} \in \mathsf{K}(A^*)^3\} = m^\sigma(\overline{u}),$$

and hence $m^*(\overline{u}) \subseteq m^\sigma(\overline{u})$.
Case 2: If $P \in u_3$, then $P \in x = m^*(\overline{x})$, where $\overline{x} = (0^*, 0^*, x) \in \mathsf{K}(A^*)^3$ and $\overline{x} \leq \overline{u}$. Then,

$$P \in \bigcup\{m^*(\overline{x}) : \overline{u} \geq \overline{x} \in \mathsf{K}(A^*)^3\} = m^\sigma(\overline{u})$$

and we obtain $m^*(\overline{u}) \subseteq m^\sigma(\overline{u})$.

In any case, we have $m^*(\overline{u}) \subseteq m^\sigma(\overline{u})$. Then $m^\sigma = m^*$. The proof of $m^\pi = m^*$ follows by a dual argumentation. Therefore, m is smooth. □

We now move on to study the extensions of embeddings and onto homomorphisms. We will show that the DN-completion commutes with respect to quotients. Let A and B be distributive nearlattices and let $f\colon A \to B$ be a homomorphism. Note that f preserves finite joins and existing finite meets. So, the extensions f^σ and f^π coincide, i.e., f is smooth. We write $f^* = f^\sigma = f^\pi$ and we can use for f^* all the properties valid for f^σ and f^π.

Proposition 5.9. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f\colon A \to B$ be an onto homomorphism. Then, the extension $f^*\colon A^* \to B^*$ is an onto homomorphism.*

Proof. Since f preserves joins and existing meets, it follows by Proposition 5.4 that f^* preserves arbitrary joins and meets and, in particular, f^* is a homomorphism. First, let us see that f^* is onto with respect to the closed elements. Let $s \in \mathsf{K}(B^*)$. So, there is $G \in \mathsf{Fi}(B)$ such that $s = \bigwedge G$. Since f preserves existing finite meets and $f(1) = 1$, $F = f^{-1}[G] \in \mathsf{Fi}(A)$. Let $x = \bigwedge F \in \mathsf{K}(A^*)$. As f is onto, $f[F] = f[f^{-1}[G]] = G$. Moreover, by Lemma 3.3, we have that $F = \{a \in A : x \leq a\}$. Then

$$f^*(x) = \bigwedge\{f(a) : x \leq a\} = \bigwedge f[F] = \bigwedge G = s.$$

Let now $v \in B^*$. Then

$$v = \bigvee\{s \in \mathsf{K}(B^*) : s \leq v\} = \bigvee\{f^*(x) : x \in \mathsf{K}(A^*), f^*(x) \leq v\}.$$

Let $u = \bigvee\{x \in \mathsf{K}(A^*) : f^*(x) \leq v\}$. Since f^* preserves arbitrary joins, we obtain that $f^*(u) = \bigvee\{f^*(x) : x \in \mathsf{K}(A^*), f^*(x) \leq v\} = v$. Therefore, f^* is onto. □

Let A be a distributive nearlattice and let A^* be the DN-completion of A. Let $\theta \in \mathrm{Con}(A)$ and let $\pi_\theta \colon A \to A/\theta$ be the natural map. It is clear that π_θ is an onto homomorphism and thus, by Proposition 5.9, $\pi_\theta^* \colon A^* \to (A/\theta)^*$ is an onto homomorphism. Let $\theta^* = \mathrm{Ker}(\pi_\theta^*) \in \mathrm{Con}(A^*)$. Hence $(A/\theta)^* \cong A^*/\theta^*$.

For the following result, we need to restrict ourselves to a smaller class of homomorphisms between distributive nearlattices.

Definition 5.10. Let A and B be distributive nearlattices and let $f \colon A \to B$ be a map. We say that f is an *s-homomorphism* if it is a homomorphism and the following conditions hold:

(1) for each $a_1, a_2 \in A$, if $f(a_1) \wedge f(a_2)$ exists, then $a_1 \wedge a_2$ exists,
(2) for each $a \in A$ and for each $b \in B$, if $f(a) \leq b$, then there is $a' \in A$ such that $a \leq a'$ and $f(a') = b$.

Proposition 5.11. *Let A and B be distributive nearlattices and let A^* and B^* be the DN-completions of A and B, respectively. Let $f \colon A \to B$ be an injective s-homomorphism. Then, the extension $f^* \colon A^* \to B^*$ is an embedding.*

Proof. We know that the extension f^* is a homomorphism. In order to show that f^* is injective, we first prove that $f^*(x) \leq f^*(y)$ implies $x \leq y$, for all $x \in \mathsf{K}(A^*)$ and $y \in \mathsf{O}(A^*)$. By Proposition 5.2, we have

$$\bigwedge\{f(a) : x \leq a\} = f^*(x) \leq f^*(y) = \bigvee\{f(a) : a \leq y\}.$$

So, by Lemma 3.7, there exist $b_1, \dots, b_n \in B$ and $a_1, \dots, a_n \in A$ such that $f(a_i) \leq b_i$ and $x \leq a_i$ for every $i \in \{1, \dots, n\}$, and there exist $a'_1, \dots, a'_m \in A$ such that $a'_j \leq y$ for every $j \in \{1, \dots, m\}$ and

$$b_1 \wedge \cdots \wedge b_n \leq f(a'_1) \vee \cdots \vee f(a'_m).$$

By (2) of Definition 5.10, for each $i \in \{1, \dots, n\}$, there exists $a''_i \in A$ such that $a_i \leq a''_i$ and $f(a''_i) = b_i$. Then, by the previous inequality and (1) of Definition 5.10, we obtain that

$$\begin{aligned} f(a''_1 \wedge \cdots \wedge a''_n) &= f(a''_1) \wedge \cdots \wedge f(a''_n) \\ &= b_1 \wedge \cdots \wedge b_n \\ &\leq f(a'_1) \vee \cdots \vee f(a'_m) \\ &= f(a'_1 \vee \dots, \vee a'_m). \end{aligned}$$

Since f is injective, it follows that $a''_1 \wedge \cdots \wedge a''_n \leq a'_1 \vee \cdots \vee a'_m$ and thus $x \leq y$.

Let now $u_1, u_2 \in A^*$ be such that $f^*(u_1) = f^*(u_2)$. By (D), $u_1 = \bigvee\{x \in \mathsf{K}(A^*) : x \leq u_1\}$ and $u_2 = \bigwedge\{y \in \mathsf{O}(A^*) : u_2 \leq y\}$. As $f^*(u_1) = f^*(u_2)$, we have

$$\bigvee\{f^*(x) : x \in \mathsf{K}(A^*), x \leq u_1\} = \bigwedge\{f^*(y) : y \in \mathsf{O}(A^*), u_2 \leq y\}.$$

Thus, $f^*(x) \leq f^*(y)$ and by what we have already proved, $x \leq y$ for every $x \in \mathsf{K}(A^*)$ such that $x \leq u_1$ and for every $y \in \mathsf{O}(A^*)$ such that $u_2 \leq y$. Then

$$u_1 = \bigvee\{x \in \mathsf{K}(A^*) : x \leq u_1\} \leq \bigwedge\{y \in \mathsf{O}(A^*) : u_2 \leq y\} = u_2,$$

i.e., $u_1 \leq u_2$. Similarly, we have $u_2 \leq u_1$. Therefore, f^* is injective. □

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