Algebraic logic for the negation fragment of classical logic

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Abstract

The general aim of this article is to study the negation fragment of classical logic within the framework of contemporary (Abstract) Algebraic Logic. More precisely, we shall find the three classes of algebras that are canonically associated with a logic in Algebraic Logic, i.e. we find the classes Alg*, Alg and the intrinsic variety of the negation fragment of classical logic. In order to achieve this, firstly, we propose a Hilbert-style axiomatization for this fragment. Then, we characterize the reduced matrix models and the full generalized matrix models of this logic. Also, we classify the negation fragment in the Leibniz and Frege hierarchies.

Keywords: Classical logic, classical negation, algebraic logic, reduced matrix models, full models.

1 Introduction

It is clear that the negation fragment of classical propositional logic (from now on CPL) is a very inexpressive logic from the syntactic point of view. One can only consider up to equivalence a proposition p and its negation $\neg p$, and no other compound sentential can be built up. However, this fragment has several algebra-based semantics (in the different senses of this term) that are not so trivial. We intend to describe these algebra-based semantics of the negation fragment of CPL from the point of view of algebraic logic.

In the framework of (abstract) algebraic logic, there are essentially three classes of algebras associated with a propositional logic. These classes are obtained from different procedures, which intend to be a kind of generalization or abstraction of the Tarski–Lindenbaum method applied to classical (intuitionistic) propositional logic. The class $\mathrm{Alg}^*(\mathcal{S})$ is the class of algebras that is canonically associated with a logic $\mathcal{S}$ according to the theory of logical matrices. More precisely, $\mathrm{Alg}^*(\mathcal{S})$ is the class of the algebraic reducts of the reduced matrix models of $\mathcal{S}$. The intrinsic variety of a logic $\mathcal{S}$ (denoted by $\mathbb{V}(\mathcal{S})$) is defined as the variety generated by a quotient algebra on the formula algebra. And the class $\mathrm{Alg}(\mathcal{S})$ is determined from the reduced generalized matrix models, i.e. $\mathrm{Alg}(\mathcal{S})$ is the class of algebraic reducts of the reduced g-models of $\mathcal{S}$. We refer the reader to [3, 5, 6] for the specific definitions. In general, for any logic $\mathcal{S}$, we always have that $\mathrm{Alg}^*(\mathcal{S}) \subseteq \mathrm{Alg}(\mathcal{S}) \subseteq \mathbb{V}(\mathcal{S})$. In many cases, the three classes coincide, and in many others cases the inclusions are proper. From the point of view of algebraic logic, the class of algebras which is more representative for a logic $\mathcal{S}$ or which can be considered as the *algebraic counterpart* of $\mathcal{S}$ is the class $\mathrm{Alg}(\mathcal{S})$. We refer the reader to [3, 5, 6] for a deeper explanation of the aims and goals of algebraic logic and the corresponding algebra-based semantics. We intend to obtain the classes Alg*, the intrinsic variety and the class Alg for the negation fragment of CPL.

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The article is organized as follows. Section 2 is about the significance of the negation fragment of CPL, and we try to justify why it is important to study algebraically this fragment. In Section 3, we introduce the very basics to start with the work. Throughout the paper, we will introduce the needed concepts and results of algebraic logic. We assume that the reader is familiar with the very basics of algebraic logic, for instance, with the notions of propositional logic (or sentential logic), logical filter, a theory of a logic, logical matrices, generalized matrices, etc. We shall provide all those notions that might be less usual for the reader. We refer the reader to [3, 5] for further information on algebraic logic. Section 4 presents a Hilbert-style system for the negation fragment of CPL and shows a completeness theorem. In Section 5, we characterize the Leibniz-reduced matrix models and we describe the class Alg* of the negation fragment. Section 6 is devoted to obtaining the intrinsic variety of the negation fragment. In Section 7, we characterize the full g-models of the negation fragment. In order to obtain this, we describe the logical filters generated by a set on an arbitrary algebra and present several properties of these logical filters. Then, we find the class Alg of the negation fragment. Finally, in Section 8, we classify the negation fragment in the Leibniz and Frege hierarchies.

2 Significance

As noted in the introduction, the negation fragment of CPL is a very simple logic since its algebraic language has only one unary connective. In this section, we want to justify and convince the reader about the importance of having an algebraic description of this fragment. So, we ask ourselves, what is the algebraic counterpart, under the Algebraic Logic ([3, 5, 6]) point of view, of the negation fragment of CPL? This question was addressed in the literature for the others fragments of CPL, see Table 1. It is also important to answer the above question for the negation fragment beyond its simplicity.

In order to study propositional logics is often important to have an axiomatization, for instance, a Hilbert-style or Gentzen-style axiomatization. By [12], it is known that the negation fragment of CPL has a finite Hilbert-style axiomatization. But there it isn't explicitly presented. However, it is known from the folklore that the Hilbert-style rules $x, \neg x \vdash y$, $x \vdash \neg\neg x$ and $\neg\neg x \vdash x$ are an axiomatization of the negation fragment of CPL. As far as we know, there isn't in the literature an argumentation of this. Here we present one.

As mentioned in the introduction, for the negation fragment of CPL, we described the three classes of algebras that are naturally associated with a propositional logic in Algebraic Logic. In spite of the syntactical simplicity of the negation fragment of CPL, we show that these three classes of algebras are different and they are not so simple. In particular, we characterize the class Alg for this fragment. We notice that this class of algebras was recently obtained in [10] using the concept of Suszko-reduced matrix. In this paper, we follow an alternative path to describe the class Alg. We use the Tarski-reduced full g-models. On the one hand, a logical matrix is a pair $\langle A, F\rangle$ where A is an algebra (over a corresponding algebraic language) and $F \subseteq A$. On the other hand, a generalized matrix (g-matrix) is a pair $\langle A, \mathcal{C}\rangle$ where A is an algebra and $\mathcal{C}$ is a closure system on A. Both structures serve to establish algebra-based semantics for propositional logics (see Sections 5 and 7). These two algebra-based semantics have their differences, and regarding the more general difference between them, we want to quote Font and Jansana (quoted from [5]):

> 'Since an abstract logic can be viewed as a "bundle" or family of matrices, one might think that the new models are essentially equivalent to the old ones; but we believe, after an overall appreciation of the work done in this area, that it is precisely the treatment of an abstract logic

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TABLE 1. The fragments of CPL and their classes of algebras

Classes Fragments	$\mathrm{Alg}^*(\mathcal{S}_N)$	$\mathrm{Alg}(\mathcal{S}_N)$	$\mathbb{V}(\mathcal{S}_N)$	For instance, see
$\{\neg\}$	$\mathbb{I}(A_1, A_2, A_3)$	$A \models x \approx \neg\neg x$ and $A \models (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)$	$A \models x \approx \neg\neg x$	
$\{\wedge\}$	$\mathbb{I}(\mathbf{1}_\wedge, \mathbf{2}_\wedge)$	(meet)-Semilattices	(meet)-Semilattices	[7] (also [3, pp. 208–209])
$\{\vee\}$	(Join)-Semilattices with 1 + $a < b \implies \exists c(a \vee c \neq 1 \;\&\; b \vee c = 1)$	(join)-Semilattices	(join)-Semilattices	[7]
$\{\to\}$ or $\{\vee, \to\}$	Implication algebras (or Tarski algebras)	Implication algebras (or Tarski algebras)	Implication algebras (or Tarski algebras)	[11] (also [3, p. 85])
$\{\wedge, \vee\}$	Distributive lattices with 1 + $a < b \implies \exists c(a \vee c \neq 1 \;\&\; b \vee c = 1)$	Distributive lattices	Distributive lattices	[4, 8]
$\{\wedge, \to\}$ or $\{\wedge, \vee, \to\}$	Relatively complemented distributive lattices with upper bound 1	Relatively complemented distributive lattices with upper bound 1	Relatively complemented distributive lattices with upper bound 1	
$\{\wedge, \neg\}$, $\{\vee, \neg\}$ or $\{\to, \neg\}$	Boolean algebras	Boolean algebras	Boolean algebras	[3]
$\{\leftrightarrow\}$	Boolean groups	Boolean groups	Boolean groups	[1, p. 57] (also [3, p. 131])

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as a single object what gives rise to a useful—and beautiful—mathematical theory, able to explain the connections, not only at the logical level but at the metalogical level, between a sentential logic and the particular class of models we associate with it, namely the class of its full models.'

Moreover, to justify why characterize the Tarski congruence and the Tarski-reduced full g-models rather than the Suszko congruence and the Suszko-reduced models, respectively, we quote Font et al. (quoted from [6, p. 73]):

'The Suszko congruences appear as particular cases of the Tarski congruence [. . .].'

In this article, we obtain a description of the full g-models for the negation fragment of CPL, showing that they are not simple at all. In order to convince the reader why to characterize the full g-models of the negation fragment, which seem to be more complex than the negation fragment itself, we quote Font and Jansana in [5, p. 3]:

'We associate with each sentential logic $\mathcal{S}$ a class of abstract logics called *the full models of* $\mathcal{S}$ [. . .] with the conviction that (some of) the interesting metalogical properties of the sentential logic are precisely those shared by its full models. [. . .] And we claim that *the notion of full model is a "right" notion of model of a sentential logic* [. . .].'

Beyond the scope of this article, we want to mention that there is also a great interest in studying negation from the philosophical, linguistics, artificial intelligence and logic programming point of view. We refer the reader to [9] where there is a compendium of articles studying negation from different perspectives addressed to the question: What is negation? For instance, in [2] Dunn discusses several properties that a negation can have: $\varphi \vdash \psi$ only if $\neg\psi \vdash \neg\varphi$ (*contraposition)*; $\varphi \vdash -\neg\psi$ (*Galois double negation*); $\varphi \vdash \neg\neg\varphi$ (*constructive double negation*); $\varphi \vdash \neg\psi$ only if $\psi \vdash \neg\varphi$ (*constructive contraposition*); $\varphi \vdash \psi$ and $\varphi \vdash \neg\psi$ only if $\varphi \vdash \chi$ (*absurdity*); $\neg\neg\varphi \vdash \varphi$ (*classical double negation*); $\neg\varphi \vdash \psi$ only if $\neg\psi \vdash \varphi$ (*classical contraposition*). These properties are considered in different contexts. In [2], Dunn studies several connections between different treatments of the semantics of negation in non-classical logics: the Kripke definition of negation for intuitionistic logic, the Goldblatt's semantics for negation in orthologic, the definition of De Morgan negation in relevant logic, the four-valued semantics of De Morgan negation and the star semantics. Dunn provides a detailed correspondence-theoretic classification of various notions of negation in terms of properties of a binary relation interpreted as incompatibility.

3 The $\{\neg\}$-fragment of classical propositional logic (CPL)

Throughout what follows, we establish the following simple conventions. Given a function $f : X \to Y$ and $A \cup \{x\} \subseteq X$, we denote fx instead of $f(x)$ and $fA = \{fa : a \in A\}$.

Let $\mathcal{L} = \{\neg\}$ be an algebraic language of type (1) and let Fm be the algebra of formulas over the language $\mathcal{L}$ and generated by a countably infinite set Var. Unless otherwise stated, all the algebras considered in the paper are defined over the algebraic language $\mathcal{L}$. Let us denote by $\mathcal{S}_N = \langle \mathrm{Fm}, \vdash_N \rangle$ the $\{\neg\}$-fragment of CPL, where $\vdash_N$ is the corresponding consequence relation. Let $\mathbf{2}_\neg = \langle \{0, 1\}, \neg \rangle$ be the $\{\neg\}$-reduct of the two-element Boolean algebra $\mathbf{2}_B$. Then, it is clear that for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_N \varphi \quad \Longleftrightarrow \quad \forall h \in \mathrm{Hom}(\mathrm{Fm}, \mathbf{2}_\neg)(h\Gamma \subseteq \{1\} \implies h\varphi = 1).$$

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Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $x \in \mathrm{Var}$. For each $n \in \mathbb{N}_0$, we define recursively $\neg^n x$ as usual:

$$\begin{cases} \neg^0 x & = x \\ \neg^n x & = \neg\neg^{n-1} x \quad \forall n \geq 1. \end{cases}$$

LEMMA 3.1
For every $\alpha \in \mathrm{Fm}$, there is $x \in \mathrm{Var}$ and $n \in \mathbb{N}_0$ such that $\alpha = \neg^n x$.

PROOF. It is straightforward because Fm is the absolutely free algebra of type $\{\neg\}$ over Var. □

4 Hilbert-style axiomatization for the $\{\neg\}$-fragment of CPL

In [12, Theorem 1], it is claimed that every two-valued logic (a logic defined as usual by a two-valued matrix $M = \langle A, \{1\}\rangle$, i.e. $A = \{0, 1\}$ is a two-element algebra) has a finite Hilbert-style axiomatization. Hence, it follows that $\mathcal{S}_N$ has a finite Hilbert-style axiomatization. In [12, p. 322], it is mentioned that a Hilbert-style axiomatization for $\mathcal{S}_N$ is easily established, but it isn't explicitly presented. It is part of the folklore that the negation fragment of CPL is axiomatized by the following rules: $x, \neg x \vdash y$, $x \vdash \neg\neg x$ and $\neg\neg x \vdash x$. We give a proof of it for the lack of proper reference.

DEFINITION 4.1
Let $\mathcal{S}_\neg = \langle \mathrm{Fm}, \vdash_\neg\rangle$ be the propositional logic defined, as usual, by the following Hilbert-style system:

(R1) $x, \neg x \vdash y$

(R2) $x \vdash \neg\neg x$

(R3) $\neg\neg x \vdash x$.

Then, our goal is to show that the logics $\mathcal{S}_N$ and $\mathcal{S}_\neg$ coincide. To this end, we need some auxiliary results. Given a propositional logic $\mathcal{S} = \langle \mathrm{Fm}, \vdash\rangle$, a subset of formulas Γ is said to be *inconsistent* if $\Gamma \vdash \alpha$ for all $\alpha \in \mathrm{Fm}$. Otherwise, Γ is said to be *consistent*.

PROPOSITION 4.2
Let $\Gamma \subseteq \mathrm{Fm}$ be consistent. If $\Gamma \vdash_\neg \alpha$, then there is $\gamma \in \Gamma$ such that $\gamma \vdash_\neg \alpha$.

PROOF. Suppose that $\Gamma \vdash_\neg \alpha$. We proceed by induction on the length of the proofs from Γ. That is, we prove that for all $n \in \mathbb{N}$, if $\alpha_1, \dots, \alpha_n$ is a proof from Γ, then there is $\gamma \in \Gamma$ such that $\gamma \vdash_\neg \alpha_n$.

If $n = 1$, then α_1 is a proof from Γ. Then $\alpha_1 \in \Gamma$. Hence, there is $\gamma := \alpha_1 \in \Gamma$ such that $\gamma \vdash_\neg \alpha_1$. Now suppose that for each proof $\alpha_1, \dots, \alpha_k$ from Γ of length $k < n$, there is $\gamma \in \Gamma$ such that $\gamma \vdash_\neg \alpha_k$. Let $\alpha_1, \dots, \alpha_n$ be a proof from Γ. Then, $\alpha_n \in \Gamma$ or there is $i < n$ such that α_n is derivable from the rules (R2) or (R3) and α_i (notice that α_n cannot be derivable from the rule (R1) because Γ is consistent). Thus, in any case, $\alpha_i \vdash_\neg \alpha_n$. By inductive hypothesis, there is $\gamma \in \Gamma$ such that $\gamma \vdash_\neg \alpha_i$. Then $\gamma \vdash_\neg \alpha_n$. □

PROPOSITION 4.3
Let $\alpha, \beta \in \mathrm{Fm}$. Then, $\alpha \vdash_\neg \beta$ if and only if there is $x \in \mathrm{Var}$ and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\beta = \neg^k x$ and $n \equiv_2 k$.

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PROOF. ($\Rightarrow$) Assume that $\alpha \vdash_\neg \beta$. We proceed by induction on the length of the proofs from α. That is, we prove that for all $m \in \mathbb{N}$, if $\alpha_1, \ldots, \alpha_m$ is a proof from α, then there is $x \in$ Var and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\alpha_m = \neg^k x$ and $n \equiv_2 k$. If $m = 1$, then $\alpha = \alpha_1$. From Lemma 3.1, we have $\alpha = \neg^n x = \alpha_1$ for some $x \in$ Var and $n \in \mathbb{N}_0$. Now suppose that it holds for all $i < m$. Let $\alpha_1, \ldots, \alpha_m$ be a proof from α. Then, $\alpha_m = \alpha$ or α_m is derivable from α_i with $i < m$ by an application of (R2) or (R3) (it cannot be derivable by (R1); otherwise, there are $\alpha_i = \beta$ and $\alpha_j = \neg\beta$ with $i, j < m$. Then by inductive hypothesis there is a variable x and $n, m, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\alpha_i = \beta = \neg^m x$, $\alpha_j = \neg\beta = \neg^k x$, $n \equiv_2 m$ and $n \equiv_2 k$. Hence, $m \equiv_2 k$ and $m + 1 = k$, a contradiction). It is straightforward when $\alpha_m = \alpha$. Suppose that α_m is derivable from α_i with $i < m$ by an application of (R2) or (R3). By inductive hypothesis, there is $x \in$ Var and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\alpha_i = \neg^k x$ and $n \equiv_2 k$. On the one hand, if α_m is derivable from (R2), then $\alpha_m = \neg\neg\alpha_i$. Hence, $\alpha_m = \neg\neg\alpha_i = \neg^{k+2} x$ and $n \equiv_2 k + 2$. On the other hand, if α_m is derivable from (R3), then $\alpha_i = \neg\neg\alpha_m$. Since $\neg\neg\alpha_m = \alpha_i = \neg^k x$, we have that $k \geq 2$. Then $\alpha_m = \neg^{k-2} x$ and $k - 2 \geq 0$. Hence, $\alpha = \neg^n x$, $\alpha_m = \neg^{k-2} x$ and $n \equiv_2 k - 2$.

($\Leftarrow$) Suppose that there are $x \in$ Var and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\beta = \neg^k x$ and $n \equiv_2 k$. Suppose that $n \leq k$. Thus $0 \leq k - n = 2q$, for some $q \in \mathbb{N}_0$. Then, we have $\alpha = \neg^n x \vdash_\neg \neg^{n+2} x \vdash_\neg \cdots \vdash_\neg \neg^{n+2q} x = \neg^k x = \beta$ (this can be proved by induction). If $n > k$, then $n - 2q = k$. Hence, $\alpha = \neg^n x \vdash_\neg \neg^{n-2} x \vdash_\neg \cdots \vdash_\neg \neg^{n-2q} x = \beta$. ☐

REMARK 4.4
Notice that for all $\alpha, \beta \in$ Fm, $\alpha \vdash_\neg \beta$ if and only if $\beta \vdash_\neg \alpha$. In other words, $\alpha \vdash_\neg \beta$ if and only if $\alpha \dashv\vdash_\neg \beta$.

COROLLARY 4.5
Let $\Gamma \cup \{\alpha, \beta\} \subseteq$ Fm.

(1) $\alpha \vdash_\neg \beta \implies \neg\alpha \nvdash_\neg \beta$.
(2) $\alpha \vdash_\neg \beta \iff \neg\beta \vdash_\neg \neg\alpha$.
(3) $\Gamma, \alpha \vdash_\neg \beta$ and $\Gamma, \neg\alpha \vdash_\neg \beta \implies \Gamma \vdash_\neg \beta$.

PROOF. (1) Suppose $\alpha \vdash_\neg \beta$. Thus, there are $x \in$ Var and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\beta = \neg^k x$ and $n \equiv_2 k$. Then $\neg\alpha = \neg^{n+1} x$ and $n + 1 \not\equiv_2 k$. Hence $\neg\alpha \nvdash_\neg \beta$.

(2) ($\Rightarrow$) Suppose that $\alpha \vdash_\neg \beta$. Then, there are $x \in$ Var and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$, $\beta = \neg^k x$ and $n \equiv_2 k$. Thus $\neg\alpha = \neg^{n+1} x$, $\neg\beta = \neg^{k+1} x$ and $n + 1 \equiv_2 k + 1$. Hence $\neg\beta \vdash_\neg \neg\alpha$. ($\Leftarrow$) It is straightforward by the above and by rules (R2) and (R3).

(3) Assume that $\Gamma, \alpha \vdash_\neg \beta$ and $\Gamma, \neg\alpha \vdash_\neg \beta$. Suppose that $\Gamma \nvdash_\neg \beta$. Then, by Proposition 4.2, it follows that $\alpha \vdash_\neg \beta$ and $\neg\alpha \vdash_\neg \beta$. This is a contradiction by property (1). Hence, $\Gamma \vdash_\neg \beta$. ☐

PROPOSITION 4.6
Let $\alpha \in$ Fm and let Δ be a maximal theory relative to α of the logic $\mathcal{S}_\neg$ (i.e. Δ is a maximal theory among all the consistent theories not containing α). Then, for all $\beta \in$ Fm, $\beta \in \Delta$ iff $\neg\beta \notin \Delta$.

PROOF. Let $\beta \in$ Fm.

($\Rightarrow$) Assume $\beta \in \Delta$. Since Δ is consistent ($\alpha \notin \Delta$), it follows by rule (R1) that $\neg\beta \notin \Delta$.

($\Leftarrow$) Suppose that $\neg\beta \notin \Delta$. We suppose, towards a contradiction, that $\beta \notin \Delta$. Let Γ_β and $\Gamma_{\neg\beta}$ be the theories generated by $\Delta \cup \{\beta\}$ and $\Delta \cup \{\neg\beta\}$, respectively. Since $\Delta \subset \Gamma_\beta$ and $\Delta \subset \Gamma_{\neg\beta}$, it

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follows by the maximality of Δ that $\alpha \in \Gamma_\beta \cap \Gamma_{\neg\beta}$. Then

$$\Delta, \beta \vdash \alpha \qquad \text{and} \qquad \Delta, \neg\beta \vdash \alpha.$$

Hence, by (3) of Corollary 4.5, we obtain that $\Delta \vdash \alpha$, which is a contradiction. Therefore, $\beta \in \Delta$.$\square$

Now we are ready to show that the rules (R1)–(R3) are an axiomatization for $\mathcal{S}_N$.

THEOREM 4.7
The Hilbert calculus formed by the rules (R1), (R2) and (R3) is an axiomatization for the $\{\neg\}$-fragment of CPL.

PROOF. We need to show that $\vdash_N = \vdash_\neg$. Recall that $\vdash_N$ is defined by the matrix $\langle \mathbf{2}_\neg, \{1\}\rangle$.

($\vdash_\neg \subseteq \vdash_N$) (Soundness) This is a routine argument.

($\vdash_N \subseteq \vdash_\neg$) Suppose that $\Gamma \nvdash_\neg \alpha$. By [3, Lem. 1.43], there is a theory Δ such that $\Gamma \subseteq \Delta$, $\alpha \notin \Delta$, and Δ is maximal relative to α. We define $v\colon \mathrm{Fm} \to \mathbf{2}_\neg$ as follows: for all $\varphi \in \mathrm{Fm}$, $v\varphi = 1 \iff \varphi \in \Delta$. By Proposition 4.6, we obtain that v is a homomorphism such that $v\Gamma \subseteq \{1\}$ and $v\alpha = 0$. Hence, $\Gamma \nvdash_N \alpha$. $\square$

5 Reduced models and the class $\mathbf{Alg}^*(\mathcal{S}_N)$

In this section, we characterize the reduced matrix models of the logic $\mathcal{S}_N$ and we describe the class $\mathrm{Alg}^*(\mathcal{S}_N)$. We recall some needed notions.

Recall that a *logical matrix* (*matrix* for short) is a pair $\langle A, F\rangle$ where A is an algebra and $F \subseteq A$. The *Leibniz congruence*, denoted by $\Omega^A F$, of a matrix $\langle A, F\rangle$ can be defined as follows (see [3, Theo. 4.23]): for all $a, b \in A$,

$$(a,b) \in \Omega^A F \iff \text{for all } \delta(x, \vec{z}) \in \text{ Fm and all } \vec{c} \in \vec{A}, \quad [\delta^A(a, \vec{c}) \in F \iff \delta^A(b, \vec{c}) \in F]. \tag{5.1}$$

A matrix $\langle A, F\rangle$ is said to be *reduced* when $\Omega^A F = \mathrm{Id}_A$. Recall that a matrix $\langle A, F\rangle$ is a *model* of a logic $\mathcal{S}$ when for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{S}} \alpha \implies \forall h \in \mathrm{Hom}(\mathrm{Fm}, A)(h\Gamma \subseteq F \implies h\alpha \in F). \tag{5.2}$$

Then, the class $\mathrm{Alg}^*(\mathcal{S})$ is defined as follows;

$$\mathrm{Alg}^*(\mathcal{S}) = \{A : \text{ there is some} F \subseteq A \text{ such that } \langle A, F\rangle \text{ is a reduced model of } \mathcal{S}\}.$$

Recall also that a subset F of an algebra A is called an $\mathcal{S}$-*filter* of A if condition (5.2) is satisfied. That is, a subset $F \subseteq A$ is an $\mathcal{S}$-filter of A if and only if the matrix $\langle A, F\rangle$ is a model of $\mathcal{S}$.

By (5.1) and taking into account that every formula $\alpha \in \mathrm{Fm}$ is of the form $\alpha = \neg^n x$ for some $x \in \mathrm{Var}$ and $n \in \mathbb{N}_0$, we obtain the following characterization of the Leibniz congruences.

PROPOSITION 5.1
For every algebra A and every $F \subseteq A$,

$$(a,b) \in \Omega^A F \iff \forall n \in \mathbb{N}_0(\neg^n a \in F \iff \neg^n b \in F)$$

for all $a, b \in A$.

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PROPOSITION 5.2
Let A be an algebra and $F \subseteq A$. Then, F is an $\mathcal{S}_N$-filter if and only if the following conditions hold:

(1) $a, \neg a \in F \implies F = A$;
(2) $a \in F \iff \neg\neg a \in F$.

PROOF. It is straightforward by rules (R1)–(R3). □

We are ready to characterize the reduced models of $\mathcal{S}_N$.

THEOREM 5.3
Let $\langle A, F\rangle$ be a matrix. Then, $\langle A, F\rangle$ is a reduced model of $\mathcal{S}_N$ if and only if the following conditions hold:

(1) $A \models x \approx \neg\neg x$,
(2) $F = \{a_0\}$ for some $a_0 \in A$ such that $a_0 \neq \neg a_0$, and
(3) $2 \leq |A| \leq 3$.

PROOF. ($\Rightarrow$) Assume that $\langle A, F\rangle$ is a reduced model of $\mathcal{S}_N$. Thus, F is an $\mathcal{S}_N$-filter and $\Omega^A F = \mathrm{Id}_A$.
(1) Let $a \in A$. From Proposition 5.2, we obtain

$$\forall n \in \mathbb{N}_0(\neg^n a \in F \iff \neg^n(\neg\neg a) \in F).$$

Then, $(a, \neg\neg a) \in \Omega^A F$. Hence $a = \neg\neg a$.
(2) We are assumed that the algebra A is not trivial, i.e. $|A| \geq 2$. Then, $F \neq \emptyset$ (otherwise $\Omega^A F = A \times A \neq \Delta_A$). Let $a, b \in F$. By (1), we have that $a = \neg^{2k} a$ and $b = \neg^{2k} b$ for all $k \in \mathbb{N}_0$. Thus, $\neg^{2k} a, \neg^{2k} b \in F$, for all $k \in \mathbb{N}_0$. On the other hand, since $a, b \in F$ and F is proper (otherwise $\langle A, F\rangle$ is not reduced), it follows by Proposition 5.2 that $\neg a, \neg b \notin F$. Then, $\neg^{2k+1} a, \neg^{2k+1} b \notin F$, for all $k \in \mathbb{N}_0$. Hence, we have that $\forall n \in \mathbb{N}_0(\neg^n a \in F \iff \neg^n b \in F)$. Thus, $(a, b) \in \Omega^A F$. Then, $a = b$. Therefore, $F = \{a_0\}$ for some $a_0 \in A$. Moreover, $a_0 \neq \neg a_0$. Otherwise, $a_0 = \neg a_0 \in F$, and by Proposition 5.2 we have $F = A$, which is a contradiction because A is not trivial.
(3) Since A is not trivial, we have $|A| \geq 2$. By (2), we have that $F = \{a_0\}$ and $a_0 \neq \neg a_0$. Let $a, b \in A$ be such that $a, b \notin \{a_0, \neg a_0\}$. Then, $a, b, \neg a, \neg b \notin F = \{a_0\}$. Thus, it holds that $a \in F \iff b \in F$ and $\neg a \in F \iff \neg b \in F$. Now by (1), it follows that $\forall n \in \mathbb{N}_0(\neg^n a \in F \iff \neg^n b \in F)$. Then $(a, b) \in \Omega^A F$. Hence, $a = b$. Therefore, $|A| \leq 3$.

($\Leftarrow$) Let $\langle A, F\rangle$ be a matrix such that satisfies (1)–(3). By (1) and (2), it follows that $F = \{a_0\}$ is an $\mathcal{S}_N$-filter, i.e. $\langle A, F\rangle$ is a model of $\mathcal{S}_N$. Let us see that $\langle A, F\rangle$ is reduced. Let $a, b \in A$ and assume that $(a, b) \in \Omega_A F$. Thus, we have $a \in F \iff b \in F$ and $\neg a \in F \iff \neg b \in F$. If $a \in F$ or $\neg a \in F$, then $a = b$. Similarly, if $b \in F$ or $\neg b \in F$, then $a = b$. Now suppose that $a, b, \neg a, \neg b \notin F = \{a_0\}$. Thus $a, b \notin \{a_0, \neg a_0\}$. Since $|A| \leq 3$, we obtain that $a = b$. Hence, $\Omega^A F = \Delta_A$. Therefore, $\langle A, F\rangle$ is a reduced model of $\mathcal{S}_N$. □

Therefore, we have that all non-trivial reduced matrix models $\langle A, F\rangle$ are of the form, up to isomorphism:

1. $A_1 = \{a\}$ and $F = \emptyset$;
2. $A_2 = \{a, b\}$ such that $\neg a = b$ and $\neg b = a$, with $F = \{a\}$; or
3. $A_3 = \{a, b, c\}$ such that $\neg a = b$, $\neg b = a$ and $\neg c = c$, with $F = \{a\}$.

COROLLARY 5.4
$\mathrm{Alg}^*(\mathcal{S}_N) = \mathbb{I}(\{A_1, A_2, A_3\})$.

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6 The intrinsic variety of $\mathcal{S}_N$

If one insists on having a variety associated with a logic $\mathcal{S}$, the *intrinsic variety* of $\mathcal{S}$ might be the adequate choice. The significance of the intrinsic variety as an algebra-based semantics for a logic is in general weak, because no general theory asserts that the algebraic counterpart of a logic should be a variety. In order to define the intrinsic variety and for what follows in the article, we recall some needed concepts.

Recall that a *generalized matrix* (*g-matrix*) is a pair $\langle A, \mathcal{C}\rangle$ where A is an algebra and $\mathcal{C}$ is a closure system on A. By the correspondence between closure systems and closure operators, we can also consider a g-matrix as a pair $\langle A, C\rangle$ where C is a closure operator on A.

The *Tarski congruence* of a g-matrix $\langle A, C\rangle$, denoted by $\widetilde{\Omega}^A C$, is defined as the largest congruence θ on A satisfying the following

$$(a, b) \in \theta \implies C(a) = C(b).$$

Given a logic $\mathcal{S}$, let us denote by $\mathcal{T}h(\mathcal{S})$ the closure system of all theories of $\mathcal{S}$. Consider the Tarski congruence $\widetilde{\Omega}^{\mathrm{Fm}}\mathcal{T}h(\mathcal{S})$ of the g-matrix $\langle \mathrm{Fm}, \mathcal{T}h(\mathcal{S})\rangle$. The *intrinsic variety* of $\mathcal{S}$, denoted by $\mathbb{V}(\mathcal{S})$, is defined as the variety generated by the quotient algebra $\mathrm{Fm}/\widetilde{\Omega}^{\mathrm{Fm}}\mathcal{T}h(\mathcal{S})$. That is, $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\mathrm{Fm}/\widetilde{\Omega}^{\mathrm{Fm}}\mathcal{T}h(\mathcal{S}))$. Now, since the interderivability relation $\dashv\vdash_N$ is a congruence on Fm (i.e. the logic $\mathcal{S}_N$ is self-extensional [3]), we have that

$$\mathbb{V}(\mathcal{S}_N) \models \alpha \approx \beta \iff \mathrm{Fm}/\widetilde{\Omega}^{\mathrm{Fm}}\mathcal{T}h(\mathcal{S}) \models \alpha \approx \beta \iff (\alpha, \beta) \in \widetilde{\Omega}^{\mathrm{Fm}}\mathcal{T}h(\mathcal{S}) \iff \alpha \dashv\vdash_N \beta.$$

Theorem 6.1
$\mathbb{V}(\mathcal{S}_N) = \{A : A \models x \approx \neg\neg x\}$.

Proof. Let $V := \{A : A \models x \approx \neg\neg x\}$. Notice that if $\alpha = \neg^n x$, then for all $h \in \mathrm{Hom}(\mathrm{Fm}, A)$ with $A \in V$, it follows that

$$\begin{cases} h\alpha = h(\neg^n x) = hx & \text{if } n \text{ is even} \\ h\alpha = h(\neg^n x) = \neg hx & \text{if } n \text{ is odd.} \end{cases} \tag{6.1}$$

Let $\alpha, \beta \in \mathrm{Fm}$. Let us prove that

$$V \models \alpha \approx \beta \iff \exists x \in \mathrm{Var},\ \exists n, k \in \mathbb{N}_0 \text{ such that } \alpha = \neg^n x, \beta = \neg^k x \text{ and } n \equiv_2 k.$$

Suppose that $V \models \alpha \approx \beta$. By Lemma 3.1, there are $x, y \in \mathrm{Var}$ and $n, k \in \mathbb{N}_0$ such that $\alpha = \neg^n x$ and $\beta = \neg^k y$. If $x \neq y$, then taking $A = \{a, b\}$ with $\neg a = a$ and $\neg b = b$, we obtain that $A \in V$ and $A \not\models \alpha \approx \beta$. A contradiction. Hence, $x = y$. Thus, $\alpha = \neg^n x$ and $\beta = \neg^k x$. Suppose that $n \not\equiv_2 k$. Thus, n is even and k is odd (or vice versa). Taking $A_2 = \{a, b\}$ as in page 8 and $hx = a$, we obtain by (6.1) that

$$h\alpha = h(\neg^n x) = hx = a \quad \text{and} \quad h\beta = h(\neg^k x) = \neg hx = \neg a = b.$$

Hence $A \not\models \alpha \approx \beta$, which is a contradiction. Then $n \equiv_2 k$.

Now assume that $\alpha = \neg^n x$, $\beta = \neg^k x$ and $n \equiv_2 k$, for some $x \in \mathrm{Var}$ and $n, k \in \mathbb{N}_0$. Let $A \in V$ and $h \in \mathrm{Hom}(\mathrm{Fm}, A)$. We have that n and k are both even or odd. In any case, by (6.1), we obtain that $h\alpha = h\beta$. Hence $V \models \alpha \approx \beta$.

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Hence, for all $\alpha, \beta \in \mathrm{Fm}$, we have

$$\begin{aligned}\mathbb{V}(\mathcal{S}_N) \models \alpha \approx \beta &\iff \alpha \dashv\vdash_N \beta \\ &\iff \exists x \in \mathrm{Var},\ \exists n, k \in \mathbb{N}_0 \ \text{ s.t. } \alpha = \neg^n x,\ \beta = \neg^k x \text{ and } n \equiv_2 k \\ &\iff V \models \alpha \approx \beta.\end{aligned}$$

Therefore, $\mathbb{V}(\mathcal{S}_N) = \{A : A \models x \approx \neg\neg x\}$. □

7 Full g-models for $\mathcal{S}_N$ and the class Alg($\mathcal{S}_N$)

In this section, we obtain the class Alg($\mathcal{S}_N$), which is considered in the framework of algebraic logic as *the* algebraic counterpart of $\mathcal{S}_N$. To this end, we need first to obtain a characterization of the full g-models of $\mathcal{S}_N$. The class of full g-models of a logic $\mathcal{S}$ is a particular class of g-models of $\mathcal{S}$ that behave particularly well, in the sense that some interesting metalogical properties of a propositional logic are precisely those shared by its full g-models. Then, the class Alg($\mathcal{S}$) is obtained as the algebraic reducts of the reduced full g-models of $\mathcal{S}$. It is claimed (see [5, p. 3]) that the notion of full g-model is the 'right' notion of model of a propositional logic and that the class Alg($\mathcal{S}$) is the 'right' class of algebras to be canonically associated with a propositional logic. We refer the reader to [3, 5, 6] for a more detailed argumentation about the full g-models of a logic $\mathcal{S}$ and the class Alg($\mathcal{S}$).

We notice that the class Alg($\mathcal{S}_N$) was already described in [10] but using the notion of Suszko-reduced matrix models. As we mentioned before, we follow a different path to find the class Alg($\mathcal{S}_N$).

We start presenting some needed notions. A g-matrix $\langle A, C\rangle$ is said to be a *g-model* of a logic $\mathcal{S}$ when for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{S}} \alpha \implies \text{for all } h \in \mathrm{Hom}(\mathrm{Fm}, A),\ h\alpha \in C(h\Gamma).$$

Recall from the previous section the notion of Tarski congruence of a g-matrix. A g-matrix $\langle A, \mathcal{C}\rangle$ is said to be *reduced* if $\widetilde{\Omega}^A \mathcal{C} = \mathrm{Id}_A$. Then, the class Alg($\mathcal{S}$) is defined as follows:

$$\mathrm{Alg}(\mathcal{S}) = \{A : \text{ there is a closure system } \mathcal{C} \text{ on} A \text{ such that } \langle A, \mathcal{C}\rangle \text{ is a reduced g-model of } \mathcal{S}\}.$$

An $h \in \mathrm{Hom}(A_1, A_2)$ is called a *strict homomorphism* from the g-matrix $\langle A_1, C_1\rangle$ to the g-matrix $\langle A_2, C_2\rangle$ when

$$a \in C_1(X) \iff ha \in C_2(hX) \qquad \text{for all } X \cup \{a\} \subseteq A_1.$$

Let A be an algebra. We denote by $\mathcal{F}i_{\mathcal{S}}(A)$ the closure system of all $\mathcal{S}$-filters of the algebra A of a logic $\mathcal{S}$. Given a subset $X \subseteq A$, let us denote by $\mathrm{Fig}^A_{\mathcal{S}}(X)$ the $\mathcal{S}$-filter of A generated by X, i.e. $\mathrm{Fig}^A_{\mathcal{S}}(X)$ is the least $\mathcal{S}$-filter of A containing X.

A g-matrix $\langle A, \mathcal{C}\rangle$ is said to be a *full g-model* of a logic $\mathcal{S}$ if there is an algebra A_1 and a surjective strict homomorphism $h\colon \langle A, \mathcal{C}\rangle \to \langle A_1, \mathcal{F}i_{\mathcal{S}}(A_1)\rangle$. The above g-matrix $\langle A_1, \mathcal{F}i_{\mathcal{S}}(A_1)\rangle$ can be chosen in a particular, significant way:

PROPOSITION 7.1 ([3, Prop. 5.85]).
A g-matrix $\langle A, \mathcal{C}\rangle$ is a full g-model of a logic $\mathcal{S}$ if and only if

$$\{F/\widetilde{\Omega}^A \mathcal{C} : F \in \mathcal{C}\} = \mathcal{F}i_{\mathcal{S}}\left(A/\widetilde{\Omega}^A \mathcal{C}\right).$$

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PROPOSITION 7.2 ([3, Corollary 5.88(3)]).
Given a logic $\mathcal{S}$,

$$\mathrm{Alg}(\mathcal{S}) = \{A : \text{there is a closure system } \mathcal{C} \text{ on } A \text{ such that } \langle A, \mathcal{C}\rangle \text{ is a reduced full g-model of } \mathcal{S}\}.$$

In order to characterize the full g-models of $\mathcal{S}_N$, it will be useful to obtain a characterization of the $\mathcal{S}_N$-filters generated. We start with some basic properties.

PROPOSITION 7.3
Let A be an algebra. Then:

(B1) $\mathrm{Fig}^{A}_{\mathcal{S}_N}(\emptyset) = \emptyset$.
(B2) $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a, \neg a) = A$, for all $a \in A$.
(B3) $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = \mathrm{Fig}^{A}_{\mathcal{S}_N}(\neg\neg a)$.

PROOF. (B1) is a consequence from the fact that the logic $\mathcal{S}_N$ have no theorems, and (B2) and (B3) are straightforward by rules (R1)–(R3). □

Now let us to obtain a characterization of $\mathrm{Fig}^{A}_{\mathcal{S}_N}(X)$. Let $\mathcal{E} = \{n \in \mathbb{N}_0 : n \text{ is even}\}$ and $\mathcal{O} = \{n \in \mathbb{N}_0 : n \text{ is odd}\}$. The next is a key proposition.

PROPOSITION 7.4
Let A be an algebra. Let $B \subseteq A$ be such that the following condition holds:

$$\forall b, b' \in B, \forall (s,t) \in \mathcal{E} \times \mathcal{O} (\neg^s b \neq \neg^t b'). \tag{7.1}$$

Then,

$$\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) = \{a \in A : \exists b \in B, \exists s,t \in \mathcal{E}\, (\neg^s a = \neg^t b)\} \quad \text{and} \quad \mathrm{Fig}^{A}_{\mathcal{S}_N}(B) \neq A.$$

PROOF. Let

$$F = \{a \in A : \exists b \in B, \exists s,t \in \mathcal{E}\, (\neg^s a = \neg^t b)\}.$$

We prove in several steps that $\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) = F$.

• $F \neq A$. If $B = \emptyset$, then $F = \emptyset \neq A$. Suppose that $B \neq \emptyset$, and let $b \in B$. We show that $\neg b \notin F$. Suppose that $\neg b \in F$. Thus, there is $b' \in B$ and $s,t \in \mathcal{E}$ such that $\neg^s(\neg b) = \neg^t b'$. Then $\neg^{s+1} b = \neg^t b'$ with $(t, s+1) \in \mathcal{E} \times \mathcal{O}$, which is a contradiction by (7.1). Hence $\neg b \notin F$. Therefore, $F \neq A$.

• We show that F is an $\mathcal{S}_N$-filter. We need to verify conditions (1) and (2) of Proposition 5.2. (1) Since $F \neq A$, we need to show that for all $a \in A$, $a \notin F$ or $\neg a \notin F$. Let $a \in A$. Suppose that $a, \neg a \in F$. Thus, there are $b, b' \in B$ and $s,t,s',t' \in \mathcal{E}$ such that $\neg^s a = \neg^t b$ and $\neg^{s'}(\neg a) = \neg^{t'} b'$. Then, $\neg^{s+s'+1} a = \neg^{t+s'+1} b$ and $\neg^{s+s'+1} a = \neg^{s+t'} b'$. Hence, $\neg^{t+s'+1} b = \neg^{s+t'} b'$ with $b, b' \in B$ and $(s+t', t+s'+1) \in \mathcal{E} \times \mathcal{O}$, which is a contradiction by (7.1). Thus, $a \notin F$ or $\neg a \notin F$. (2) Let $a \in A$. If $a \in F$, then there is $b \in B$ and $s,t \in \mathcal{E}$ such that $\neg^s a = \neg^t b$. Hence, $\neg^s(\neg\neg a) = \neg^{t+2} b$. By definition of F, we obtain that $\neg\neg a \in F$. Conversely, suppose that $\neg\neg a \in F$. Thus, there is $b \in B$ and $s,t \in \mathcal{E}$ such that $\neg^s(\neg\neg a) = \neg^t b$. Hence, $\neg^{s+2} a = \neg^t b$ and $s+2, t \in \mathcal{E}$. Then $a \in F$. Therefore, it follows by Proposition 5.2 that F is an $\mathcal{S}_N$-filter.

• $B \subseteq F$. It is obvious by definition of F.

• F is the least $\mathcal{S}_N$-filter of A containing B. Let G be an $\mathcal{S}_N$-filter of A such that $B \subseteq G$. Let $a \in F$. Thus, there is $b \in B$ and $s,t \in \mathcal{E}$ such that $\neg^s a = \neg^t b$. By Proposition 4.3, we have that $x \vdash_N \neg^t x$.

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Since $b \in B \subseteq G$ and G is an $\mathcal{S}_N$-filter, it follows that $\neg^t b \in G$. Thus, $\neg^s a \in G$. Since $\neg^s x \vdash_N x$, it follows that $a \in G$. Hence, $F \subseteq G$. Therefore, we have proved that $\mathrm{Fig}^A_{\mathcal{S}_N}(B) = F$. □

PROPOSITION 7.5
Let A be an algebra and $B \subseteq A$. The following conditions are equivalent.

(1) There exist $b, b' \in B$ and $(s,t) \in \mathcal{E} \times \mathcal{O}$ such that $\neg^s b = \neg^t b'$.
(2) $\mathrm{Fig}^A_{\mathcal{S}_N}(B) = A$.

PROOF. (1) $\Rightarrow$ (2) Let $b, b' \in B$ and $(s,t) \in \mathcal{E} \times \mathcal{O}$ such that $\neg^s b = \neg^t b'$. We have that $b, b' \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$. Since s is even, it follows that $x \vdash_N \neg^s x$. Then, $\neg^s b \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$. Thus, $\neg^t b' \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$. Since t is odd, it follows by Proposition 4.3 that $\neg^t x \vdash_N \neg x$. Then $\neg b' \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$. That is, $b', \neg b' \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$. Hence $\mathrm{Fig}^A_{\mathcal{S}_N}(B) = A$.
(2) $\Rightarrow$ (1) It follows by Proposition 7.4. □

From Propositions 7.4 and 7.5, we have that for every algebra A and every $B \subseteq A$,

$$\mathrm{Fig}^A_{\mathcal{S}_N}(B) = A \quad \text{or} \quad \mathrm{Fig}^A_{\mathcal{S}_N}(B) = \{a \in A : \exists b \in B, \exists s,t \in \mathcal{E}\,(\neg^s a = \neg^t b)\}.$$

PROPOSITION 7.6
Let A be an algebra and $b \in A$ such that

$$\forall (s,t) \in \mathcal{E} \times \mathcal{O}\,(\neg^s b \neq \neg^t b). \tag{7.2}$$

Then,

(1) $\mathrm{Fig}^A_{\mathcal{S}_N}(b) = \{a \in A : \exists s,t \in \mathcal{E}\,(\neg^s a = \neg^t b)\}$;
(B4) $a \in \mathrm{Fig}^A_{\mathcal{S}_N}(b) \implies b \in \mathrm{Fig}^A_{\mathcal{S}_N}(a)$;
(B5) $a \in \mathrm{Fig}^A_{\mathcal{S}_N}(b) \implies \neg b \in \mathrm{Fig}^A_{\mathcal{S}_N}(\neg a)$.

PROOF. (1) is an immediate consequence of Proposition 7.4. (B4) and (B5) follow by (1) and from Proposition 4.3. □

Let A be an algebra. Notice from Proposition 7.5 that an element $b \in A$ satisfies condition (7.2) if and only if $\mathrm{Fig}^A_{\mathcal{S}_N}(b) \neq A$. Moreover, if $B \subseteq A$ satisfying condition (7.1), then every element $b \in B$ satisfies condition (7.2). Thus, the next proposition is a consequence of Proposition 7.4 and by (1) of Proposition 7.6.

PROPOSITION 7.7
Let A be an algebra and $B \subseteq A$. Then,

(B6) if $a \in \mathrm{Fig}^A_{\mathcal{S}_N}(B)$ and $\mathrm{Fig}^A_{\mathcal{S}_N}(B) \neq A$, then there is $b \in B$ such that $a \in \mathrm{Fig}^A_{\mathcal{S}_N}(b)$.

PROPOSITION 7.8
Let $\langle A, C \rangle$ be a g-matrix. Then

$$(a,b) \in \widetilde{\Omega}^A C \iff \forall n \in \mathbb{N}_0 (C(\neg^n a) = C(\neg^n b)).$$

PROOF. It follows from the fact that $\widetilde{\Omega}^A C = \bigcap_{F \in C} \Omega^A F$ (see [3, Lem. 5.31]) and by Proposition 5.1. □

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PROPOSITION 7.9
Let $\langle A, C\rangle$ be a g-matrix such that for each $a \in A$, $C(a) = C(\neg\neg a)$. Then, for all $a, b \in A$,

$$(a, b) \in \widetilde{\Omega}^A C \iff C(a) = C(b) \text{ and} C(\neg a) = C(\neg b).$$

PROOF. The implication $\Rightarrow$ follows by the previous proposition.

($\Leftarrow$) Let $a, b \in A$ be such that $C(a) = C(b)$ and $C(\neg a) = C(\neg b)$. We prove that $\forall n \in \mathbb{N}_0 (C(\neg^n a) = C(\neg^n b))$ by induction on n. It is clear that holds for $n = 0$ and $n = 1$. Let $n \geq 2$ and suppose that for all $k < n$, $C(\neg^k a) = C(\neg^k b)$. Since $n \geq 2$, it follows that

$$C(\neg^n a) = C(\neg^{n-2} a) \overset{I.H.}{=} C(\neg^{n-2} b) = C(\neg^n b).$$

Hence, $\forall n \in \mathbb{N}_0 (C(\neg^n a) = C(\neg^n b))$. Therefore $(a, b) \in \widetilde{\Omega}^A C$. □

Now we are ready to present a characterization for the full g-models of the negation fragment $\mathcal{S}_N$.

THEOREM 7.10
Let $\langle A, C\rangle$ be a g-matrix. Then, $\langle A, C\rangle$ is a full g-model of the logic $\mathcal{S}_N$ if and only if it satisfies:

(F1) $C(a, \neg a) = A$, for all $a \in A$.
(F2) $C(a) = C(\neg\neg a)$, for all $a \in A$.
(F3) For all $a, b \in A$, if $a \in C(b)$ and $C(b) \neq A$, then $b \in C(a)$.
(F4) For all $a, b \in A$, if $a \in C(b)$ and $C(b) \neq A$, $\neg b \in C(\neg a)$.
(F5) For all $B \cup \{a\} \subseteq A$, if $a \in C(B)$ and $C(B) \neq A$, then there is $b \in B$ such that $a \in C(b)$.
(F6) For every $B \subseteq A$, if $C(B) = A$, then there are $b, b' \in B$ such that $C(b) = C(\neg b')$.

PROOF. Let $\langle A, C\rangle$ be a g-matrix and let $\mathcal{C}$ be the closure system associated with C.

($\Rightarrow$) Assume that $\langle A, C\rangle$ is a full g-model of the logic $\mathcal{S}_N$. Thus, there is an algebra A_1 and a surjective strict homomorphism $h\colon \langle A, \mathcal{C}\rangle \to \langle A_1, \mathcal{F}i_{\mathcal{S}_N}(A_1)\rangle$. Conditions (F1) and (F2) follow by Proposition 7.3 and from the fact that theses conditions are preserved by surjective strict homomorphisms (see [3, Proposition 5.90]).
(F3) Let $a \in C(b)$ and suppose that $C(b) \neq A$. Since h is a strict homomorphism, it follows that $ha \in \mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hb)$ and $\mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hb) \neq A$. Thus, by Proposition 7.6, we have that $hb \in \mathrm{Fig}^{A_1}_{\mathcal{S}_N}(ha)$. Then $b \in C(a)$.
(F4) It is similar to the proof of (F3).
(F5) Let $a \in C(B)$ and suppose that $C(B) \neq A$. Then, we have that $ha \in \mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hB)$ and $\mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hB) \neq A$. By Proposition 7.7, there is $b \in B$ such that $ha \in \mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hb)$. Then $a \in C(b)$ with $b \in B$.
(F6) Let $B \subseteq A$ be such that $C(B) = A$. Since h is a surjective strict homomorphism, it follows that $\mathrm{Fig}^{A_1}_{\mathcal{S}_N}(hB) = A_1$. By Proposition 7.5, there are $b, b' \in B$ and $(s, t) \in \mathcal{E} \times \mathcal{O}$ such that $\neg^s hb = \neg^t hb'$. Then, $\mathrm{Fig}^{A_1}_{\mathcal{S}_N}(h(\neg^s b)) = \mathrm{Fig}^{A_1}_{\mathcal{S}_N}(h(\neg^t b'))$. Thus, $C(\neg^s b) = C(\neg^t b')$. By (F2), it follows that $C(b) = C(\neg^s b) = C(\neg^t b') = C(\neg b')$.

($\Leftarrow$) Assume that $\langle A, C\rangle$ is a g-matrix satisfying conditions (F1)–(F6). Let $\langle A^*, C^*\rangle$ be the reduction of $\langle A, C\rangle$. That is, $A^* = A/\widetilde{\Omega}^A C$ and $\mathcal{C}^* = \{F/\widetilde{\Omega}^A C : F \in \mathcal{C}\}$. By Proposition 7.1, it is enough to prove that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}_N}(A^*)$. Notice that the natural homomorphism $\pi : A \to A^*$ is a surjective strict homomorphism from the g-matrix $\langle A, C\rangle$ onto its reduction $\langle A^*, C^*\rangle$. For each $a \in A$, let $\overline{a} = a/\widetilde{\Omega}^A \mathcal{C}$.

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($\subseteq$) By (F1) and (F2), we have that the g-matrix $\langle A, C\rangle$ is a g-model of $\mathcal{S}_N$. Then, $\langle A^*, C^*\rangle$ is a g-model of $\mathcal{S}_N$. Hence, $\mathcal{C}^* \subseteq \mathcal{F}i_{\mathcal{S}_N}(A^*)$.

($\supseteq$) In order to prove the inverse inclusion, let us show that $C^*(B^*) \subseteq \mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*)$, for all $B \subseteq A$ where $B^* = \{\overline{b} : b \in B\}$. Let $B \subseteq A$. If $\mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*) = A^*$, then $C^*(B^*) \subseteq \mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*)$. Assume that $\mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*) \neq A^*$. Let us show that $C^*(B^*) \neq A^*$. Suppose by contradiction that $C^*(B^*) = A^*$. Thus, $C(B) = A$. By (F6), there are $a, a' \in B$ such that $C(a) = C(\neg a')$. Now we show that $C(a) \neq A$. Suppose that $C(a) = A$. Thus, by (F6), we obtain that $C(a) = C(\neg a)$. By (F2), we have that $C(\neg a) = C(\neg(\neg a))$. Then, by Proposition 7.9, it follows that $\overline{a} = \overline{\neg a} = \neg\overline{a}$. Since $\overline{a} \in B^*$, we have that $\overline{a}, \neg\overline{a} \in \mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*)$. Then, $\mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*) = A^*$, which is a contradiction. Hence, $C(\neg a') = C(a) \neq A$. It follows by (F4) and (F2) that $C(\neg a) = C(a')$. Then $\overline{a} = \neg\overline{a'}$. Given that $\overline{a}, \overline{a'} \in B^*$, we obtain that $\mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*) = A^*$. A contradiction. Hence, $C^*(B^*) \neq A^*$. And thus $C(B) \neq A$. Now, let $\overline{a} \in C^*(B^*)$. So $a \in C(B)$. By (F5), there is $a_0 \in B$ such that $a \in C(a_0)$. Since $C(B) \neq A$, it follows that $C(a_0) \neq A$. Thus, since $a \in C(a_0) \neq A$, it follows by (F3) and (F4) that $a_0 \in C(a)$ and $\neg a_0 \in C(\neg a)$. Thus, $C(a) = C(a_0)$. Since $C(a) = C(a_0) \neq A$ and $a_0 \in C(a)$, we have by (F4) that $\neg a \in C(\neg a_0)$. Then, we obtain that $C(a) = C(a_0)$ and $C(\neg a) = C(\neg a_0)$. Thus, $\overline{a} = \overline{a_0} \in B^*$. Hence, $\overline{a} \in \mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*)$. Hence, $C^*(B^*) \subseteq \mathrm{Fig}^{A^*}_{\mathcal{S}_N}(B^*)$.

We have proved that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}_N}(A^*)$. Therefore, $\langle A, C\rangle$ is a full g-model of $\mathcal{S}_N$. □

REMARK 7.11
Notice that for every full g-model $\langle A, C\rangle$, it follows that $C(\emptyset) = \emptyset$. Indeed, condition (F6) implies that $C(\emptyset) \neq A$. Then condition (F5) implies that $a \notin C(\emptyset)$ for all $a \in A$. Hence, $C(\emptyset) = \emptyset$.

Notice that conditions $C(\emptyset) = \emptyset$ and (F1)–(F5) coincide respectively with conditions (B1)–(B6) when they are considered on the g-matrices $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$. And condition (F6) coincides with the implication (2)⇒(1) of Proposition 7.5.

Let $\langle A, C\rangle$ be a g-matrix. The *Frege relation*, denoted by $\Lambda_A C$, of C on A is defined as follows:

$$(a, b) \in \Lambda_A C \iff C(a) = C(b)$$

for all $a, b \in A$. Notice that the Tarski congruence is the largest congruence below $\Lambda_A C$.

PROPOSITION 7.12
Let $\langle A, C\rangle$ be a g-matrix satisfying conditions (F4) and (F6). Then, the Frege relation $\Lambda_A C$ is a congruence on A.

PROOF. Let $a, b \in A$ be such that $(a, b) \in \Lambda_A C$. Thus $C(a) = C(b)$. If $C(a) = C(b) \neq A$, then we have $a \in C(b) \neq A$. Then, we obtain by (F4) that $\neg b \in C(\neg a)$. Analogously, $\neg a \in C(\neg b)$. Hence, $C(\neg a) = C(\neg b)$. On the other hand, if $C(a) = C(b) = A$, it follows by (F6) that $C(a) = C(\neg a)$ and $C(b) = C(\neg b)$. Then $C(\neg a) = C(\neg b)$. Hence $(\neg a, \neg b) \in \Lambda_A C$. □

The following proposition tells us that for an algebra A in the intrinsic variety of $\mathcal{S}_N$, the proper generated $\mathcal{S}_N$-filters on A are quite simple.

PROPOSITION 7.13
Let $A \in \mathbb{V}(\mathcal{S}_N)$. For all $B \subseteq A$, if $\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) \neq A$, then $\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) = B$.

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PROOF. Let $A \in \mathbb{V}(\mathcal{S}_N)$. Suppose that $\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) \neq A$ and let $a \in \mathrm{Fig}^{A}_{\mathcal{S}_N}(B)$. By Proposition 7.4, there is $b \in B$ and $s, t \in \mathcal{E}$ such that $\neg^n a = \neg^t b$. Since $A \models x \approx \neg\neg x$, it follows that $a = b \in B$. Hence $\mathrm{Fig}^{A}_{\mathcal{S}_N}(B) = B$. □

THEOREM 7.14

$$\mathrm{Alg}(\mathcal{S}_N) = \{A : A \models x \approx \neg\neg x \text{ and } A \models (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)\}.$$

PROOF. ($\subseteq$) Let $A \in \mathrm{Alg}(\mathcal{S}_N)$. Thus, $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$ is a reduced full g-model of $\mathcal{S}_N$. Then, $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$ satisfies conditions (F1)–(F6). By the previous proposition, it follows that $\Lambda_A \mathcal{F}i_{\mathcal{S}_N}(A) = \widetilde{\Omega}^A \mathcal{F}i_{\mathcal{S}_N}(A) = \mathrm{Id}_A$. Then, by (B3) (or (F2)), we obtain that $a = \neg\neg a$, for all $a \in A$. Hence, $A \models x \approx \neg\neg x$. Now let $a, b \in A$ and assume that $a = \neg a$ and $b = \neg b$. By (B2), we have $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = A = \mathrm{Fig}^{A}_{\mathcal{S}_N}(b)$. Thus, $(a, b) \in \Lambda_A \mathcal{F}i_{\mathcal{S}_N}(A)$. Then $a = b$. Hence, $A \models (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)$.

($\supseteq$) Let A be an algebra such that $A \models x \approx \neg\neg x$ and $A \models (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)$. It is straightforward that the g-matrix $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$ is a full g-model of $\mathcal{S}_N$. Let us show that the Frege relation of $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$ is the identity relation. Let $(a, b) \in \Lambda_A \mathcal{F}i_{\mathcal{S}_N}(A)$. Thus, $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = \mathrm{Fig}^{A}_{\mathcal{S}_N}(b)$. If $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = \mathrm{Fig}^{A}_{\mathcal{S}_N}(b) \neq A$, it follows by Proposition 7.13 that $a = b$. Assume that $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = \mathrm{Fig}^{A}_{\mathcal{S}_N}(b) = A$. From $\mathrm{Fig}^{A}_{\mathcal{S}_N}(a) = A$, it follows by Proposition 7.5 that there is $(s, t) \in \mathcal{E} \times \mathcal{O}$ such that $\neg^s a = \neg^t a$. Then, $a = \neg a$. Similarly, $b = \neg b$. Since $A \models (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)$, we obtain that $a = b$. Hence, $\Lambda_A \mathcal{F}i_{\mathcal{S}_N}(A) = \mathrm{Id}_A$. Therefore, the full g-model $\langle A, \mathcal{F}i_{\mathcal{S}_N}(A)\rangle$ is reduced, and thus $A \in \mathrm{Alg}(\mathcal{S}_N)$. □

8 Concluding remarks

Throughout this article, we have obtained the three classes of algebras that are canonically associated with the logic $\mathcal{S}_N$ in the context of algebraic logic:

$$\mathrm{Alg}^*(\mathcal{S}_N) = \mathbb{I}(A_1, A_2, A_3) \qquad \mathbb{V}(\mathcal{S}) = \{A : A \models x \approx \neg\neg x\}$$

$$\mathrm{Alg}(\mathcal{S}_N) = \{A : A \models (x \approx \neg\neg x) \;\&\; (x \approx \neg x \;\&\; y \approx \neg y \implies x \approx y)\}.$$

With this, we can complete the table with all the fragments of CPL and their corresponding classes of algebras canonically associated, see Table 1.

Notice that it is clear that the inclusions $\mathrm{Alg}^*(\mathcal{S}_N) \subset \mathrm{Alg}(\mathcal{S}_N) \subset \mathbb{V}(\mathcal{S}_N)$ are proper. For instance, the algebra $A_2 = \{a, b\}$, with $\neg a = a$ and $\neg b = b$, belongs to $\mathbb{V}(\mathcal{S}_N)$ but not to $\mathrm{Alg}(\mathcal{S}_N)$. The $\{\neg\}$-reduct of the four-element Boolean algebra belongs to $\mathrm{Alg}(\mathcal{S}_N)$ but not to $\mathrm{Alg}^*(\mathcal{S}_N)$. Also notice that the class $\mathrm{Alg}(\mathcal{S}_N)$ is a quasi-variety but not a variety.

From the results that we have obtained, we can classify the $\{\neg\}$-fragment of CPL in the Leibniz hierarchy and in the Frege hierarchy (the two hierarchies of algebraic logic, see [3, Chap. 6 and 7]) and respond to an open problem proposed in [3, p. 418]. The $\{\neg\}$-fragment $\mathcal{S}_N$ of CPL is outside of the Leibniz hierarchy because it is neither protoalgebraic nor truth-equational: Notice that the logic $\mathcal{S}_N$ is non-trivial, and thus $\mathcal{S}_N$ is not almost inconsistent. Since the unique protoalgebraic logic without theorems is the almost inconsistent one, it follows that $\mathcal{S}_N$ is not protoalgebraic. Moreover,

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since the logic $\mathcal{S}_N$ has no theorems, it follows that $\mathcal{S}_N$ is not truth-equational. Now we turn out our attention to the Frege hierarchy (see [3, p. 414]). Since being Fregean is a property that is preserved by fragments and CPL is clearly Fregean, it follows that $\mathcal{S}_N$ is Fregean. Moreover, the logic $\mathcal{S}_N$ is also fully self-extensional: Let $A \in \mathrm{Alg}(\mathcal{S}_N)$. In the proof of Theorem 7.14, we have proved that the Frege relation $\varLambda_A \mathcal{F}i_{\mathcal{S}_N}(A)$ is the identity relation. Hence, $\mathcal{S}_N$ is fully self-extensional. However, we will show that $\mathcal{S}_N$ is not fully Fregean. In order to show this, consider the following relation: let $\mathcal{S}$ be a logic, A an algebra and $F \subseteq A$,

$$\varLambda^A_{\mathcal{S}} F = \{(a,b) \in A \times A : \mathrm{Fig}^A_{\mathcal{S}}(F,a) = \mathrm{Fig}^A_{\mathcal{S}}(F,b)\}.$$

PROPOSITION 8.1 ([3, Prop. 7.56]).
A logic $\mathcal{S}$ is fully Fregean if and only if $\varLambda^A_{\mathcal{S}} F \subseteq \varOmega^A F$ for every $F \in \mathcal{F}i_{\mathcal{S}}(A)$ and every algebra A.

Let $A_3 = \{a,b,c\}$ be the algebra given on page 8. Let $F = \{a\}$. We know that $F \in \mathcal{F}i_{\mathcal{S}_N}(A_3)$. We have that $\mathrm{Fig}^{A_3}_{\mathcal{S}_N}(F,b) = A$ (because $b, \neg b \in \mathrm{Fig}^{A_3}_{\mathcal{S}_N}(F,b)$) and $\mathrm{Fig}^{A_3}_{\mathcal{S}_N}(F,c) = A$ (because $c, \neg c \in \mathrm{Fig}^{A_3}_{\mathcal{S}_N}(F,c)$). Hence, $(b,c) \in \varLambda^A_{\mathcal{S}_N} F$. But $(b,c) \notin \varOmega^A F$ because $\neg b \in F$ and $\neg c \notin F$ (see Proposition 5.1). Thus, $\varLambda^A_{\mathcal{S}_N} F \nsubseteq \varOmega^A F$. Therefore, the $\{\neg\}$-fragment of CPL is not fully Fregean. This answers negatively the question: Is the class of fully Fregean logics the intersection of the classes of the Fregean and the fully self-extensional ones? That is, $\mathcal{S}_N$ is a Fregean and fully self-extensional logic but is not fully Fregean one. It is worth mentioning that Tommaso Moraschini and Ramon Jansana found this example before but they don't publish it (c.p.).

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References

[1] W. J. Blok and D. Pigozzi. Algebraizable logics. In *Algebraizable Logics*, vol. 77. Amer. Math. Soc., 1989.

[2] J. M. Dunn. A comparative study of various model-theoretic treatments of negation: a history of formal negation. In *What Is Negation*? D. M. Gabbay and H. Wansing., eds, pp. 23–51. Springer, Dordrecht, 1999.

[3] J. M. Font. *Abstract Algebraic Logic. An Introductory Textbook*. Studies in Logic, vol. 60. College Publications, London, 2016.

[4] J. M. Font, F. Guzmán and V. Verdú. Characterization of the reduced matrices for the $\{\wedge, \vee\}$-fragment of classical logic. *Bulletin of the Section of Logic*, **20**, 124–128, 1991.

[5] J. M. Font and R. Jansana. *A General Algebraic Semantics for Sentential Logics*. Lecture Notes in Logic, vol. 7. A.S.L., 2nd edn., 2009.

[6] J. M. Font, R. Jansana and D. Pigozzi. A survey of abstract algebraic logic. *Studia Logica*, **74**, 13–97, 2003.

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[7] J. M. Font and T. Moraschini. Logics of varieties, logics of semilattices and conjunction. *Logic Journal of the IGPL*, **22**, 818–843, 2014.

[8] J. M. Font and V. Verdú. Algebraic logic for classical conjunction and disjunction. *Studia Logica*, **50**, 391–419, 1991.

[9] D. M. Gabbay and H. Wansing., eds. *What is Negation*? Springer, Dordrecht, 1999.

[10] R. Jansana and T. Moraschini. The poset of all logics I: interpretations and lattice structure. *The Journal of Symbolic Logic*, **86**, 935–964, 2021.

[11] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, 1974.

[12] W. Rautenberg. 2-element matrices. *Studia Logica*, **40**, 315–353, 1981.

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