Log. Univers.

DOI 10.1007/s11787-017-0181-4

Logica Universalis

An Alternative Definition of Quantifiers on Four-Valued Łukasiewicz Algebras

L. J. González, M. B. Lattanzi and A. G. Petrovich

Abstract. An alternative notion of an existential quantifier on four-valued Łukasiewicz algebras is introduced. The class of four-valued Łukasiewicz algebras endowed with this existential quantifier determines a variety which is denoted by $\mathbb{M}_{\frac{2}{3}}\mathbb{L}_4$. It is shown that the alternative existential quantifier is interdefinable with the standard existential quantifier on a four-valued Łukasiewicz algebra. Some connections between the new existential quantifier and the existential quantifiers defined on bounded distributive lattices and Boolean algebras are given. Finally, a completeness theorem for the monadic four-valued Łukasiewicz predicate calculus corresponding to the dual of the alternative existential quantifier is proven.

Mathematics Subject Classification. Primary 03B50; Secondary 06D30, 06D35, 06D20.

Keywords. Four-valued Łukasiewicz algebras, quantifiers, monadic logics.

1. Introduction

In [9] Halmos introduced the monadic Boolean algebras as the algebraic counterpart of the monadic predicate calculus. After this, several generalizations of monadic algebras were obtained for some classes of algebras associated to non-classic logics [1,2,7,8]. In particular, the monadic four-valued Łukasiewicz algebras were studied in [1,7,8] as a four-valued Łukasiewicz algebra endowed with an existential (universal) quantifier, here called standard existential (universal) quantifier. However, it is important to point out that although the underlying propositional logic is non-classical, the corresponding quantifier is interpreted in the context of classical logic.

In this paper, we present an alternative notion of an existential quantifier on four-valued Łukasiewicz algebras, whose interpretation is non-classical, and

This work was partially supported by Universidad Nacional de La Pampa (Fac. de Cs. Exactas y Naturales) under the Grant P.I. 64 M, Res. 432/14 CD.

Published online: 07 November 2017 Birkhäuser

which turns out to be a generalization of the usual concept of an existential quantifier on Boolean algebras. This notion of quantifier arises as a generalization of the middle existential quantifiers on three-valued Łukasiewicz algebras introduced by Petrovich in [16]. In [17] a quantifier is associated to each element of the three-valued Łukasiewicz chain $\mathbf{3} = \{0, \frac{1}{2}, 1\}$ in the following sense. Let X be a nonempty set and let $\mathbf{3}^X$ be the three-valued Łukasiewicz algebra where the operations are defined pointwise. If $\nu \in \{0, \frac{1}{2}, 1\}$ then the ν-existential quantifier $\exists_\nu : \mathbf{3}^X \to \mathbf{3}^X$ is characterized by the following property, for each $f \in \mathbf{3}^X$:

$$\begin{array}{l}\exists_\nu(f) \text{ is the constant function taking the value } \nu \text{ if and only if} \\ f \text{ takes the value } \nu \text{ in some element } x \in X.\end{array} \tag{P$_\nu$}$$

Thus $\exists_1$ and $\exists_0$ are the standard existential and universal quantifiers, respectively, defined on a three-valued Łukasiewicz algebra [14] while $\exists_{\frac{1}{2}}$ is the middle quantifier considered in [17].

Now, let $\mathbf{4} = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ be the four-valued Łukasiewicz chain and $\mathbf{4}^X$ be the four-valued Łukasiewicz algebra where the operations are defined pointwise. So, in a similar way, we can define four quantifiers on $\mathbf{4}^X$ which satisfy Property (P$_\nu$) for all $f \in \mathbf{4}^X$ and every $\nu \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Again $\exists_1$ and $\exists_0$ are the standard existential and universal quantifiers, respectively, defined on a four-valued Łukasiewicz algebra [1,2]. We can see that $\exists_{\frac{1}{3}}$ is the dual operator of $\exists_{\frac{2}{3}}$ as well as $\exists_0$ is the dual of $\exists_1$, i.e., $\exists_{\frac{1}{3}} = \neg\exists_{\frac{2}{3}}\neg$ and $\exists_0 = \neg\exists_1\neg$. For this reason we will investigate only the properties of the existential quantifier $\exists_{\frac{2}{3}}$.

The paper is organized as follows. In Sect. 2, we define an operator on the functional four-valued Łukasiewicz algebras $\mathbf{4}^X$ and, we study its main properties. Then, we introduce the general and abstract definition of the 2/3-existential quantifiers on four-valued Łukasiewicz algebras and show several properties from this definition. The main aim in Sect. 3 is to prove that the four-valued Łukasiewicz algebras endowed with a 2/3-existential quantifier are polynomially equivalent to the four-valued Łukasiewicz algebras endowed with a standard existential quantifier (in the sense of [2]). In Sect. 4, we study some connections between the 2/3-existential quantifiers on four-valued Łukasiewicz algebras and the Boolean existential quantifiers [9] defined on the Boolean elements and the lattice existential quantifiers [5] defined on certain distributive sub-lattices of four-valued Łukasiewicz algebras. In fact, we prove that from a 2/3-existential quantifier it can be defined a Boolean existential quantifier and a lattice existential quantifier and reciprocally, under certain restrictions, from a Boolean existential quantifier and a lattice existential quantifier it is possible to define a 2/3-existential quantifier. Finally, in Sect. 5, we propose a monadic four-valued Łukasiewicz predicate calculus, which correspond to the 2/3-universal quantifier. Then we prove a completeness theorem for this logic. To attain this, we need to consider a monadic four-valued Łukasiewicz predicate calculus corresponding to the standard universal quantifier [2] and we show a completeness theorem for it.

The variety of four-valued Łukasiewicz algebras is the algebraic counterpart of Łukasiewicz four-valued propositional calculus, but the variety of n-valued Łukasiewicz algebras for $n \geq 5$ does not correspond to Łukasiewicz n-valued propositional logic. Four-valued Łukasiewicz algebras are polynomially equivalent to four-valued MV-algebras and to four-valued Wajsberg algebras [2,4,6,11]. In [7] it is proved that monadic n-valued MV-algebras are polynomially equivalent to monadic n-valued Łukasiewicz algebras for $n = 3$ and $n = 4$.

A *four-valued Łukasiewicz algebra*, for short L_4-*algebra*, (see for instance [2] and [11]) is an algebra $\langle L, \vee, \wedge, \neg, \sigma_1, \sigma_2, \sigma_3, 0, 1\rangle$ of type (2,2,1,1,1,1,0,0) satisfying the following conditions, for all $x, y \in L$ and any $i, j \in \{1, 2, 3\}$:

(L1) $\langle L, \vee, \wedge, \neg, 0, 1\rangle$ is a De Morgan algebra,
(L2) $\sigma_i(x \vee y) = \sigma_i x \vee \sigma_i y$,
(L3) $\sigma_i(x \wedge y) = \sigma_i x \wedge \sigma_i y$,
(L4) $\sigma_i x \vee \neg\sigma_i x = 1$,
(L5) $\neg\sigma_i x = \sigma_{4-i}\neg x$,
(L6) $\sigma_i\sigma_j x = \sigma_j x$,
(L7) $\sigma_1 x \leq \sigma_2 x \leq \sigma_3 x$,
(MDP) if $\sigma_i x = \sigma_i y$ for all $i \in \{1, 2, 3\}$, then $x = y$ (*Moisil's Determination Principle*).

The following identity holds in every L_4-algebra and will be used frequently

$$x \wedge \neg x = (\neg x \wedge \sigma_2 x) \vee (x \wedge \neg\sigma_2 x). \tag{L8}$$

For brief, we will denote an L_4-algebra $\langle L, \vee, \wedge, \neg, \sigma_1, \sigma_2, \sigma_3, 0, 1\rangle$ by its support set L. We refer the reader to [2] and [11] for the basic properties of four-valued Łukasiewicz algebras. Let L be an L_4-algebra. An element $x \in L$ is a Boolean element if $\sigma_1 x = x$; we denote by $B(L)$ the set of Boolean elements of L.

Let $\langle \mathbf{4}, \vee, \wedge, \sigma_1, \sigma_2, \sigma_3, 0, 1\rangle$ be the L_4-algebra of four elements where the operations are defined as follows: $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $\neg x = 1 - x$ and

$$\sigma_i\left(\frac{k}{3}\right) = \begin{cases} 0 & \text{if} \quad i + k \leq 3 \\ 1 & \text{if} \quad i + k > 3 \end{cases} \qquad \text{for all } i \in \{1, 2, 3\} \text{ and } k \in \{0, 1, 2, 3\}.$$

Let X be a nonempty set. It is clear that $\mathbf{4}^X$ is also a four-valued Łukasiewicz algebra where the operations are defined pointwise. The constant functions having the values 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1 will be denote by $\mathbf{0}$, $\mathbf{\frac{1}{3}}$, $\mathbf{\frac{2}{3}}$ and $\mathbf{1}$, respectively.

The standard existential and universal quantifiers on $\mathbf{4}^X$ are defined by $\exists_1 f = \bigvee_{x \in X} f(x)$ and $\exists_0 f = \bigwedge_{x \in X} f(x)$, respectively, and it should be clear that $\exists_0$ and $\exists_1$ hold Property (P_ν). In what follows we will define an existential quantifier associated to the element $\frac{2}{3} \in \mathbf{4}$.

2. 2/3-Existential Quantifiers on L_4-Algebras

Using the above considerations, we give an alternative notion of an existential quantifier on a functional L_4-algebra, and we show the main properties of this operator. Then, we will introduce an abstract definition of 2/3-existential quantifiers on four-valued Łukasiewicz algebras.

Definition 2.1. Let X be a nonempty set. We define the unary operator $\exists_{\frac{2}{3}}$ on $\mathbf{4}^X$ by

$$\exists_{\frac{2}{3}} f = \left(\bigvee_{x \in X} f(x) \right) \wedge \bigwedge_{x \in X} \left(f(x) \vee \sigma_2 \neg f(x) \right).$$

It is clear that $\exists_{\frac{2}{3}} f$ is a constant function for all $f \in \mathbf{4}^X$.

Notice that in the previous definition $f(x) \vee \sigma_2 \neg f(x) = f(x) \vee 2\neg f(x)$, for all $f \in \mathbf{4}^X$ and every $x \in X$, where $2a = a \oplus a$, being $\oplus$ the sum of MV-algebras.

Proposition 2.2. *The operator $\exists_{\frac{2}{3}}$ defined above on $\mathbf{4}^X$ satisfies the property*

$$\exists_{\frac{2}{3}} f = \frac{2}{3} \text{ if and only if there is } x_0 \in X \text{ such that } f(x_0) = \frac{2}{3}. \qquad (\mathrm{P}_{\frac{2}{3}})$$

Proof. First assume that there is $x_0 \in X$ such that $f(x_0) = \frac{2}{3}$. Since $f(x) \vee \sigma_2(\neg f(x)) \geq \frac{2}{3}$ for all $x \in X$ we have

$$\bigvee_{x \in X} f(x) \geq \frac{2}{3} \quad \text{and} \quad \bigwedge_{x \in X} \left(f(x) \vee \sigma_2 \left(\neg f(x) \right) \right) = \frac{2}{3}.$$

Then, $\exists_{\frac{2}{3}} f = \frac{2}{3}$. Conversely, assume that $\exists_{\frac{2}{3}} f = \frac{2}{3}$. Then we have two possibilities $\bigvee_{x \in X} f(x) = \frac{2}{3}$ or $\bigwedge_{x \in X} \left(f(x) \vee \sigma_2 \left(\neg f(x) \right) \right) = \frac{2}{3}$. In the first case, there is $x_0 \in X$ such that $f(x_0) = \frac{2}{3}$. Otherwise, there is $x_0 \in X$ such that $f(x_0) \vee \sigma_2(\neg f(x_0)) = \frac{2}{3}$. Since the image of σ_2 is $\{0, 1\}$, we obtain $f(x_0) = \frac{2}{3}$. □

The following proposition tells us what the behaviour of $\exists_{\frac{2}{3}}$ is; the proof is not hard, and we leave the details to the reader.

Proposition 2.3. *The operator $\exists_{\frac{2}{3}}$ introduced in* Definition 2.1 *can be expressed as follows:*

$$\exists_{\frac{2}{3}} f = \begin{cases} 0 & \text{if } f = 0, \\ \frac{1}{3} & \text{if there is } x_0 \in X \text{ such that } f(x_0) = \frac{1}{3} \text{ and } f \leq \frac{1}{3}, \\ \frac{2}{3} & \text{if there is } x_0 \in X \text{ such that } f(x_0) = \frac{2}{3}, \\ 1 & \text{if there is } x_0 \in X \text{ such that } f(x_0) = 1 \text{ and} \\ & f(x) \neq \frac{2}{3} \text{ for all } x \in X. \end{cases}$$

The following proposition follows from Definition 2.1 and Proposition 2.3.

Proposition 2.4. *The operator $\exists_{\frac{2}{3}}$ satisfies, for all $f, g \in \mathbf{4}^X$, the following properties:*

(1) $\exists_{\frac{2}{3}} 0 = \mathbf{0}$;
(2) $\sigma_i f \leq \exists_{\frac{2}{3}} \sigma_i f$, *for* $i \in \{1,2,3\}$,
(3) $\sigma_i \exists_{\frac{2}{3}} f = \exists_{\frac{2}{3}} \sigma_i f$, *for* $i \in \{2,3\}$,
(4) $\exists_{\frac{2}{3}} \left(f \wedge \exists_{\frac{2}{3}} g\right) = \exists_{\frac{2}{3}} f \wedge \exists_{\frac{2}{3}} g$,
(5) $\exists_{\frac{2}{3}} f \leq f \vee \sigma_2 \neg f$,
(6) $\exists_{\frac{2}{3}} (f \vee \neg\sigma_2 f) \leq \exists_{\frac{2}{3}} f \vee \neg\exists_{\frac{2}{3}} \sigma_2 f$.

Now, taking into account the properties established in the previous proposition, we introduce the following definition.

Definition 2.5. Let $\langle L, \vee, \wedge, \neg, \sigma_1, \sigma_2, \sigma_3, 0, 1\rangle$ be an L_4-algebra. An operator $\exists_{\frac{2}{3}} : L \to L$ is a *2/3-existential quantifier* if satisfies the following conditions, for all $x, y \in L$:
(A1) $\exists_{\frac{2}{3}} 0 = 0$,
(A2) $\sigma_i x \leq \exists_{\frac{2}{3}} \sigma_i x$, for $i \in \{1,2,3\}$,
(A3) $\exists_{\frac{2}{3}} \sigma_i x = \sigma_i \exists_{\frac{2}{3}} x$, for $i \in \{2,3\}$,
(A4) $\exists_{\frac{2}{3}} (x \wedge \exists_{\frac{2}{3}} y) = \exists_{\frac{2}{3}} x \wedge \exists_{\frac{2}{3}} y$,
(A5) $\exists_{\frac{2}{3}} x \leq x \vee \sigma_2 \neg x$,
(A6) $\exists_{\frac{2}{3}} (x \vee \neg\sigma_2 x) \leq \exists_{\frac{2}{3}} x \vee \neg\exists_{\frac{2}{3}} \sigma_2 x$.

The pair $\langle L, \exists_{\frac{2}{3}}\rangle$ is a *2/3-monadic L_4-algebra* if L is an L_4-algebra and $\exists_{\frac{2}{3}}$ is a 2/3-existential quantifier on L. For short, hereinafter we will write $\exists_2$ instead of $\exists_{\frac{2}{3}}$. We denote by $\mathbb{M}_{\frac{2}{3}}\mathbb{L}_4$ the class of all 2/3-monadic L_4-algebras. It is clear that if we consider the language $\langle \vee, \wedge, \neg, \sigma_1, \sigma_2, \sigma_3, \exists_2, 0, 1\rangle$ of type (2, 2, 1, 1, 1, 1, 1, 0, 0), then the class $\mathbb{M}_{\frac{2}{3}}\mathbb{L}_4$ is a variety.

Example 1. Let X be a nonempty set. Then $\langle \mathbf{4}^X, \exists_{\frac{2}{3}}\rangle$ is a 2/3-monadic L_4-algebra, where $\exists_{\frac{2}{3}}$ is defined as in Definition 2.1.

Example 2. Let L be an L_4-algebra and let c be an element of L satisfying the identities $\sigma_2 c = 0$ and $\sigma_3 c = 1$.

(i) The operator $\exists_* : L \to L$ given by the formula:

$$\exists_* x = \begin{cases} 1 & if\ x \in B(L)\ and\ x \neq 0, \\ 0 & if\ x = 0, \\ c & if\ x \neq 0\ and\ \sigma_2 x = 0, \\ \neg c & otherwise \end{cases}$$

satisfies axioms (A1) to (A5) but not (A6).

We leave to the reader the task of verifying that $\exists_*$ satisfies axioms (A1) to (A5). Let L be an L_4-algebra (for instance $L = 4 \times 4$) and let $x \in B(L)$ be such that $x \neq 0$ and $x \neq 1$. Let $z = x \vee c$. Note that $\sigma_2 z = \sigma_2 x = x$. Then $\exists_*(z \vee \neg\sigma_2 z) = \exists_*(x \vee c \vee \neg x) = \exists_*(1) = 1$. On the other hand we claim that $\exists_* z = \neg c$. Indeed, it is clear that $z \neq 0$, so $\exists_* z \neq 0$. Since $x \neq 1$ then $z \notin B(L)$ because $z \geq c$ implies $\sigma_3 z \geq \sigma_3 c = 1$, and if $z \in B(L)$ then $z = 1 = x \vee c$ and hence $1 = \sigma_2 x = x$, a contradiction. So $z \notin B(L)$. Since $x \neq 0$ it follows that $\sigma_2 z = \sigma_2 x = x \neq 0$. Therefore $\exists_* z = \neg c$. Finally, $\exists_* \sigma_2 z = \exists_* x = 1$ because x is a boolean element different from 0, so $\neg\exists_*\sigma_2 z = 0$ and then $\exists_* z \vee \neg\exists_*\sigma_2 z = \neg c$ and $\exists_*(z \vee \neg\sigma_2 z) = 1$, so the inequality given in (A6) does not hold.

(ii) The operator $\exists_2 : L \to L$ given by the prescription:

$$\exists_2(x) = \begin{cases} 0 & if\ x = 0 \\ c & if\ x \neq 0\ and\ \sigma_2 x = 0 \\ \neg c & if\ \sigma_2 x \neq \sigma_1 x \\ 1 & otherwise \end{cases}$$

is a $\frac{2}{3}$-existential quantifier.

Let L be an L_4-algebra, S an L_4-subalgebra of L and $q\colon L \to L$ a function. We denote by $q_{/S}$ the restriction of q to S and by $q(S)$ the image of S by q. Next we give some basic properties of 2/3-monadic L_4-algebras.

Proposition 2.6. *Let $\langle L, \exists_2\rangle$ be a 2/3-monadic L_4-algebra and let $x, y \in L$. Then, the following properties hold:*

(P1) $\exists_2 1 = 1$,
(P2) $\exists_2\exists_2 x = \exists_2 x$,
(P3) $x \leq \exists_2 y$ *implies* $\exists_2 x \leq \exists_2 y$,
(P4) $\exists_2 x \leq y$ *implies* $\exists_2 x \leq \exists_2 y$,
(P5) $x \in B(L)$ *implies* $\exists_2 x \in B(L)$,
(P6) $x \in \exists_2(L)$ *if and only if* $\exists_2 x = x$,
(P7) $\langle B(L), \exists_{/B(L)}\rangle$ *is a monadic Boolean algebra and* $\exists_2(B(L)) = \exists_2(L) \cap B(L)$,
(P8) $\exists_2\sigma_1\exists_2 x = \sigma_1\exists_2 x$,
(P9) $\sigma_1\exists_2 x \leq \exists_2\sigma_1 x$.

Proof. (P1) It is straightforward from (A2).

(P2) $\exists_2(1 \wedge \exists_2 x) = \exists_2 1 \wedge \exists_2 x$. So, $\exists_2\exists_2 x = \exists_2 x$.

(P3) It follows from (A4).

(P4) It follows by applying (A4) and (P2).

(P5) Let $x \in B(L)$. So, $\sigma_2 x = x$ and then, from (A3), we have $\sigma_2\exists_2 x = \exists_2\sigma_2 x = \exists_2 x$. Thus $\exists_2 x \in B(L)$.

(P6) It follows from (P2).

(P7) It is clear from (P5), (A1), (A2) and (A4).

(P8) Since $\exists_2$ is an existential quantifier on $B(L)$ and $\sigma_1\exists_2 x \in B(L)$, it follows that $\sigma_1\exists_2 x \leq \exists_2\sigma_1\exists_2 x$. Notice that $\sigma_1\exists_2 x \leq \exists_2 x$. So, by (P3) and (P2) we have $\exists_2\sigma_1\exists_2 x \leq \exists_2 x$. Then, by (P5), $\exists_2\sigma_1\exists_2 x \leq \sigma_1\exists_2 x$. Therefore $\exists_2\sigma_1\exists_2 x = \sigma_1\exists_2 x$.

(P9) By (A5) we have $\exists_2 x \leq x \vee \sigma_2\neg x$. So, $\sigma_1\exists_2 x \leq \sigma_1 x \vee \sigma_2\neg x$. Then $\sigma_1\exists_2 x \wedge \sigma_2 x \leq (\sigma_1 x \vee \neg\sigma_2 x) \wedge \sigma_2 x = \sigma_1 x$. Since $\exists_2$ is an existential quantifier on $B(L)$, it follows that $\exists_2(\sigma_1\exists_2 x \wedge \sigma_2 x) \leq \exists_2\sigma_1 x$. Now, from (P8), (A4) and (A3) we have $\exists_2(\sigma_1\exists_2 x \wedge \sigma_2 x) = \sigma_1\exists_2 x$. Hence, $\sigma_1\exists_2 x \leq \exists_2\sigma_1 x$.

Proposition 2.7. *Let $\langle L, \exists_2\rangle$ be a 2/3-monadic L_4-algebra. Then $\exists_2(L)$ is an L_4-subalgebra of L.*

Proof. By Definition 2.5 and Proposition 2.6, it is clear that $\exists_2(L)$ is closed under $\wedge$, σ_1, σ_2, and σ_3, and includes 0 and 1. To prove that $\exists_2(L)$ is closed under $\neg$ we will show that $\exists_2\neg\exists_2 x = \neg\exists_2 x$ using condition (MDP). First, notice that $\neg\sigma_3\exists_2 x \in \exists_2(L)$ follows from (A3) and (P7). Hence, $\neg\sigma_3\exists_2 x = \exists_2\neg\sigma_3\exists_2 x$. Now, by (P9) we have $\sigma_1\exists_2\neg\exists_2 x \leq \exists_2\sigma_1\neg\exists_2 x = \exists_2\neg\sigma_3\exists_2 x = \exists_2\neg\exists_2\sigma_3 x = \neg\exists_2\sigma_3 x = \neg\sigma_3\exists_2 x = \sigma_1\neg\exists_2 x$. Since $\exists_2 x \leq \sigma_3\exists_2 x$ we obtain $\neg\sigma_3\exists_2 x \leq \neg\exists_2 x$ and thus, by applying (P4), we have $\sigma_1\neg\exists_2 x = \neg\sigma_3\exists_2 x \leq \exists_2\neg\exists_2 x$. Then $\sigma_1\neg\exists_2 x \leq \sigma_1\exists_2\neg\exists_2 x$. Hence, $\sigma_1\neg\exists_2 x = \sigma_1\exists_2\neg\exists_2 x$. Equalities $\sigma_i\neg\exists_2 x = \sigma_i\exists_2\neg\exists_2 x$ follow from (A3) and (P8), for $i \in \{2, 3\}$. Therefore, by (MDP), it results $\exists_2\neg\exists_2 x = \neg\exists_2 x$.

Proposition 2.8. *Let $\langle L, \exists_2\rangle$ be a 2/3-monadic L_4-algebra. Then the following properties hold:*

(P10) $\exists_2 x = 0$ *implies* $x = 0$,
(P11) $\neg\exists_2 x \leq \exists_2\neg x$,
(P12) $\exists_2(x \vee \neg\sigma_2 x) \leq x \vee \neg\sigma_2 x$,
(P13) $\exists_2(\neg x \wedge \sigma_2 x) = \neg\exists_2(x \vee \neg\sigma_2 x)$,
(P14) $\exists_2(x \vee \neg\sigma_2 x) = \exists_2 x \vee \neg\exists_2\sigma_2 x$,
(P15) $\exists_2(\neg x \wedge \sigma_2 x) = \neg\exists_2 x \wedge \exists_2\sigma_2 x$,
(P16) $\sigma_1\exists_2 x = \exists_2\sigma_1 x \wedge \neg\exists_2(\neg\sigma_1 x \wedge \sigma_2 x)$,
(P17) $\exists_2(\neg x \wedge \sigma_2 x) \leq \exists_2(\neg x \vee \sigma_2 x)$,
(P18) $\exists_2 x = [\exists_2\sigma_2 x \wedge \exists_2(x \vee \neg\sigma_2 x)] \vee \exists_2(x \wedge \neg\sigma_2 x)$.

Proof. (P10) Suppose $\exists_2 x = 0$. By (A2) we have $x \leq \sigma_3 x \leq \exists_2\sigma_3 x = \sigma_3\exists_2 x = 0$.

(P11) We will use condition (MDP). By (A2), $x \leq \sigma_3 x \leq \exists_2\sigma_3 x$. So, $\neg\exists_2\sigma_3 x \leq \neg x$. Since $\neg\exists_2\sigma_3 x \in \exists_2(L)$, by using (P4) and (A3), we have $\sigma_1\neg\exists_2 x = \neg\exists_2\sigma_3 x \leq \exists_2\neg x$. Hence $\sigma_1\neg\exists_2 x \leq \sigma_1\exists_2\neg x$. Now, for σ_2, using (A3) and (P7) we obtain $\sigma_2\neg\exists_2 x = \neg\exists_2\sigma_2 x \leq \exists_2\neg\sigma_2 x = \exists_2\sigma_2\neg x = \sigma_2\exists_2\neg x$. Finally, since $\neg x \leq \sigma_3\neg x \leq \exists_2\sigma_3\neg x$, it follows that $\neg\exists_2\sigma_3\neg x \leq x$. Thus, by (P4) and (P7) it results $\neg\exists_2\sigma_3\neg x \leq \exists_2 x$. So, by applying σ_1 to both sides of the inequality we obtain $\neg\exists_2\sigma_3\neg x \leq \sigma_1\exists_2 x$. Then $\neg\sigma_1\exists_2 x \leq \exists_2\sigma_3\neg x$. Hence, $\sigma_3\neg\exists_2 x \leq \sigma_3\exists_2\neg x$. Therefore, by (MDP), $\neg\exists_2 x \leq \exists_2\neg x$.

(P12) It is clear from (A5), taking $x \vee \neg\sigma_2 x$ instead of x.

(P13) From (P12) it follows $\neg x \wedge \sigma_2 x \leq \neg\exists_2(x \vee \neg\sigma_2 x)$. Then, by applying Proposition 2.7, (P3) and (P11) we have $\exists_2(\neg x \wedge \sigma_2 x) \leq \neg\exists_2(x \vee \neg\sigma_2 x) \leq \exists_2(\neg x \wedge \sigma_2 x)$. Hence, $\exists_2(\neg x \wedge \sigma_2 x) = \neg\exists_2(x \vee \neg\sigma_2 x)$.

(P14) From (A6) we have $\exists_2(x \vee \neg\sigma_2 x) \leq \exists_2 x \vee \neg\exists_2\sigma_2 x$. To prove the other inequality, first we use (A5) and (P4) to obtain $\exists_2 x \leq \exists_2(x \vee \neg\sigma_2 x)$. On the other hand, by (A2), $\sigma_2 x \leq \exists_2\sigma_2 x$. Thus, $\neg\exists_2\sigma_2 x \leq \neg\sigma_2 x \leq x \vee \neg\sigma_2 x$. Then, by Proposition 2.7 and (P4) we obtain $\neg\exists_2\sigma_2 x \leq \exists_2(x \vee \neg\sigma_2)$. Therefore $\exists_2 x \vee \neg\exists_2\sigma_2 x \leq \exists_2(x \vee \neg\sigma_2 x)$.

(P15) It follows from properties (P13) and (P14).

(P16) Using (A3), (P15), (P9) and the fact that the restriction of $\exists_2$ to $B(L)$ is an order preserving map, we have $\exists_2\sigma_1 x \wedge \neg\exists_2(\neg\sigma_1 x \wedge \sigma_2 x) = \exists_2\sigma_1 x \wedge \neg\exists_2\sigma_3(\neg x \wedge \sigma_2 x) = \exists_2\sigma_1 x \wedge \sigma_1\neg(\neg\exists_2 x \wedge \exists_2\sigma_2 x) = \exists_2\sigma_1 x \wedge (\sigma_1\exists_2 x \vee \neg\exists_2\sigma_2 x) = \sigma_1\exists_2 x$.

(P17) It follows easily using (MDP).

(P18) From (P14), (P15) and (P11) we have $[\exists_2\sigma_2 x \wedge \exists_2(x \vee \neg\sigma_2 x)] \vee \exists_2(x \wedge \neg\sigma_2 x) = [\exists_2\sigma_2 x \wedge (\exists_2 x \vee \neg\exists_2\sigma_2 x)] \vee [\neg\exists_2\neg x \wedge \exists_2\sigma_2\neg x] \leq (\exists_2\sigma_2 x \wedge \exists_2 x) \vee (\exists_2 x \wedge \exists_2\sigma_2\neg x) \leq \exists_2 x$. So, now we need to prove the reverse inequality: $\exists_2 x \leq [\exists_2\sigma_2 x \wedge \exists_2(x \vee \neg\sigma_2 x)] \vee \exists_2(x \wedge \neg\sigma_2 x)$. Equivalently, by applying the distributive law and (P17), we will show that $\exists_2 x \leq [\exists_2\sigma_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)] \wedge \exists_2(x \vee \neg\sigma_2 x)$. For this, first we show that $\exists_2 x \leq \exists_2\sigma_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)$ using (MDP). Since $\sigma_1\exists_2 x \leq \sigma_2\exists_2 x \leq \sigma_2\exists_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)$ we have the following two inequalities $\sigma_1 x \leq \sigma_1(\sigma_2\exists_2 x \vee \exists_2(x \wedge \neg\sigma_2 x))$ and $\sigma_2\exists_2 x \leq \sigma_2(\sigma_2\exists_2 x \vee \exists_2(x \wedge \neg\sigma_2 x))$. On the other hand, $\sigma_3(\exists_2\sigma_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)) = \exists_2\sigma_2 x \vee \exists_2(\sigma_3 x \wedge \neg\sigma_2 x) = \exists_2(\sigma_2 x \vee (\sigma_3 x \wedge \neg\sigma_2 x)) = \exists_2\sigma_3 x = \sigma_3\exists_2 x$. Hence, by (MDP), $\exists_2 x \leq \exists_2\sigma_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)$. Moreover, by (A5) and (P4), $\exists_2 x \leq \exists_2(x \vee \neg\sigma_2 x)$. Therefore $\exists_2 x \leq [\exists_2\sigma_2 x \vee \exists_2(x \wedge \neg\sigma_2 x)] \wedge \exists_2(x \vee \neg\sigma_2 x)$. □

3. Connection Between the 2/3-Existential Quantifier and the Existential Quantifier on L_4-Algebras

In this section we establish the main connection between the class $\mathbb{M}_{\frac{2}{3}}\mathbb{L}_4$ and the class of monadic four-valued Łukasiewicz algebras (see [7,11]), denoted by $\mathbb{ML}_4$. We will prove that these classes are polynomially equivalent [3]. A *monadic four-valued Łukasiewicz algebra* (or *ML_4-algebra* , for short) is a pair $\langle L, \exists\rangle$ where L is an L_4-algebra and $\exists\colon L \to L$ is a mapping, called *existential quantifier*, which satisfies the following conditions:

(M1) $\exists 0 = 0$,
(M2) $x \leq \exists x$,
(M3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,
(M4) $\exists\sigma_i x = \sigma_i\exists x$, for $i \in \{1, 2, 3\}$.

In this work, we refer to existential quantifiers as standard existential quantifiers. Notice that these structures are abstractions of monadic functional algebras $\langle L_4^X, \exists\rangle$, where X is a nonempty set and, for each $f \in L_4^X$, $\exists f = \bigvee_{x\in X} f(x) = \exists_1 f$. Under this setting, these monadic structures are natural generalizations of monadic Boolean algebras introduced and developed by Halmos in [9]. The class of monadic four-valued Łukasiewicz algebras is clearly equational; many properties of these algebras can be found in [1,2,7]. Recall that if $\langle L, \exists\rangle$ is a monadic four-valued Łukasiewicz algebra, then $\langle B(L), \exists_{/B(L)}\rangle$ is a monadic Boolean algebra. Moreover, monadic L_4-algebras are semisimple and the simple monadic L_4-algebras are the subalgebras of the monadic functional L_4-algebras $\langle L_4^X, \exists\rangle$.

For each four-valued Łukasiewicz algebra L we denote by $\mathcal{E}(L)$ and $\mathcal{E}_{\frac{2}{3}}(L)$ the sets of existential quantifiers and 2/3-existential quantifiers defined on L, respectively.

Theorem 3.1. *Let L be a four-valued Łukasiewicz algebra.*

(1) *If* $\exists_2\colon L \to L$ *is a 2/3-existential quantifier, then the operator* $\exists_2^*\colon L \to L$ *defined by*

$$\exists_2^* x = \exists_2 \sigma_1 x \vee \exists_2(x \wedge \neg\sigma_1 x) \tag{3.1}$$

for all $x \in L$*, is an existential quantifier.*

(2) *If* $\exists\colon L \to L$ *is an existential quantifier, then the operator* $\exists_2\colon L \to L$ *defined by*

$$\exists_2 x = \exists x \wedge \neg\exists(\neg x \wedge \sigma_2 x) \tag{3.2}$$

for all $x \in L$*, is a 2/3- existential quantifier.*

(3) *The maps* $\psi\colon \mathcal{E}_{\frac{2}{3}}(L) \to \mathcal{E}(L)$ *and* $\varphi\colon \mathcal{E}(L) \to \mathcal{E}_{\frac{2}{3}}(L)$ *defined by* $\psi(\exists_2) = \exists_2^*$ *and* $\varphi(\exists) = \exists_2$ *are mutually inverse, i.e.* $\varphi \circ \psi$ *is the identity function on* $\mathcal{E}_{\frac{2}{3}}(L)$ *and* $\psi \circ \varphi$ *is the identity function on* $\mathcal{E}(L)$*.*

Proof. Let L be a four-valued Łukasiewicz algebra and let $x, y \in L$.

(1) Let $\exists_2\colon L \to L$ be a 2/3-existential quantifier. We check that $\exists_2^*$ satisfies conditions (M1)–(M4).

(M1) It is trivial because $\exists_2 0 = 0$.

(M2) We show that $x \leq \exists_2^* x$ using (MDP). From (A2) it is clear that $\sigma_1 x \leq \exists_2\sigma_1 x \leq \exists_2^* x$. Then $\sigma_1 x \leq \sigma_1\exists_2^* x$. Using (A3), (P7) and (A2) we have $\sigma_i\exists_2^* x = \exists_2\sigma_1 x \vee \exists_2(\sigma_i x \wedge \neg\sigma_1 x) = \exists_2\left(\sigma_1 x \vee (\sigma_i x \wedge \neg\sigma_1 x)\right) = \exists_2\sigma_i x \geq \sigma_i x$, for $i \in \{2,3\}$. Hence, by (MDP), $x \leq \exists_2^* x$.

(M3) We must show that $\exists_2^*\left(x \wedge \exists_2^* y\right) = \exists_2^* x \wedge \exists_2^* y$. From above, we can assure that $\sigma_i\exists_2^* x = \exists_2\sigma_i x$, for $i \in \{2,3\}$. Using (P9) it is easy to see that $\sigma_1\exists_2^* x = \exists_2\sigma_1 x$ holds. So, for $i \in \{1,2,3\}$ we have $\sigma_i\exists_2^*\left(x \wedge \exists_2^* y\right) = \exists_2\sigma_i\left(x \wedge \exists_2^* y\right) = \exists_2\left(\sigma_i x \wedge \sigma_i\exists_2^* y\right) = \exists_2\left(\sigma_i x \wedge \exists_2\sigma_i y\right) = \exists_2\sigma_i x \wedge \exists_2\sigma_i y = \sigma_i\exists_2^* x \wedge \sigma_i\exists_2^* y = \sigma_i\left(\exists_2^* x \wedge \exists_2^* y\right)$. Hence, by (MDP), condition (M3) holds.

(M4) We already have shown that $\sigma_i\exists_2^* x = \exists_2\sigma_i x$ for $i \in \{1,2,3\}$. Since $\exists_2^*\sigma_i x = \exists_2\sigma_i x \vee \exists_2(\sigma_i x \wedge \neg\sigma_i x) = \exists_2\sigma_i x$ for $i \in \{1,2,3\}$, it follows that $\sigma_i\exists_2^* x = \exists_2^*\sigma_i x$ for all $i \in \{1,2,3\}$.

(2) Let $\exists\colon L \to L$ be a standard existential quantifier. We need to check that $\exists_2$ satisfies conditions (A1)–(A6) of Definition 2.5.

(A1) $\exists_2 0 = \exists 0 \wedge \neg\exists(\neg 0 \wedge \sigma_2 0) = 0$.

(A2) Using (M2) we have $\exists_2\sigma_i x = \exists\sigma_i x \wedge \neg\exists(\neg\sigma_i x \wedge \sigma_i x) = \exists\sigma_i x \geq \sigma_i x$, for $i \in \{1,2,3\}$.

(A3) Notice that in the above item we have proved that $\exists_2\sigma_i x = \exists\sigma_i x$ for $i \in \{1,2,3\}$. So, using (M4) we have $\sigma_i\exists_2 x = \sigma_i\exists x \wedge \sigma_i\neg\exists(\neg x \wedge \sigma_2 x) = \exists\sigma_i x \wedge \neg\exists\left(\neg\sigma_i x \wedge \sigma_2 x\right) = \exists\sigma_i x \wedge \neg\exists 0 = \exists\sigma_i x = \exists_2\sigma_i x$, for $i \in \{2,3\}$.

(A4) We prove that $\exists_2(x \wedge \exists_2 y) = \exists_2 x \wedge \exists_2 y$ using (MDP). Let $i \in \{1,2,3\}$. Then

$$\sigma_i\exists_2(x \wedge \exists_2 y) = \sigma_i\left[\exists(x \wedge \exists_2 y) \wedge \neg\exists(\neg(x \wedge \exists_2 y) \wedge \sigma_2(x \wedge \exists_2 y))\right].$$

By (M4) we have

$$\begin{aligned}\sigma_i\exists_2(x \wedge \exists_2 y) = {} & \exists(\sigma_i x \wedge \sigma_i\exists_2 y) \wedge \\ & \wedge\neg\exists((\neg\sigma_i x \wedge \sigma_2 x \wedge \sigma_2\exists_2 y) \vee (\neg\sigma_i\exists_2 y \wedge \sigma_2 x \wedge \sigma_2\exists_2 y)).\end{aligned} \tag{3.3}$$

Since $\exists_2\sigma_i x = \exists\sigma_i x$ for all $i \in \{1,2,3\}$, by (A3) it follows that $\sigma_i\exists_2 x = \exists_2\sigma_i x = \exists\sigma_i x$, for all $i \in \{2,3\}$. Hence, from (3.3) and (M3) we obtain

$$\sigma_i\exists_2(x \wedge \exists_2 y) = \exists\sigma_i x \wedge \sigma_i\exists_2 y \wedge \neg\exists(0 \vee 0) = \sigma_i\exists_2 x \wedge \sigma_i\exists_2 y = \sigma_i(\exists_2 x \wedge \exists_2 y),$$

for $i \in \{2,3\}$. For $i = 1$, note that $\sigma_1\exists_2 y = \exists\sigma_1 y \wedge \neg\exists(\neg\sigma_1 y \wedge \sigma_2 y)$, thus $\sigma_1\exists_2 y \in \exists(L)$. So, from (3.3), (M3) and the fact that $\exists$ preserves the join, we can write $\sigma_1\exists_2(x \wedge \exists_2 y)$ as

$$\exists\sigma_1 x \wedge \sigma_1\exists_2 y \wedge (\neg\exists(\neg\sigma_1 x \wedge \sigma_2 x) \vee \neg\sigma_2\exists_2 y) \wedge (\sigma_1\exists_2 y \vee \neg\exists\sigma_2 x \vee \neg\sigma_2\exists_2 y).$$

Then, by applying the distributive law and taking into account that $\sigma_1\exists_2 x = \exists\sigma_1 x \wedge \neg\exists(\neg\sigma_1 x \wedge \sigma_2 x)$ we obtain

$$\begin{aligned}\sigma_1\exists_2(x \wedge \exists_2 y) &= ((\sigma_1\exists_2 y \wedge \sigma_1\exists_2 x) \vee 0) \wedge (\sigma_1\exists_2 y \vee \neg\exists\sigma_2 x \vee \neg\sigma_2\exists_2 y)\\ &= \sigma_1\exists_2 y \wedge \sigma_1\exists_2 x = \sigma_1(\exists_2 y \wedge \exists_2 x).\end{aligned}$$

(A5) By (M2), $\neg\exists x \leq \neg x$ and so $\exists_2 x \leq \neg\exists(\neg x \wedge \sigma_2 x) \leq \neg(\neg x \wedge \sigma_2 x) \leq x \vee \sigma_2(\neg x)$.

(A6) We will prove $\exists_2(x \vee \neg\sigma_2 x) \leq \exists_2 x \vee \neg\exists_2\sigma_2 x$. By definition of $\exists_2$ we obtain

$$\begin{aligned}\exists_2(x \vee \neg\sigma_2 x) &= \exists(x \vee \neg\sigma_2 x) \wedge \neg\exists\left(\neg(x \vee \neg\sigma_2 x) \wedge \sigma_2(x \vee \neg\sigma_2 x)\right)\\ &= \exists(x \vee \neg\sigma_2 x) \wedge \neg\exists(\neg x \wedge \sigma_2 x) = \neg\exists(\neg x \wedge \sigma_2 x)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\exists_2 x \vee \neg\exists_2\sigma_2 x &= [\exists x \wedge \neg\exists(\neg x \wedge \sigma_2 x)] \vee \neg\exists\sigma_2 x\\ &= [\exists x \vee \neg\exists\sigma_2 x] \wedge [\neg\exists(\neg x \wedge \sigma_2 x) \vee \neg\exists\sigma_2 x]\\ &= [\exists x \vee \neg\exists\sigma_2 x] \wedge \neg\exists(\neg x \wedge \sigma_2 x).\end{aligned} \tag{3.5}$$

It is clear that $\neg\exists x \wedge \exists\sigma_2 x = \exists(\neg\exists x \wedge \sigma_2 x) \leq \exists(\neg x \wedge \sigma_2 x)$, then $\neg\exists(\neg x \wedge \sigma_2 x) \leq \exists x \vee \neg\exists\sigma_2 x$ which completes the proof.

(3) Let $\exists_2 : L \to L$ be a 2/3-existential quantifier. Then, for each $x \in L$,

$$\varphi\psi(\exists_2)(x) = (\exists_2^*)_2\, x = [\exists_2\sigma_1 x \vee \exists_2(x \wedge \neg\sigma_1 x)] \wedge \neg\exists_2(\neg x \wedge \sigma_2 x). \tag{3.6}$$

We will use condition (MDP). First, by applying in (3.6) the distributivity property, we obtain

$$\varphi\psi(\exists_2)(x) = (\exists_2\sigma_1 x \wedge \neg\exists_2(\neg x \wedge \sigma_2 x)) \vee (\exists_2(x \wedge \neg\sigma_1 x) \wedge \neg\exists_2(\neg x \wedge \sigma_2 x)).$$

Then, by (P16) it follows that $\sigma_1\varphi\psi(\exists_2)(x) = (\exists_2\sigma_1 x \wedge \neg\exists_2(\neg\sigma_1 x \wedge \sigma_2 x)) \vee 0 = \sigma_1\exists_2 x$. From (3.6), (P9), (A3) and (P7) we have $\sigma_i\varphi\psi(\exists_2)(x) = \exists_2\sigma_1 x \vee \exists_2(\sigma_i x \wedge \neg\sigma_1 x) = \exists_2\left(\sigma_1 x \vee (\sigma_i x \wedge \neg\sigma_1 x)\right) = \exists_2\sigma_i x = \sigma_i\exists_2 x$, for $i \in \{2,3\}$. Therefore, by (MDP), $\varphi\psi(\exists_2) = \exists_2$.

Now, let $\exists : L \to L$ be an existential quantifier. So, for each $x \in L$,

$$\psi\varphi(\exists)(x) = \exists_2^*(x) = \exists_2\sigma_1 x \vee \exists_2(x \wedge \neg\sigma_1 x).$$

Since $\exists_2\sigma_1 x = \exists\sigma_1 x$ and $\exists_2(x \wedge \neg\sigma_1 x) = \exists(x \wedge \neg\sigma_1 x) \wedge \neg\exists(\neg x \wedge \sigma_2 x)$, using the distributive law and the fact that $\exists$ preserves the join, we obtain

$$\psi\varphi(\exists)(x) = \exists x \wedge [\exists\sigma_1 x \vee \neg\exists(\neg x \wedge \sigma_2 x)]. \tag{3.7}$$

Then, by applying (MDP) to expression (3.7), we have $\psi\varphi(\exists) = \exists$. □

4. Connection Between 2/3-Quantifiers, Lattice Quantifiers and Boolean Quantifiers

Given a bounded distributive lattice A, an operator $\exists\colon A \to A$ is an *existential quantifier* (see Cignoli [5]) provided it satisfies conditions (M1), (M2), (M3) and

(M5) $\exists(x \vee y) = \exists x \vee \exists y$.

Let $\langle L, \exists_2 \rangle$ be a 2/3-monadic L_4-algebra and let us consider the set

$$I_L := \{a \in L : \ \sigma_2 a = 0\} = \{a \in L : a \leq \neg a\} = \{a \wedge \neg a : a \in L\}.$$

It is clear that the set I_L is a lattice ideal of L and therefore $I_L \cup \{1\}$ is a bounded distributive lattice. Moreover, I_L is closed under the operator $\exists_2$. Notice that if $a \in I_L \cup \{1\}$ and $\exists_2 a = 1$ then $a = 1$.

Proposition 4.1. *Let $\langle L, \exists_2 \rangle$ be a 2/3-monadic L_4-algebra. Let $\exists_I$ be the restriction of $\exists_2$ to $I_L \cup \{1\}$. Then $\exists_I$ is an existential quantifier of bounded distributive lattices.*

Proof. We need to prove that the operator $\exists_I$ satisfies conditions (M1)–(M3) and (M5). Conditions (M1) and (M3) are straightforward. To show (M2), let $x \in I_L \cup \{1\}$. If $x = 1$, then $1 = \exists 1$. Suppose that $x \in I_L$. So, $\sigma_2 x = 0$ and then $\sigma_1 x = 0$. Thus, $\sigma_i x \leq \sigma_i \exists_2 x$ for $i = 1, 2$. Now, by (A2) and (A3) we have $\sigma_3 x \leq \exists_2 \sigma_3 x = \sigma_3 \exists_2 x = \sigma_3 \exists_I x$. Hence, by (MDP), $x \leq \exists_I x$. For (M5) it is easy to check that if $x \leq y$ then $\exists_I x \leq \exists_I y$. So, it is clear that $\exists_I x \vee \exists_I y \leq \exists_I (x \vee y)$. Now, for the reverse inequality we use again condition (MDP). If either $x = 1$ or $y = 1$ then (M5) holds. Suppose $x, y \in I_L$. So $x \vee y \in I_L$ and then $\exists_I (x \vee y) \in I_L$. Thus, $\sigma_1 \exists_I (x \vee y) = \sigma_2 \exists_I (x \vee y) = 0$. Hence, $\sigma_i \exists_I (x \vee y) \leq \sigma_i \left(\exists_I x \vee \exists_I y\right)$ for $i = 1, 2$. Now, for σ_3 we have $\sigma_3 \exists_I (x \vee y) = \sigma_3 \exists_2 (x \vee y) = \exists_2 (\sigma_3 x \vee \sigma_3 y) = \exists_2 \sigma_3 x \vee \exists_2 \sigma_3 y = \sigma_3 \exists_I x \vee \sigma_3 \exists_I y = \sigma_3 \left(\exists_I x \vee \exists_I y\right)$. Therefore $\exists_I (x \vee y) = \exists_I x \vee \exists_I y$. □

Thus, we can conclude that the 2/3-existential quantifier $\exists_2$ induces both a Boolean existential quantifier $\exists_{2/B(L)}$ on $B(L)$ and a lattice existential quantifier $\exists_I$ on I_L. Let $\langle L, \exists_2 \rangle$ be a 2/3-monadic L_4-algebra. Then $\neg x \wedge \sigma_2 x, x \wedge \neg \sigma_2 x \in I_L$, for all $x \in L$. Using (P13) we can rewrite the equality (P18) as follows

$$\begin{aligned} \exists_2 x &= [\exists_2 \sigma_2 x \wedge \neg \exists_2 (\neg x \wedge \sigma_2 x)] \vee \exists_2 (x \wedge \neg \sigma_2 x) \\ &= \left[\exists_{B(L)} \sigma_2 x \wedge \neg \exists_I (\neg x \wedge \sigma_2 x)\right] \vee \exists_I (x \wedge \neg \sigma_2 x). \end{aligned}$$

Hence, $\exists_2$ can be expressed in terms of $\exists_{B(L)}$ and $\exists_I$. Now, we want to prove a reciprocal statement of the previous result. In other words, if L is an L_4-algebra, $\exists_B\colon B(L) \to B(L)$ is a Boolean existential quantifier and $\exists_I\colon I_L \cup \{1\} \to I_L \cup \{1\}$ is a lattice existential quantifier, can we define a 2/3-existential quantifier from them? The following proposition answers to this question.

Proposition 4.2. *Let L be an L_4-algebra. Let $\exists_I\colon I_L \cup \{1\} \to I_L \cup \{1\}$ be a lattice existential quantifier and let $\exists_B\colon B(L) \to B(L)$ be a Boolean existential quantifier such that the following conditions hold:*

(1) $\sigma_3\exists_I x = \exists_B\sigma_3 x$, *for all* $x \in I_L$,
(2) $\exists_I x = 1$ *implies* $x = 1$.

Then the operator $\exists_2 \colon L \to L$ *defined by*

$$\exists_2 x = [\exists_B\sigma_2 x \wedge \neg\exists_I(\neg x \wedge \sigma_2 x)] \vee \exists_I(x \wedge \neg\sigma_2 x)$$

is a 2/3-existential quantifier.

Proof. We need to prove that $\exists_2$ satisfies conditions (A1)–(A6) in Definition 2.5. Let $x, y \in L$.

(A1) It is immediate by definition.

(A2) Let $i \in \{1,2,3\}$. So $\exists_2\sigma_i x = [\exists_B\sigma_i x \wedge \neg\exists_I(\neg\sigma_i x \wedge \sigma_i x)] \vee \exists_I(\sigma_i x \wedge \neg\sigma_i x) = \exists_B\sigma_i x \geq \sigma_i x$. Notice that if $a \in B(L)$, then $\exists_2 a = \exists_B a$.

(A3) We need to prove that $\sigma_j\exists_2 x = \exists_2\sigma_j x$ for $j = 2, 3$. By definition of $\exists_2$ we have $\sigma_j\exists_2 x_2 = [\exists_B\sigma_2 x \wedge \sigma_j\neg\exists_I(\neg x \wedge \sigma_2 x)] \vee \sigma_j\exists_I(x \wedge \neg\sigma_2 x) = [\exists_B\sigma_2 x \wedge \neg\sigma_{4-j}\exists_I(\neg x \wedge \sigma_2 x)] \vee \sigma_j\exists_I(x \wedge \neg\sigma_2 x)$. For $j = 2$ or 3, it follows that $4 - j = 2$ or 1. Since $\neg x \wedge \sigma_2 x \in I_L$, it follows by condition (2) that $\sigma_{4-j}\exists_I(\neg x \wedge \sigma_2) = 0$. So, we obtain $\sigma_j\exists_2 x = \exists_B\sigma_2 x \vee \sigma_j\exists_I(x \wedge \neg\sigma_2 x)$. Now, if $j = 2$ then $\sigma_2\exists_2 x = \exists_B\sigma_2 x \vee \sigma_2\exists_I(x \wedge \neg\sigma_2 x) = \exists_B\sigma_2 x = \exists_2\sigma_2 x$. On the other hand, for $j = 3$, it follows from condition (1) that $\sigma_3\exists_2 x = \exists_B\sigma_2 x \vee \sigma_3\exists_I(x \wedge \neg\sigma_2 x) = \exists_B\sigma_2 x \vee \exists_B(\sigma_3 x \wedge \neg\sigma_2 x) = \exists_B(\sigma_2 x \vee (\sigma_3 x \wedge \neg\sigma_2 x)) = \exists_B\sigma_3 x = \exists_2\sigma_3 x$.

(A4) To prove $\exists_2(x \wedge \exists_2 y) = \exists_2 x \wedge \exists_2 y$ we use (MDP). On the one hand, by (A3) and using that $\exists_2 a = \exists_B a$ for all $a \in B(L)$, we have

$$\begin{aligned}\exists_2(x \wedge \exists_2 y) =& [\exists_B(\sigma_2 x \wedge \exists_B\sigma_2 y) \wedge \neg\exists_I((\neg x \vee \neg\exists_2 y) \wedge (\sigma_2 x \wedge \exists_B\sigma_2 y))] \vee \\ & \vee \exists_I(x \wedge \exists_2 y \wedge (\neg\sigma_2 x \vee \neg\exists_B\sigma_2 y)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\exists_2 x \wedge \exists_2 y =& [[\exists_B\sigma_2 x \wedge \neg\exists_I(\neg x \wedge \sigma_2 x)] \vee \exists_I(x \wedge \neg\sigma_2 x)] \wedge \\ & \wedge [[\exists_B\sigma_2 y \wedge \neg\exists_I(\neg y \wedge \sigma_2 y)] \vee \exists_I(y \wedge \neg\sigma_2 y)].\end{aligned}$$

Now, we apply the endomorphisms σ_1, σ_2 and σ_3. First,

$$\begin{aligned}\sigma_1\exists_2(x \wedge \exists_2 y) =& \exists_B\sigma_2 x \wedge \exists_B\sigma_2 y \wedge \neg\exists_B[\neg\sigma_1 x \wedge \sigma_2 x \wedge \exists_B\sigma_2 y] \wedge \\ & \wedge \neg\exists_B[\neg\sigma_1\exists_2 y \wedge \sigma_2 x \wedge \exists_B\sigma_2 y].\end{aligned}$$

Replacing $\exists_2 y$ by its definition we have

$$\begin{aligned}\sigma_1\exists_2(x \wedge \exists_2 y) &= \exists_B\sigma_2 x \wedge \exists_B\sigma_2 y \wedge \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x) \wedge \neg\exists_B(\neg\sigma_1 y \wedge \sigma_2 y) \\ &= \sigma_1(\exists_2 x \wedge \exists_2 y).\end{aligned}$$

Now, let $i \in \{2,3\}$. Then, $\sigma_i(\exists_2 x \wedge \exists_2 y) = \exists_2\sigma_i x \wedge \exists_2\sigma_i y = \exists_B\sigma_i x \wedge \exists_B\sigma_i y = \exists_B(\sigma_i x \wedge \exists_B\sigma_i y) = \exists_B(\sigma_i x \wedge \sigma_i\exists_2 y) = \exists_B\sigma_i(x \wedge \exists_2 y) = \sigma_i\exists_2(x \wedge \exists_2 y)$.

(A5) We prove $\exists_2 x \leq x \vee \neg\sigma_2 x$ using (MDP). So, applying σ_1 we obtain $\sigma_1\exists_2 x = \exists_B\sigma_2 x \wedge \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x) \leq \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x) \leq \neg(\neg\sigma_1 x \wedge \sigma_2 x) = \sigma_1(x \vee \neg\sigma_2 x)$. Now, let $i \in \{2,3\}$. Since $\sigma_i(x \vee \neg\sigma_2 x) = 1$ it follows that $\sigma_i\exists_2 x \leq \sigma_i(x \vee \neg\sigma_2 x)$.

(A6) By definition of $\exists_2$ we have $\exists_2(x \vee \neg\sigma_2 x) = \neg\exists_I(\neg x \wedge \sigma_2 x)$. Thus, by condition (1), we obtain $\sigma_1\exists_2(x \vee \neg\sigma_2 x) = \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x)$. On the other hand, $\sigma_1(\exists_2 x \vee \neg\exists_2\sigma_2 x) = \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x) \vee \neg\exists_2\sigma_2 x \geq \neg\exists_B(\neg\sigma_1 x \wedge \sigma_2 x)$.

Then $\sigma_1\exists_2(x\vee\neg\sigma_2 x) \leq \sigma_1(\exists_2 x\vee\neg\exists_2\sigma_2 x)$. It is straightforward to check directly that $\sigma_i\exists_2(x \vee \neg\sigma_2 x) \leq \sigma_i(\exists_2 x \vee \neg\exists_2\sigma_2 x) = 1$, for $i \in \{2, 3\}$. Hence, by (MDP), we conclude $\exists_2(x \vee \neg\sigma_2 x) \leq \exists_2 x \vee \neg\exists_2\sigma_2 x$. □

Proposition 4.3. *Let $\exists_2$ and $\exists_2'$ be two 2/3-existential quantifiers on an L_4-algebra L such that coincide on $B(L)$. Then $\exists_2 = \exists_2'$.*

Proof. It is a consequence of (A3) and (P16). □

Let L be an L_4-algebra. It is clear that $\sigma_3(I_L)$ is an ideal of $B(L)$. Let $I_B := \sigma_3(I_L)$. Notice that the following property holds:

$$\text{for each } p \in I_B \text{ there exists a unique } t \in I_L \text{ such that } p = \sigma_3 t. \tag{u}$$

Indeed, if $p \in I_B$ there exists $t \in I_L$ such that $p = \sigma_3 t$. Suppose that there exists $t' \in I_L$ such that $p = \sigma_3 t'$. Thus $\sigma_3 t' = \sigma_3 t$ and since both elements are in I_L we have $\sigma_i t' = \sigma_i t$ for $i \in \{1, 2\}$. Then $t = t'$ follows by (MDP).

Theorem 4.4. *Let L be an L_4-algebra.*

(1) *Let $\exists_2 \colon L \to L$ be a 2/3-existential quantifier. Then, for all $p \in B(L)$, $p \in I_B$ implies $\exists_2(p) \in I_B$.*
(2) *Let $\exists \colon B(L) \to B(L)$ be an existential quantifier of Boolean algebras. Then $\exists$ can be extended to a (necessarily unique) 2/3-existential quantifier on L if and only if $\exists p \in I_B$ whenever that $p \in I_B$.*

Proof. Item (1) follows immediately from (A3).

(2) Let $\exists \colon B(L) \to B(L)$ be an existential quantifier of Boolean algebras which satisfies the following property: $p \in I_B$ implies $\exists p \in I_B$. Notice that for each $x \in L$, by (L9), $x \wedge \neg x = (\neg x \wedge \sigma_2 x) \vee (x \wedge \neg\sigma_2 x)$ with $\neg x \wedge \sigma_2 x \in I_L$ and $x \wedge \neg\sigma_2 x \in I_L$. Hence, $\exists(\sigma_3(\neg x \wedge \sigma_2 x)) \in I_B$ and $\exists(\sigma_3(x \wedge \neg\sigma_2 x)) \in I_B$. Therefore, by Property (u), there exist unique elements $t_x \in I_L$ and $v_x \in I_L$ such that

$$\exists\sigma_3(\neg x \wedge \sigma_2 x) = \sigma_3 t_x \quad \text{and} \quad \exists\sigma_3(x \wedge \neg\sigma_2 x) = \sigma_3 v_x. \tag{4.1}$$

We define the operator $\exists_2 \colon L \to L$ as follows:

$$\exists_2 x := (\exists\sigma_2 x \wedge \neg t_x) \vee v_x.$$

We claim that $\exists_2$ is a 2/3-existential quantifier that extends $\exists$. Indeed, if $x \in B(L)$ then $x = \sigma_2 x$ and $\sigma_3 t_x = 0 = \sigma_3 v_x$, hence $t_x = v_x = 0$ and $\exists_2 x = (\exists x \wedge 1) \vee 0 = \exists x$. Therefore, $\exists_2 x = \exists x$ for all $x \in B(L)$. Now, we want to prove conditions (A1)-(A6) of Definition 2.5.

Since $\exists_2$ and $\exists$ coincide on $B(L)$, it follows immediately that conditions (A1), (A2) and (A3) hold. For (A4), let $x, y \in L$ and let $z := x \wedge \exists_2 y$. Then

$$\exists_2(x \wedge \exists_2 y) = \exists_2 z = (\exists\sigma_2 z \wedge \neg t_z) \vee v_z,$$

where $t_z, v_z \in I_L$ are the unique elements such that $\sigma_3 t_z = \exists\sigma_3(\neg z \wedge \sigma_2 z)$ and $\sigma_3 v_z = \exists\sigma_3(z\wedge\neg\sigma_2 z)$. Moreover, $\exists_2 x = (\exists\sigma_2 x\wedge\neg t_x)\vee v_x$ where $t_x, v_x \in I_L$ are the unique elements such that $\sigma_3 t_x = \exists\sigma_3(\neg x\wedge\sigma_2 x)$ and $\sigma_3 v_x = \exists\sigma_3(x\wedge\neg\sigma_2 x)$.

Now, we use the (MDP) principle to prove $\exists_2(x\wedge\exists_2 y) = \exists_2 x\wedge\exists_2 y$. First,

$$\sigma_1\exists_2 z = (\exists\sigma_2 z \wedge \sigma_1\neg t_z) \vee 0 = \exists\sigma_2 z \wedge \neg\exists\sigma_3(\neg z \wedge \sigma_2 z). \tag{4.2}$$

Notice that $\exists\sigma_2 z = \exists\sigma_2(x \wedge \exists_2 y) = \exists(\sigma_2 x \wedge \exists\sigma_2 y) = \exists\sigma_2 x \wedge \exists\sigma_2 y$. Since

$$\sigma_3(\neg z \wedge \sigma_2 z) = \sigma_3((\neg x \vee \neg\exists_2 y) \wedge \sigma_2 x \wedge \sigma_2\exists_2 y) = \\ [\sigma_3(\neg x \wedge \sigma_2 x) \wedge \exists\sigma_2 y] \vee [\sigma_2 x \wedge \neg\sigma_1\exists_2 y \wedge \sigma_2\exists_2 y],$$

it follows that $\exists\sigma_3(\neg z \wedge \sigma_2 z) = \exists[\sigma_3(\neg x \wedge \sigma_2 x) \wedge \exists\sigma_2 y] \vee \exists[\sigma_2 x \wedge \neg\sigma_1\exists_2 y \wedge \exists\sigma_2 y]$. Now, notice that $\sigma_1\exists_2 y = \sigma_1[\exists\sigma_2 y \wedge \neg t_y] = \exists\sigma_2 y \wedge \neg\sigma_3 t_y = \exists\sigma_2 y \wedge \neg\exists\sigma_3(\neg y \wedge \sigma_2 y) \in \exists(B(L))$. Then,

$$\exists\sigma_3(\neg z \wedge \sigma_2 z) = [\exists\sigma_3(\neg x \wedge \sigma_2 x) \wedge \exists\sigma_2 y] \vee [\exists\sigma_2 x \wedge \neg\sigma_1\exists_2 y \wedge \exists\sigma_2 y] = \\ = [\exists\sigma_3(\neg x \wedge \sigma_2 x) \vee (\exists\sigma_2 x \wedge \neg\sigma_1\exists_2 y)] \wedge \exists\sigma_2 y.$$

Hence, $\neg\exists\sigma_3(\neg z \wedge \sigma_2 z) = [\neg\sigma_3 t_x \wedge (\neg\exists\sigma_2 x \vee \sigma_1\exists_2 y)] \vee \neg\exists\sigma_2 y$. Therefore, by replacing this term in (4.2) and taking into account that $\sigma_1\exists_2 y \leq \sigma_2\exists_2 y = \exists\sigma_2 y$, we obtain $\sigma_1\exists_2 z = \exists\sigma_2 x \wedge \exists\sigma_2 y \wedge [(\neg\sigma_3 t_x \wedge (\neg\exists\sigma_2 x \vee \sigma_1\exists_2 y)) \vee \neg\exists\sigma_2 y] = \exists\sigma_2 x \wedge \neg\sigma_3 t_x \wedge \sigma_1\exists_2 y$. It is easy to cheek that $\sigma_1(\exists_2 x \wedge \exists_2 y) = [(\exists\sigma_2 x \wedge \neg\sigma_3 t_x) \vee \sigma_1 v_x] \wedge \sigma_1\exists_2 y = \exists\sigma_2 x \wedge \neg\sigma_3 t_x \wedge \sigma_1\exists_2 y = \sigma_1\exists_2 z$. On the other hand, $\sigma_2\exists_2 z = \sigma_2(\exists\sigma_2 z \wedge \neg t_z) \vee 0 = \exists\sigma_2 z \wedge 1 = \exists\sigma_2 z = \exists\sigma_2(x \wedge \exists_2 y) = \exists\sigma_2 x \wedge \exists\sigma_2 y = \sigma_2(\exists_2 x \wedge \exists_2 y)$, and similarly we have $\sigma_3\exists_2 z = \exists\sigma_3 z = \exists\sigma_3(x \wedge \exists_2 y) = \sigma_3(\exists_2 x \wedge \exists_2 y)$.

To prove (A5) we will use (MDP). Let $x \in L$. So $\exists_2 x = (\exists\sigma_2 x \wedge \neg t_x) \vee v_x$ where $t_x, v_x \in I_L$ are the unique elements such that $\sigma_3 t_x = \exists\sigma_3(\neg x \wedge \sigma_2 x)$ and $\sigma_3 v_x = \exists\sigma_3(x \wedge \neg\sigma_2 x)$. Then $\sigma_1\exists_2 x = (\exists\sigma_2 x \wedge \neg\sigma_3 t_x) \vee \sigma_1 v_x = \exists\sigma_2 x \wedge \neg\sigma_3 t_x = \exists\sigma_2 x \wedge \neg\exists(\neg\sigma_1 x \wedge \sigma_2 x)$. Since $\neg\sigma_1 x \wedge \sigma_2 x \leq \exists(\neg\sigma_1 x \wedge \sigma_2 x)$ we have $\neg\exists(\neg\sigma_1 x \wedge \sigma_2 x) \leq \sigma_1(x \vee \neg\sigma_2 x))$ and thus $\sigma_1\exists_2 x = \exists\sigma_2 x \wedge \neg\exists(\neg\sigma_1 x \wedge \sigma_2 x) \leq \neg\exists(\neg\sigma_1 x \wedge \sigma_2 x) \leq \sigma_1(x \vee \neg\sigma_2 x)$. Since $\sigma_j(x \vee \neg\sigma_2 x) = 1$ for $j \in \{2, 3\}$ the proof is complete.

Finally, to prove (A6) let $x \in L$ and let $z := x \vee \neg\sigma_2 x$. Then

$$\exists_2(x \vee \neg\sigma_2 x) = \exists_2 z = (\exists\sigma_2 z \wedge \neg t_z) \vee v_z,$$

where $t_z, v_z \in I_L$ are the unique elements such that $\sigma_3 t_z = \exists\sigma_3(\neg z \wedge \sigma_2 z)$ and $\sigma_3 v_z = \exists\sigma_3(z \wedge \neg\sigma_2 z)$. Moreover

$$\exists_2 x = (\exists\sigma_2 x \wedge \neg t_x) \vee v_x$$

where $t_x, v_x \in I_L$ are the unique elements such that $\sigma_3 t_x = \exists\sigma_3(\neg x \wedge \sigma_2 x)$ and $\sigma_3 v_x = \exists\sigma_3(x \wedge \neg\sigma_2 x)$. It is easy to check that $\sigma_2 z = 1$. Then $v_z = 0$ and hence $\exists_2 z = \neg t_z$. Moreover $\sigma_3 t_z = \sigma_3 t_x$, which implies $t_z = t_x$. Then $\exists_2 x = (\exists\sigma_2 x \wedge \neg t_z) \vee v_x$. Therefore, to prove (A6) we must show that

$$\neg t_z \leq [(\exists\sigma_2 x \wedge \neg t_z) \vee v_x] \vee \neg\exists_2\sigma_2 x,$$

which is true because $[(\exists\sigma_2 x \wedge \neg t_z) \vee v_x] \vee \neg\exists_2\sigma_2 x = \neg t_z \vee v_x \vee \neg\exists_2\sigma_2 x \geq \neg t_z$. □

5. Completeness Theorems for Monadic Four-Valued Łukasiewicz Logics

In this section, we will prove a completeness theorem for the 2/3-monadic four-valued Łukasiewicz predicate logic. To this purpose, we follow a similar approach to that used by Krongold in [13] for the development of the classical monadic functional calculus of first order. To define a 2/3-monadic four-valued Łukasiewicz predicate logic, we use the axiomatization of the n-valued Łukasiewicz propositional calculus given by Cignoli in [4].

Firstly, we give a brief overview of a completeness theorem for the monadic four-valued Łukasiewicz predicate calculus corresponding to the standard universal quantifier. Due to a matter of simplicity throughout this section, we shall deal with universal quantifiers instead of existential quantifiers. We prove a completeness theorem for the monadic four-valued Łukasiewicz predicate calculus corresponding to the dual of the quantifier $\exists_{\frac{2}{3}}$, using the fact that both quantifiers, the standard and the alternative one, are interdefinable.

In this part of the paper we shall consider an equivalent definition of four-valued Łukasiewicz algebras to that given in Sect. 1. We use the characterization of Łukasiewicz algebras in terms of symmetric Heyting algebras given by Iturrioz in [12]. A *four-valued Łukasiewicz algebra* (see [12] and [4]) can be also defined as an algebra $\langle A, \vee, \wedge, \Rightarrow, \neg, \sigma_1, \sigma_2, \sigma_3, 0, 1\rangle$ such that $\langle A, \vee, \wedge, \Rightarrow, \neg, 0, 1\rangle$ is a symmetric Heyting algebra [15] and σ_1, σ_2 and σ_3 are unary operations that satisfy conditions (L2), (L4), (L6), (L7) and

(L10) $\sigma_1 x \vee x = x$,

(L11) $\sigma_i(x \Rightarrow y) = \bigwedge_{j=i}^{3}(\sigma_j x \Rightarrow \sigma_j y)$, for $1 \leq i \leq 3$.

In Sect. 3, a monadic four-valued Łukasiewicz algebra was defined as pairs $\langle L, \exists\rangle$ where L is an L_4-algebra and $\exists\colon L \to L$ is an existential quantifier. The dual quantifier associated with the existential quantifier $\exists$ is defined as usual (i.e. $\forall = \neg\exists\neg$). Thus, the class of monadic four-valued Łukasiewicz algebras can be also defined as pairs $\langle A, \forall\rangle$ [1,2,7], where A is an L_4-algebra and $\forall\colon A \to A$ is an operator, called *universal quantifier*, such that satisfies the following properties: for every $x, y \in A$,

(ML1) $\forall 1 = 1$;
(ML2) $\forall x \leq x$;
(ML3) $\forall(x \vee \forall y) = \forall x \vee \forall y$;
(ML4) $\forall \sigma_i x = \sigma_i \forall x$, for $1 \leq i \leq 3$.

It is not hard to check that if $\forall\colon A \to A$ is an operator on an L_4-algebra A and $\exists\colon A \to A$ is defined by $\exists x = \neg\forall\neg x$ for all $x \in A$, then $\forall$ is an universal quantifier if and only if $\exists$ is an existential quantifier. Moreover $\forall x = \neg\exists\neg x$ for all $x \in A$. Without loss of generality, we also denote by $\mathbb{ML}_4$ the class of monadic L_4-algebras (ML_4-algebra) endowed with a universal quantifier. The following proposition gives several classical properties on ML_4-algebras.

Proposition 5.1. *Let $\langle A, \forall\rangle$ be an ML_4-algebra. For every $a, b \in A$, the following conditions hold:*

(1) $\forall 0 = 0$,
(2) $a \in \forall(A)$ *if and only if* $\forall a = a$,
(3) $\forall(a \wedge b) = \forall a \wedge \forall b$,
(4) $\langle B(L), \forall_{/B(L)}\rangle$ *is a monadic Boolean algebra and* $\forall(B(L)) = \forall(L) \cap B(L)$,

(5) $\forall(\forall a \Rightarrow b) = \forall a \Rightarrow \forall b$,
(6) $\forall(A)$ *is an* L_4*-subalgebra of* A.

Proof. The proof is standard and it is ommited. □

A *language* $\mathcal{L}$ of a monadic four-valued Łukasiewicz predicate logic consists of a single variable x, a countable set of unary predicate letters $\{P_n : n \in \omega\}$, a countable set of constant letters $\{a_n : n \in \omega\}$, the binary logic connectives $\Rightarrow, \wedge, \vee$, the unary logic connectives $\neg, \sigma_1, \sigma_2, \sigma_3$, the universal quantifier $\forall$ and the punctuation symbols (,). We denote by $\mathcal{F}m(\mathcal{L})$ the set of all formulas of $\mathcal{L}$ defined as usual. We shall sometimes omit parentheses as long as no ambiguity is caused. Given a formula α, we shall denote by $\alpha(t)$ the result of substituting t in α for every free occurrence of x, where $t = x$ or $t = a_n$, for some constant letter a_n of $\mathcal{L}$.

Let $\langle A, \forall\rangle$ be a monadic L_4-algebra. A *constant* of $\langle A, \forall\rangle$ is an L_4-homomorphism $c\colon A \to \forall(A)$ such that the restriction of c to $\forall(A)$ is the identity function. We denote by $C_\forall(A)$ the set of all constants of $\langle A, \forall\rangle$. An *interpretation* M of the language $\mathcal{L}$ is a system $M = \langle A, \forall, v, h\rangle$ where $\langle A, \forall\rangle$ is a monadic L_4-algebra, v is a map from $\{P_n : n \in \omega\}$ into A and h is a map from $\{a_n : n \in \omega\}$ into $C_\forall(A)$. Each interpretation $M = (A, \forall, v, h)$ induces a map $\widehat{v}\colon \mathcal{F}m(\mathcal{L}) \to A$ defined recursively as follows:

- $\widehat{v}(P_n(x)) = v(P_n)$, for all $n \in \omega$;
- $\widehat{v}(P_n(a_m)) = h(a_m)(v(P_n)) = h(a_m)(\widehat{v}(P_n(x)))$, for all $n, m \in \omega$;
- $\widehat{v}(\alpha \circ \beta) = \widehat{v}(\alpha) \circ \widehat{v}(\beta)$ where $\circ \in \{\wedge, \vee \Rightarrow\}$;
- $\widehat{v}(\sigma_i\alpha) = \sigma_i\widehat{v}(\alpha)$, for $i \in \{1, 2, 3\}$;
- $\widehat{v}(\neg\alpha) = \neg\widehat{v}(\alpha)$;
- $\widehat{v}(\forall x\alpha) = \forall\widehat{v}(\alpha)$.

Notice that $\widehat{v}(\alpha(a_m)) = h(a_m)(\widehat{v}(\alpha))$ for every constant letter a_m of $\mathcal{L}$ and every $\alpha \in \mathcal{F}m(\mathcal{L})$.

Definition 5.2. A formula $\alpha \in \mathcal{F}m(\mathcal{L})$ is said to be:

(a) *true in an interpretation* $M = \langle A, \forall, v, h\rangle$ of $\mathcal{L}$ (in symbols $M \Vdash \alpha$) if $\widehat{v}(\alpha) = 1$;

(b) *logically valid in* $\mathcal{L}$ if $M \Vdash \alpha$ for every interpretation M of $\mathcal{L}$.

Let $\Gamma \subseteq \mathcal{F}m(\mathcal{L})$. If $M \Vdash \beta$ for every $\beta \in \Gamma$ we write $M \Vdash \Gamma$.

Definition 5.3. A formula $\alpha \in \mathcal{F}m(\mathcal{L})$ is *logical consequence* of a set $\Gamma \subseteq \mathcal{F}m(\mathcal{L})$ (in symbols $\Gamma \vDash \alpha$) if for every interpretation M of $\mathcal{L}$, $M \Vdash \Gamma$ implies $M \Vdash \alpha$.

Proposition 5.4. *Let* $\langle A, \forall_A\rangle$ *and* $\langle B, \forall_B\rangle$ *be monadic* L_4*-algebras and let* $f\colon A \to B$ *be an homomorphism. If* c *is a constant of* A*, then* $c^*\colon f(A) \to \forall_B(f(A))$ *is a constant of* $f(A)$ *where* $c^*(f(a)) = f(c(a))$.

Proof. It is clear that $(f(A), \forall_B)$ is a monadic subalgebra of $(B, \forall_B)$. Moreover, $c(a) = \forall_A(c(a))$, thus $f(c(a)) = f(\forall_A(c(a))) = \forall_B f(c(a)) \in \forall_B(f(A))$. To see that c^* is well defined (f can not be injective) it is enough to prove that for every $a \in A$, $f(a) = 1$ implies $f(c(a)) = 1$. Let us suppose that $f(a) = 1$.

Then $1 = \forall_B(f(a)) = f(\forall_A(a)) \leq f(c(a))$ because $\forall_A a \leq c(a)$. It easy to see that c^* is an L_4-homomorphism and the restriction of c^* to $\forall_B(f(A))$ is the identity map, because $c^*(\forall_B(f(a))) = c^*(f(\forall_A(a))) = f(c(\forall_A(a))) = f(\forall_A(a)) = \forall_B(f(a))$, for all $a \in A$. □

Proposition 5.5. *A formula $\alpha \in \mathcal{F}m(\mathcal{L})$ is logically valid if and only if α is true in each interpretation $\langle \mathbf{4}^X, \forall, v, h\rangle$ where*

$$(\forall f)(x) = \bigwedge_{y \in X} f(y) \tag{5.1}$$

for every $f \in \mathbf{4}^X$ and each $x \in X$.

Proof. Let $\alpha \in \mathcal{F}m(\mathcal{L})$. Suppose that α is not logically valid. So, there is an interpretation $M = \langle A, \forall, v, h\rangle$ such that $\widehat{v}(\alpha) < 1$. Then, there exists a maximal monadic implicative filter U of $\langle A, \forall\rangle$ such that $\widehat{v}(\alpha) \notin U$ [1, pp. 78]. Then A/U is a simple algebra and so it is isomorphic to a subalgebra of $\mathbf{4}^X$ for some nonempty set X [1]. Thus, there is a monadic homomorphism $\chi\colon A \to \mathbf{4}^X$ such that $U = \chi^{-1}(\{1\})$. Let $v^* = \chi \circ v\colon \{P_n : n \in \omega\} \to \mathbf{4}^X$. Now, let a_n be a constant letter of $\mathcal{L}$. By Proposition 5.4, $h(a_n)^*$ is a constant of $\chi(A)$ where $h(a_n)^*(\chi(a)) = \chi(h_A(a_n)(a))$. Since $\mathbf{4}$ is an injective algebra [2, p. 371] and $\forall(\chi(A)) \subseteq L_4$, it follows that $h(a_n)^*$ can be extended to a homomorphism $\overline{h}(a_n)\colon \mathbf{4}^X \to \mathbf{4}$ such that $\overline{h}(a_n)$ is a constant of $\mathbf{4}^X$. Hence, we can define the interpretation $M^* = \langle \mathbf{4}^X, \forall, v^*, \overline{h}\rangle$ of $\mathcal{L}$ where $\forall$ is defined as in (5.1). It is not hard to check that $\widehat{v^*} = \chi \circ \widehat{v}$. Then, we have that $\widehat{v^*}(\alpha) = \chi(\widehat{v}(\alpha)) \neq 1$. Therefore, α is not true in the interpretation $M^* = \langle \mathbf{4}^X, \forall, v^*, \overline{h}\rangle$. □

Now, we propose the following set of axioms and rules of inference for the monadic four-valued Łukasiewicz predicate calculus $Luk_4^*(\forall)$. The axioms (A1)–(A16) correspond to an axiomatization of the four-valued Łukasiewicz propositional calculus given by Cignoli in [4], which is an extension of the classical intuitionistic calculus. Let us consider the following axiom-schemes, where $\alpha \Leftrightarrow \beta$ is an abbreviation for $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

(A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$
(A2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
(A3) $\alpha \Rightarrow (\alpha \vee \beta)$
(A4) $\beta \Rightarrow (\alpha \vee \beta)$
(A5) $(\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$
(A6) $(\alpha \wedge \beta) \Rightarrow \alpha$
(A7) $(\alpha \wedge \beta) \Rightarrow \beta$
(A8) $(\alpha \Rightarrow \beta) \Rightarrow ((\alpha \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \wedge \gamma)))$
(A9) $\alpha \Leftrightarrow \neg\neg\alpha$
(A10) $\sigma_1(\alpha \Rightarrow \beta) \Leftrightarrow \sigma_1(\neg\beta \Rightarrow \neg\alpha)$
(A11) $\sigma_i(\alpha \vee \beta) \Leftrightarrow \sigma_i\alpha \vee \sigma_i\beta$, for $1 \leq i \leq 3$
(A12) $\sigma_i(\alpha \Rightarrow \beta) \Leftrightarrow \bigwedge_{j=i}^{3} (\sigma_j\alpha \Rightarrow \sigma_j\beta)$, for $1 \leq i \leq 3$
(A13) $\sigma_i\sigma_j\alpha \Leftrightarrow \sigma_j\alpha$, for $1 \leq i, j \leq 3$
(A14) $\sigma_1\alpha \Rightarrow \alpha$

(A15) $\sigma_i\alpha \Leftrightarrow \neg\sigma_{4-i}\neg\alpha$, for $1 \leq i \leq 3$
(A16) $\sigma_1\alpha \vee \neg\sigma_1\alpha$
(A17) $\forall x\alpha \Rightarrow \alpha(t)$, with $t = x$ or $t = a_n$
(A18) $\forall x(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \forall x\beta)$, if x is not free in α
(A19) $\forall x\sigma_i\alpha \Rightarrow \sigma_i\forall x\alpha$, for $1 \leq i \leq 3$.

The rules of inference are: Modus Ponens (MP), Generalization (G) and (R1): $\alpha/\sigma_1\alpha$. The notions of *proof* and *proof from a set of formulas* are the usual ones. We write $\Gamma \vdash_{\mathcal{L}} \alpha$ if there exists a proof of α from Γ (we also say that Γ *implies syntactically* α) and, if $\Gamma = \emptyset$ we write simply $\vdash_{\mathcal{L}} \alpha$.

Let $\mathcal{L}_0$ be the language corresponding to the four-valued Łukasiewicz propositional calculus given by Cignoli in [4] (see also [2]) with a countable set $Var = \{q_n : n \in \omega\}$ of propositional variables and let $\mathcal{F}m(\mathcal{L}_0)$ be the algebra of formulas of $\mathcal{L}_0$. A *substitution of* $\mathcal{L}_0$ *in* $\mathcal{L}$ is a function $s\colon Var \to \mathcal{F}m(\mathcal{L})$. Notice that s can be uniquely extended to a function $\overline{s}\colon \mathcal{F}m(\mathcal{L}_0) \to \mathcal{F}m(\mathcal{L})$ preserving propositional connectives. Thus, for every interpretation $M = \langle A, \forall, v, h\rangle$ of $\mathcal{L}$ and every substitution s of $\mathcal{L}_0$ in $\mathcal{L}$, the map $\widehat{v} \circ \overline{s}\colon \mathcal{F}m(\mathcal{L}_0) \to A$ is an L_4-homomorphism. A formula $\alpha \in \mathcal{F}m(\mathcal{L})$ is said to be an *instance of a tautology* if there exists a tautology $\tau \in \mathcal{F}m(\mathcal{L}_0)$ and a substitution s of $\mathcal{L}_0$ in $\mathcal{L}$ such that $\alpha = \overline{s}(\tau)$.

The proofs of the following two propositions are usual by an inductive argument.

Proposition 5.6. *Let* $\alpha \in \mathcal{F}m(\mathcal{L})$ *and let* $M = \langle A, \forall, v, h\rangle$ *be an interpretation of* $\mathcal{L}$. *Then,*

(i) *if* α *is an instance of tautology, then* α *is logically valid;*
(ii) *if* α *is a sentence, then* $\widehat{v}(\alpha) \in \forall(A)$.

Proposition 5.7. (Soundness) *Let* $\Gamma \cup \{\alpha\} \subseteq \mathcal{F}m(\mathcal{L})$. *If* α *is provable from* Γ, *then* α *is logical consequence of* Γ. *In symbols,* $\Gamma \vdash_{\mathcal{L}} \alpha$ *implies* $\Gamma \vDash \alpha$.

Proposition 5.8. ([2, p. 480]) *Let* $\Gamma \subseteq \mathcal{F}m(\mathcal{L})$ *and* $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in \mathcal{F}m(\mathcal{L})$. *Then,*

(D1) *If* $\Gamma \vdash_{\mathcal{L}} (\alpha_1 \Leftrightarrow \beta_1)$ *and* $\Gamma \vdash_{\mathcal{L}} (\alpha_2 \Leftrightarrow \beta_2)$ *then* $\Gamma \vdash_{\mathcal{L}} ((\alpha_1 \circ \alpha_2) \Leftrightarrow (\beta_1 \circ \beta_2))$, *for* $\circ \in \{\wedge, \vee, \Rightarrow\}$.
(D2) *If* $\Gamma \vdash_{\mathcal{L}} (\alpha \Leftrightarrow \beta)$ *then* $\Gamma \vdash_{\mathcal{L}} (\neg\alpha \Leftrightarrow \neg\beta)$.
(D3) *If* $\Gamma \vdash_{\mathcal{L}} (\alpha \Leftrightarrow \beta)$ *then* $\Gamma \vdash_{\mathcal{L}} (\sigma_i\alpha \Leftrightarrow \sigma_i\beta)$ *for each* $i \in \{1, 2, 3\}$.
(D4) *If* $\Gamma \vdash_{\mathcal{L}} (\sigma_i\alpha \Rightarrow \sigma_i\beta)$ *for each* $i \in \{1, 2, 3\}$, *then* $\Gamma \vdash_{\mathcal{L}} (\alpha \Rightarrow \beta)$.
(D5) *If* $\Gamma \vdash_{\mathcal{L}} \neg\alpha$ *then* $\Gamma \vdash_{\mathcal{L}} (\alpha \Rightarrow \beta)$.

The proof of the following proposition is a consequence from axioms (A1)–(A19) and properties (D1)–(D5).

Proposition 5.9. *Let* $\alpha, \beta, \gamma \in \mathcal{F}m(\mathcal{L})$. *Then,*

(E1) $\vdash_{\mathcal{L}} \sigma_1(\alpha \Rightarrow \beta) \Rightarrow (\neg\beta \Rightarrow \neg\alpha)$;
(E2) $\alpha \Rightarrow \beta \vdash_{\mathcal{L}} \neg\beta \Rightarrow \neg\alpha$;
(E3) $\vdash_{\mathcal{L}} \neg(\alpha \wedge \beta) \Leftrightarrow (\neg\alpha \vee \neg\beta)$;
(E4) $\vdash_{\mathcal{L}} \neg(\alpha \vee \beta) \Leftrightarrow (\neg\alpha \wedge \neg\beta)$;
(E5) $\vdash_{\mathcal{L}} (\alpha \Rightarrow \forall x\beta) \Rightarrow \forall x(\alpha \Rightarrow \beta)$, *if* x *is not free in* α.

(E6) $\vdash_{\mathcal{L}} \forall x(\alpha \Rightarrow \beta) \Leftrightarrow (\alpha \Rightarrow \forall x\beta)$, *if* x *is not free in* α.
(E7) $\vdash_{\mathcal{L}} \forall x(\alpha \Rightarrow \beta) \Rightarrow (\forall x\alpha \Rightarrow \forall x\beta)$.
(E8) $\vdash_{\mathcal{L}} \forall x\alpha \Rightarrow (\alpha \vee \beta)$.
(E9) $\vdash_{\mathcal{L}} (\alpha \vee \forall x\beta) \Rightarrow \forall x(\alpha \vee \beta)$, *if* x *is not free in* α.
(E10) $\vdash_{\mathcal{L}} \sigma_i(\alpha \Rightarrow \beta) \Rightarrow (\sigma_i\alpha \Rightarrow \sigma_i\beta)$, *for* $i \in \{1, 2, 3\}$.
(E11) $\vdash_{\mathcal{L}} \sigma_i\alpha \Rightarrow \sigma_j\alpha$, *for* $i, j \in \{1, 2, 3\}$ *and* $i < j$.
(E12) $\vdash_{\mathcal{L}} \forall x(\alpha \wedge \beta) \Leftrightarrow (\forall x\alpha \wedge \forall x\beta)$.
(E13) $\alpha \vdash_{\mathcal{L}} \sigma_i\alpha$, *for* $i \in \{1, 2, 3\}$.
(E14) $\vdash_{\mathcal{L}} \sigma_i\forall x\alpha \Leftrightarrow \forall x\sigma_i\alpha$, *for* $i \in \{1, 2, 3\}$.
(E15) $\alpha \Rightarrow \beta \vdash_{\mathcal{L}} \forall x\alpha \Rightarrow \forall x\beta$.

We define the following equivalence relation $\equiv$ on $\mathcal{F}m(\mathcal{L})$ given by the following prescription: for all $\alpha, \beta \in \mathcal{F}m(\mathcal{L})$,

$$\alpha \equiv \beta \quad \text{if and only if} \quad \vdash_{\mathcal{L}} (\alpha \Leftrightarrow \beta).$$

For each $\alpha \in \mathcal{F}m(\mathcal{L})$, $\overline{\alpha}$ denotes the equivalence class of α. The Lindenbaum-Tarski algebra $\mathbf{F} = \langle \mathcal{F}m(\mathcal{L})/\equiv, \wedge, \vee, \Rightarrow, \neg, \sigma_1, \sigma_2, \sigma_3, \overline{0}, \overline{1}\rangle$ is defined as usual, with $\overline{1} := \{\alpha \in \mathcal{F}m(\mathcal{L}) : \vdash_{\mathcal{L}} \alpha\}$ and $\overline{0} := \neg\overline{1}$. Hence, $\mathbf{F}$ is a four-valued Łukasiewicz algebra [2,4].

In any L_4-algebra A the following identities are satisfied (see [4]):

$$x \Rightarrow y = \sigma_1(x \Rightarrow y) \vee y \quad \text{and} \quad a \Rightarrow b = \neg a \vee b$$

for all $x, y \in A$ and for all $a, b \in B(A)$. Then, these identities are satisfied in the algebra $\mathbf{F}$ and thus we can obtain the following syntactical properties in $Luk_4^*(\forall)$:

Proposition 5.10. *Let* $\alpha, \beta \in \mathcal{F}m(\mathcal{L})$. *Then,*
(E17) $\vdash_{\mathcal{L}} (\alpha \Rightarrow \beta) \Leftrightarrow (\sigma_1(\alpha \Rightarrow \beta) \vee \beta)$;
(E18) $\vdash_{\mathcal{L}} (\sigma_i\alpha \Rightarrow \sigma_i\beta) \Leftrightarrow (\neg\sigma_i\alpha \vee \sigma_i\beta)$, *for all* $i \in \{1, 2, 3\}$.

Corollary 5.11. *Let* $\alpha, \beta \in \mathcal{F}m(\mathcal{L})$. *Then,*
(E19) $\vdash_{\mathcal{L}} \sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \sigma_i(\alpha \vee \forall x\beta)$, *if* x *is not free in* α *and* $i \in \{1, 2, 3\}$.
(E20) $\vdash_{\mathcal{L}} \forall x(\alpha \vee \beta) \Leftrightarrow (\alpha \vee \forall x\beta)$, *if* x *is not free in* α.

Proof. Let $\alpha, \beta \in \mathcal{F}m(\mathcal{L})$ be such that x is not free in α. Then, we have the following proof of $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \sigma_i(\alpha \vee \forall x\beta)$:

1. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \forall x\sigma_i(\alpha \vee \beta)$ (E14)
2. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \forall x(\sigma_i\alpha \vee \sigma_i\beta)$ (Equiv., A11, E15)
3. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \forall x(\neg\sigma_i\alpha \Rightarrow \sigma_i\beta)$ (Equiv., E18, E15)
4. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow (\neg\sigma_i\alpha \Rightarrow \forall x\sigma_i\beta)$ (Equiv., E6)
5. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow (\neg\sigma_i\alpha \Rightarrow \sigma_i\forall x\beta)$ (Equiv., E14
6. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow (\sigma_i\alpha \vee \sigma_i\forall x\beta)$ (Equiv., E18)
7. $\sigma_i\forall x(\alpha \vee \beta) \Leftrightarrow \sigma_i(\alpha \vee \forall x\beta)$ (Equiv., A11).

Hence, $\vdash_{\mathcal{L}} \sigma_i\forall x(\alpha\vee\beta) \Leftrightarrow \sigma_i(\alpha\vee\forall x\beta)$, if x is not free in α and $i \in \{1, 2, 3\}$. (E20) is a consequence of the previous fact and (D5.8). □

Proposition 5.12. *The pair* $\langle \mathbf{F}, \forall\rangle$ *is a monadic* L_4*-algebra, where* $\mathbf{F}$ *is the Lindenbaum-Tarski algebra of* $\mathcal{L}$ *and* $\forall\colon \mathcal{F}m(\mathcal{L})/\equiv \to \mathcal{F}m(\mathcal{L})/\equiv$ *is defined by* $\forall\overline{\alpha} = \overline{\forall x\alpha}$, *for each* $\alpha \in \mathcal{F}m(\mathcal{L})$.

Proof. Notice that (E15) implies that the operation $\forall$ on $\mathbf{F}$ is well defined, i.e., if $\overline{\alpha} = \overline{\beta}$, then $\forall\overline{\alpha} = \forall\overline{\beta}$. We must check that conditions (ML1)-(ML4) hold. Notice that $\overline{1} = \overline{\alpha \Rightarrow \alpha}$. Then, from (A1) and (A17), we have that $\vdash_{\mathcal{L}} \forall x(\alpha \Rightarrow \alpha) \Leftrightarrow (\alpha \Rightarrow \alpha)$. Thus, $\forall\overline{1} = \overline{1}$ and hence condition (ML1) holds. Condition (ML2) is a consequence of (A17). Condition (ML3) can be deduced from (E20). Finally, (ML4) follows from (E14). □

For every constant letter a_n of the language $\mathcal{L}$, we define the operation $c_n \colon \mathcal{F}m(\mathcal{L})/\equiv \ \to \mathcal{F}m(\mathcal{L})/\equiv$ by $c_n(\overline{\alpha}) = \overline{\alpha(a_n)}$ for each $\alpha \in \mathcal{F}m(\mathcal{L})$. The following proposition is straightforward, and thus we omit its proof.

Proposition 5.13. *Let a_n be a constant letter of $\mathcal{L}$. Then, the operation c_n is a constant of the algebra $\langle \mathbf{F}, \forall \rangle$*

Theorem 5.14. (Completeness) *Let $\alpha \in \mathcal{F}m(\mathcal{L})$. Then, α is logically valid if and only if α is provable. That is, $\vDash \alpha$ if and only if $\vdash_{\mathcal{L}} \alpha$.*

Proof. The implication $\vdash_{\mathcal{L}} \alpha$ implies $\vDash \alpha$ is a consequence of Proposition 5.7. Now, assume that α is a logically valid formula. Consider the interpretation $M_{\mathcal{L}} = \langle \mathbf{F}, \forall, v, h \rangle$ of $\mathcal{L}$, where $\langle \mathbf{F}, \forall \rangle$ is the monadic L_4-algebra defined in Proposition 5.12, $v \colon \{P_n : n \in \omega\} \to \mathcal{F}m(\mathcal{L})/\equiv$ is the map given by $v(P_n) = \overline{P_n(x)}$ and $h \colon \{a_n : n \in \omega\} \to C(\mathbf{F})$ is defined by $h(a_n) = c_n$ (see Proposition 5.13). We show that for every formula $\alpha \in \mathcal{F}m(\mathcal{L})$, $\widehat{v}(\alpha) = \overline{\alpha}$. We proceed by induction.

- If $\alpha = P_n(x)$, then $\widehat{v}(P_n(x)) = v(P_n) = \overline{P_n(x)} = \overline{\alpha}$;
- If $\alpha = P_n(a_m)$, then $\widehat{v}(P_n(a_m)) = h(a_m)(v(P_n)) = c_m(\overline{P_n(x)}) = \overline{\alpha}$;
- If $\alpha = \beta \circ \gamma$ with $\circ \in \{\wedge, \vee, \Rightarrow\}$, then $\widehat{v}(\beta \circ \gamma) = \widehat{v}(\beta) \circ \widehat{v}(\gamma) = \overline{\beta} \circ \overline{\gamma} = \overline{\beta \circ \gamma} = \overline{\alpha}$;
- If $\alpha = \neg\beta$, then $\widehat{v}(\neg\beta) = \neg\widehat{v}(\beta) = \neg\overline{\beta} = \overline{\neg\beta} = \overline{\alpha}$;
- If $\alpha = \sigma_i\beta$ for $i \in \{1, 2, 3\}$, then $\widehat{v}(\sigma_i\beta) = \sigma_i\overline{\beta} = \overline{\sigma_i\beta} = \overline{\alpha}$;
- If $\alpha = \forall x\beta$, then $\widehat{v}(\forall x\beta) = \forall\widehat{v}(\beta) = \forall\overline{\beta} = \overline{\forall x\beta} = \overline{\alpha}$.

Thus, since $M_{\mathcal{L}}$ is an interpretation of $\mathcal{L}$ and α is logically valid, we have $\overline{1} = \widehat{v}(\alpha) = \overline{\alpha}$. Hence $\vdash_{\mathcal{L}} \alpha$.

The next step is to prove a completeness theorem for the monadic predicate calculus corresponding to the 2/3-existential quantifier. To this end, we shall consider the dual 2/3-universal quantifier $\forall_{\frac{2}{3}}$ (for short, we write $\forall_2$ instead of $\forall_{\frac{2}{3}}$) of $\exists_{\frac{2}{3}}$ defined as $\forall_{\frac{2}{3}} = \neg\exists_{\frac{2}{3}}\neg$.

Let A be an L_4-algebra. An operation $\forall_2 \colon A \to A$ is said to be a 2/3-*universal quantifier* if the following conditions are satisfied, for every $x, y \in A$:

(U1) $\forall_2 1 = 1$;
(U2) $\forall_2\sigma_i x \leq \sigma_i x$, for $i \in \{1, 2, 3\}$;
(U3) $\forall_2\sigma_i x = \sigma_i\forall_2 x$, for $i \in \{1, 2\}$;
(U4) $\forall_2(x \vee \forall_2 y) = \forall_2 x \vee \forall_2 y$;
(U5) $x \wedge \sigma_2\neg x \leq \forall_2 x$;
(U6) $\forall_2 x \wedge \neg\forall_2\sigma_2 x \leq \forall_2(x \wedge \neg\sigma_2 x)$.

It is straightforward to check that an operation $\forall_2\colon A \to A$ is a 2/3-universal quantifier if and only if the operation $\exists_2\colon A \to A$ defined by $\exists_2 x := \neg\forall_2\neg x$ is a 2/3-existential quantifier, and $\forall_2 x = \neg\exists_2\neg x$. Thus, given an L_4-algebra A and a 2/3-universal quantifier $\forall_2\colon A \to A$, without loss of generality we can say that $\langle A, \forall_2\rangle$ is a 2/3-*monadic* L_4-*algebra*. Moreover, since the quantifiers $\exists_2$ and $\exists$ are interdefinable (Theorem 3.1), it follows that the quantifiers $\forall_2$ and $\forall$ are interdefinable. More precisely we have

(I1) if $\langle A, \forall\rangle$ is a monadic L_4-algebra and we define $\forall_2 x := \forall x \vee \neg\forall(\neg x \vee \sigma_2 x)$, then $\langle A, \forall_2\rangle$ is a 2/3-monadic L_4-algebra;

(I2) if $\langle A, \forall_2\rangle$ is a 2/3-monadic L_4-algebra and we define $\forall x := \forall_2\sigma_3 x \wedge \forall_2(x \vee \neg\sigma_3 x)$, then $\langle A, \forall\rangle$ is a monadic L_4-algebra.

We consider the language $\mathcal{L}_2$ of a 2/3-monadic four-valued Łukasiewicz predicate calculus that consists of the same symbols that $\mathcal{L}$, except for the symbol $\forall$ which will be replaced by $\forall_2$. We denote by $\mathcal{F}m(\mathcal{L}_2)$ the set of all formulas of $\mathcal{L}_2$ defined as usual. An *interpretation* $M = \langle A, \forall_2, v, h\rangle$ of $\mathcal{L}_2$ is defined similarly to the corresponding notion of an interpretation of $\mathcal{L}$ and the same thing happens for the notion of constant of a 2/3-monadic L_4-algebra and we denote the set of all constants of $\langle A, \forall_2\rangle$ by $C_{\forall_2}(A)$. The following proposition is a consequence of the equations given in (I1) and (I2).

Proposition 5.15. *Let A be an L_4-algebra. Let $\forall$ be an universal quantifier and $\forall_2$ be a 2/3-universal quantifier, defined on A. If $\forall$ and $\forall_2$ are interdefinable by the equations in* (I1) and (I2), then $C_\forall(A) = C_{\forall_2}(A)$.

Let us consider the map $\Phi\colon \mathcal{F}m(\mathcal{L}) \to \mathcal{F}m(\mathcal{L}_2)$ defined recursively, by using (I2), as follows. Let $\alpha \in \mathcal{F}m(\mathcal{L})$:

- if $\alpha = P_n(t)$ with $t = x$ or $t = a_n$, then $\Phi(\alpha) = \alpha$;
- if $\alpha = \neg\beta$, then $\Phi(\alpha) = \neg\Phi(\beta)$;
- if $\alpha = \sigma_i\beta$ for $i \in \{1, 2, 3\}$, then $\Phi(\alpha) = \sigma_i\Phi(\beta)$;
- if $\alpha = \beta \circ \gamma$ for $\circ \in \{\wedge, \vee, \Rightarrow\}$, then $\Phi(\alpha) = \Phi(\beta) \circ \Phi(\gamma)$;
- if $\alpha = \forall x\beta$, then $\Phi(\alpha) = \forall_2 x\sigma_3\Phi(\beta) \wedge \forall_2 x\,(\Phi(\beta) \vee \neg\sigma_3\Phi(\beta))$.

Now, using the map Φ and axioms (A1)–(A19), we propose a 2/3-monadic predicated calculus on $\mathcal{L}_2$ as follows. Axioms (B1)–(B16) are the same that axioms (A1)–(A16) on $\mathcal{L}$ taking into account the formulas $\alpha, \beta, \gamma \in \mathcal{F}m(\mathcal{L}_2)$; for $\alpha, \beta \in \mathcal{F}m(\mathcal{L}_2)$,

(B17) $(\forall_2 x\sigma_3\alpha \wedge \forall_2 x(\alpha \vee \neg\sigma_3\alpha)) \Rightarrow \alpha(t)$, with $t = x$ or $t = a_n$;

(B18) $[\forall_2 x\sigma_3(\alpha \Rightarrow \beta) \wedge \forall_2 x\,((\alpha \Rightarrow \beta) \vee \neg\sigma_3(\alpha \Rightarrow \beta))] \Rightarrow$
$\Rightarrow [\alpha \Rightarrow (\forall_2 x\sigma_3\beta \wedge \forall_2 x(\beta \vee \neg\sigma_3\beta))]$, if x is not free in α;

(B19) $[\forall_2 x\sigma_3\sigma_i\alpha \wedge \forall_2 x(\sigma_i\alpha \vee \neg\sigma_3\sigma_i\alpha)] \Rightarrow \sigma_i\,[\forall_2 x\sigma_3\alpha \wedge \forall_2 x(\alpha \vee \neg\sigma_3\alpha)]$;

(B20) $\forall_2 x\alpha \Leftrightarrow [\forall_2 x\sigma_3\alpha \wedge \forall_2 x(\alpha \vee \neg\sigma_3\alpha)] \vee \neg\forall_2 x(\neg\alpha \vee \sigma_2\alpha)$.

The rules of inference are Modus Ponens (MP), Generalization (G) and (R1). Notice that axioms (B17), (B18) and (B19) are the image by Φ of axioms (A17), (A18) and (A19), respectively. Axiom (B20) is necessary to show that Φ acts like a translator from $\mathcal{L}$ to $\mathcal{L}_2$, see Proposition 5.18.

The binary relation $\equiv_2$ is defined on $\mathcal{F}m(\mathcal{L}_2)$ in a similar way as the relation $\equiv$ was defined on $\mathcal{F}m(\mathcal{L})$.

Proposition 5.16. *Let $\alpha \in \mathcal{F}m(\mathcal{L})$. If α is provable in $\mathcal{L}$, then $\Phi(\alpha)$ is provable in $\mathcal{L}_2$.*

Proof. Let $\alpha \in \mathcal{F}m(\mathcal{L})$ be a provable formula in $\mathcal{L}$. So, there is a proof $\delta_1, \ldots, \delta_n$ of α. We show by induction that $\Phi(\delta_j)$ is provable in $\mathcal{L}_2$ for each $j = 1, \ldots, n$.

- It is clear that if δ_j is an axiom of $\mathcal{L}$ then $\Phi(\delta_j)$ is an axiom of $\mathcal{L}_2$.
- Assume that there are indexes $i, k < j$ such that δ_j is obtained from δ_i and $\delta_k = \delta_i \Rightarrow \delta_j$ by (MP). So, by inductive hypothesis $\Phi(\delta_i)$ and $\Phi(\delta_k) = \Phi(\delta_i) \Rightarrow \Phi(\delta_j)$ are provable in $\mathcal{L}_2$, then by (MP) $\Phi(\delta_j)$ is provable in $\mathcal{L}_2$.
- Assume that there is an index $i < j$ such that δ_j is obtained from δ_i by (G). That is, $\delta_j = \forall x \delta_i$. Thus, $\Phi(\delta_j) = \forall_2 x \sigma_3 \Phi(\delta_i) \wedge \forall_2 x (\Phi(\delta_i) \vee \neg \sigma_3 \Phi(\delta_i))$ where by inductive hypothesis $\Phi(\delta_i)$ is provable in $\mathcal{L}_2$. Then, we have the following proof

  1. $\Phi(\delta_i) \Rightarrow \sigma_3 \Phi(\delta_i)$ (Taut)
  2. $\Phi(\delta_i)$ (Hyp)
  3. $\sigma_3 \Phi(\delta_i)$ (1,2, MP)
  4. $\forall_2 x \sigma_3 \Phi(\delta_i)$ (3, G)
  5. $\Phi(\delta_i) \Rightarrow (\Phi(\delta_i) \vee \neg \sigma_3 \Phi(\delta_i))$ (B3)
  6. $\Phi(\delta_i) \vee \neg \sigma_3 \Phi(\delta_i)$ (2,5, MP)
  7. $\forall_2 x\, (\Phi(\delta_i) \vee \neg \sigma_3 \Phi(\delta_i))$ (6, G)
  8. $\forall_2 x \sigma_3 \Phi(\delta_i) \wedge \forall_2 x\, (\Phi(\delta_i) \vee \neg \sigma_3 \Phi(\delta_i))$ (4,7, Conjunction).

  Hence, $\Phi(\delta_j)$ is provable in $\mathcal{L}_2$.
- Assume that there exists an index $i < j$ such that δ_j is obtained from δ_i by the rule (R1). So, $\delta_j = \sigma_1 \delta_i$. By inductive hypothesis, $\Phi(\delta_i)$ is provable in $\mathcal{L}_2$. Then, by (R1) we have that $\sigma_1 \Phi(\delta_i)$ is provable in $\mathcal{L}_2$. Since $\sigma_1 \Phi(\delta_i) = \Phi(\sigma_1 \delta_i) = \Phi(\delta_j)$, it follows that $\Phi(\delta_j)$ is provable in $\mathcal{L}_2$. □

The following theorem can be proved without difficulty by the interdefinability of the quantifiers $\forall$ and $\forall_2$ and so we omit its proof.

Theorem 5.17 (Soundness). *Every provable formula in $\mathcal{L}_2$ is logically valid in $\mathcal{L}_2$.*

Let us consider the map $\Psi \colon \mathcal{F}m(\mathcal{L}_2) \to \mathcal{F}m(\mathcal{L})$ defined recursively, using (I1), as follows. Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$:

- if $\alpha = P_n(t)$ with $t = x$ or $t = a_n$, then $\Psi(\alpha) = \alpha$;
- if $\alpha = \neg\beta$, then $\Psi(\alpha) = \neg\Psi(\beta)$;
- if $\alpha = \sigma_i \beta$, then $\Psi(\alpha) = \sigma_i \Psi(\beta)$;
- if $\alpha = \beta \circ \gamma$ with $\circ \in \{\wedge, \vee, \Rightarrow\}$, then $\Psi(\alpha) = \Psi(\beta) \circ \Psi(\gamma)$;
- if $\alpha = \forall_2 x \beta$, then $\Psi(\alpha) = \forall x \Psi(\beta) \vee \neg \forall x\, (\neg \Psi(\beta) \vee \sigma_2 \Psi(\beta))$.

Proposition 5.18. *Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$ and $\beta \in \mathcal{F}m(\mathcal{L})$. Then,*

(1) $\Phi\Psi(\alpha) \equiv_2 \alpha$;
(2) $\Psi\Phi(\beta) \equiv \beta$.

Proof. We prove this proposition by induction. Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$ and $\beta \in \mathcal{F}m(\mathcal{L})$.

(1) It is straightforward to prove that $\Phi\Psi(\alpha) \equiv_2 \alpha$ when $\alpha = P_n(t)$ with $t = x$ or $t = a_n, \alpha = \neg\gamma, \alpha = \sigma_i\gamma$ with $i \in \{1,2,3\}$ or $\alpha = \gamma_1 \circ \gamma_2$ with $\circ \in \{\wedge, \vee, \Rightarrow\}$. Now, suppose that $\alpha = \forall_2 x\gamma$. Thus, $\Psi(\alpha) = \forall x\Psi(\gamma) \vee \neg\forall x\,(\neg\Psi(\gamma) \vee \sigma_2\Psi(\gamma))$. Then, we have that

$$\begin{aligned}\Phi\Psi(\alpha) = {} & [\forall_2 x\sigma_3\Phi\Psi(\gamma) \wedge \forall_2 x\,(\Phi\Psi(\gamma) \vee \neg\sigma_3\Phi\Psi(\gamma))] \vee \\ & \vee\neg[\forall_2 x\sigma_3\,(\neg\Phi\Psi(\gamma) \vee \sigma_2\Phi\Psi(\gamma)) \wedge \\ & \wedge\forall_2 x\,[(\neg\Phi\Psi(\gamma) \vee \sigma_2\Phi\Psi(\gamma)) \vee \neg\sigma_3\,(\neg\Phi\Psi(\gamma) \vee \sigma_2\Phi\Psi(\gamma))]]\,.\end{aligned}$$

By inductive hypothesis $\Phi\Psi(\gamma) \equiv_2 \gamma$, then

$$\begin{aligned}\Phi\Psi(\alpha) \equiv_2 & [\forall_2 x\sigma_3\gamma \wedge \forall_2 x(\gamma \vee \neg\sigma_3\gamma)]\vee \\ & \vee \neg[\forall_2 x\sigma_3(\neg\gamma \vee \sigma_2\gamma) \wedge \forall_2 x[(\neg\gamma \vee \sigma_2\gamma) \vee \neg\sigma_3(\neg\gamma \vee \sigma_2\gamma)]].\end{aligned}$$

Notice that $\vdash_{\mathcal{L}_2} \sigma_3(\neg\gamma \vee \sigma_2\gamma)$, then $(\neg\gamma \vee \sigma_2\gamma) \vee \neg\sigma_3(\neg\gamma \vee \sigma_2\gamma) \equiv_2 (\neg\gamma \vee \sigma_2\gamma)$. Hence, from (B20), we obtain

$$\Phi\Psi(\alpha) \equiv_2 [\forall_2 x\sigma_3\gamma \wedge \forall_2 x(\gamma \vee \neg\sigma_3\gamma)] \vee \neg\forall_2 x(\neg\gamma \vee \sigma_2\gamma) \equiv_2 \forall_2 x\gamma \equiv_2 \alpha.$$

(2) It is straightforward to prove that $\Psi\Phi(\beta) \equiv \beta$ when $\beta = P_n(t)$ with $t = x$ or $t = a_n$, $\beta = \neg\gamma$, $\beta = \sigma_i\gamma$ with $i \in \{1,2,3\}$ or $\beta = \gamma_1 \circ \gamma_2$ with $\circ \in \{\wedge, \vee, \Rightarrow\}$. Now, we assume that $\beta = \forall x\gamma$. By definition of Φ we have $\Phi(\beta) = \forall_2 x\sigma_3\Phi(\gamma) \wedge \forall_2 x(\Phi(\gamma) \vee \neg\sigma_3\Phi(\gamma))$. Then, by definition of Ψ, we obtain

$$\begin{aligned}\Psi\Phi(\beta) = {} & [\forall x\sigma_3\Psi\Phi(\gamma) \vee \neg\forall x(\neg\sigma_3\Psi\Phi(\gamma) \vee \sigma_2\sigma_3\Psi\Phi(\gamma))] \wedge \\ & \wedge[\forall x(\Psi\Phi(\gamma) \vee \neg\sigma_3\Psi\Phi(\gamma)) \vee \neg\forall x[\neg(\Psi\Phi(\gamma) \vee \neg\sigma_3\Psi\Phi(\gamma)) \\ & \vee\sigma_2(\Psi\Phi(\gamma) \vee \neg\sigma_3\Psi\Phi(\gamma))]].\end{aligned}$$

By inductive hypothesis $\Psi\Phi(\gamma) \equiv \gamma$ and thus, we have

$$\begin{aligned}\Psi\Phi(\beta) \equiv {} & [\forall x\sigma_3\gamma \vee \neg\forall x(\neg\sigma_3\gamma \vee \sigma_3\gamma)] \wedge \\ & \wedge[\forall x(\gamma \vee \neg\sigma_3\gamma) \vee \neg\forall x[\neg(\gamma \vee \neg\sigma_3\gamma) \vee (\sigma_2\gamma \vee \neg\sigma_3\gamma)]]\end{aligned}$$

and so

$$\Psi\Phi(\beta) \equiv \forall x\sigma_3\gamma \wedge [\forall x(\gamma \vee \neg\sigma_3\gamma) \vee \neg\forall x(\neg\gamma \vee \sigma_2\gamma \vee \neg\sigma_3\gamma)].$$

Then, by property (D4) of Proposition 5.8 and properties (E12) and (E14), we have $\Psi\Phi(\beta) \equiv \forall x\gamma$. Hence $\Psi\Phi(\beta) \equiv \beta$. □

Proposition 5.19. *Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$. If α is logically valid in $\mathcal{L}_2$, then $\Psi(\alpha)$ is logically valid in $\mathcal{L}$.*

Proof. Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$. Assume that α is logically valid in $\mathcal{L}_2$. Let $M = \langle A, \forall, v, h\rangle$ be an interpretation of $\mathcal{L}$. We need to prove that $M \Vdash \Psi(\alpha)$. We define the interpretation $M^* = \langle A, \forall_2, v^*, h\rangle$ of $\mathcal{L}_2$ where $\forall_2$ is the 2/3-universal quantifier defined by $\forall$ as in (I1) on page 21 and $v^* := v$. Note that $\widehat{v}\colon \mathcal{F}m(\mathcal{L}) \to A$ and $\widehat{v^*}\colon \mathcal{F}m(\mathcal{L}_2) \to A$. Now, we show that $(\widehat{v} \circ \Psi)(\beta) = \widehat{v^*}(\beta)$ for all $\beta \in \mathcal{F}m(\mathcal{L}_2)$. We proceed by induction.

- If $\beta = P_n(x)$, then $(\widehat{v} \circ \Psi)\,(P_n(x)) = \widehat{v}\,(\Psi(P_n(x))) = \widehat{v}(P_n(x)) = v(P_n) = v^*(P_n) = \widehat{v^*}(P_n(x))$;

- if $\beta = P_n(a_m)$, then $(\widehat{v} \circ \Psi)(P_n(a_m)) = \widehat{v}(\Psi(P_n(a_m))) = \widehat{v}(P_n(a_m)) = h(a_m)(v(P_n)) = h(a_m)(v^*(P_n)) = \widehat{v^*}(P_n(a_m))$;
- if $\beta = \neg\gamma$, then $(\widehat{v} \circ \Psi)(\neg\gamma) = \widehat{v}(\Psi(\neg\gamma)) = \neg\widehat{v}(\Psi(\gamma)) = \neg\widehat{v^*}(\gamma) = \widehat{v^*}(\neg\gamma)$;
- if $\beta = \sigma_i\gamma$ with $i \in \{1, 2, 3\}$, then $(\widehat{v} \circ \Psi)(\sigma_i\gamma) = \widehat{v}(\Psi(\sigma_i\gamma)) = \sigma_i\widehat{v}(\Psi(\gamma)) = \sigma_i\widehat{v^*}(\gamma) = \widehat{v^*}(\sigma_i\gamma)$;
- if $\beta = \gamma_1 * \gamma_2$ with $* \in \{\wedge, \vee, \Rightarrow\}$, then $(\widehat{v} \circ \Psi)(\gamma_1 * \gamma_2) = \widehat{v}(\Psi(\gamma_1 * \gamma_2)) = \widehat{v}(\Psi(\gamma_1)) * \widehat{v}(\Psi(\gamma_2)) = \widehat{v^*}(\gamma_1) * \widehat{v^*}(\gamma_2) = \widehat{v^*}(\gamma_1 * \gamma_2)$;
- if $\beta = \forall_2 x\gamma$, then $(\widehat{v} \circ \Psi)(\forall_2 x\gamma) = \widehat{v}(\forall x\Psi(\gamma) \vee \neg\forall x(\neg\Psi(\gamma) \vee \sigma_2\Psi(\gamma))) = \forall\widehat{v}\Psi(\gamma) \vee \neg\forall(\neg\widehat{v}\Psi(\gamma) \vee \sigma_2\widehat{v}\Psi(\gamma)) = \forall_2\widehat{v}\Psi(\gamma) = \forall_2\widehat{v^*}(\gamma) = \widehat{v^*}(\forall_2 x\gamma)$.

Now, since α is logically valid in $\mathcal{L}_2$, we have $\widehat{v^*}(\alpha) = 1$. Then $1 = \widehat{v^*}(\alpha) = (\widehat{v} \circ \Psi)(\alpha) = \widehat{v}(\Psi(\alpha))$. Therefore, $\Psi(\alpha)$ is logically valid in $\mathcal{L}$.

Theorem 5.20 (Completeness). *Let α be a formula of $\mathcal{L}_2$. Then, α is logically valid if and only if α is provable in $\mathcal{L}_2$.*

Proof. Let $\alpha \in \mathcal{F}m(\mathcal{L}_2)$ be logically valid. So, by the previous proposition, $\Psi(\alpha)$ is logically valid in $\mathcal{L}$. Then, by Theorem 5.14, we have that $\Psi(\alpha)$ is provable in $\mathcal{L}$. Now, by Proposition 5.16, we have that $\Phi\Psi(\alpha)$ is provable in $\mathcal{L}_2$ and, from Proposition 5.18 we have $\Phi\Psi(\alpha) \equiv_2 \alpha$. Therefore α is provable in $\mathcal{L}_2$.

References

[1] Abad, M.: Estructuras cíclica y monádica de un álgebra de Łukasiewicz n-valente. Notas de Lógica Matemática, Vol. 36. Universidad Nacional del Sur (1988)

[2] Boicescu, V., Filipoiu, A., Georgescu, G., Rudeanu, S.: Łukasiewicz-Moisil Algebras. North-Holland (1991)

[3] Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra. Springer, Berlin (1981)

[4] Cignoli, R.: Proper n-valued Łukasiewicz algebras as S-algebras of Łukasiewicz n-valued prepositional calculi. Studia Logica **41**, 3–16 (1982)

[5] Cignoli, R.: Quantifiers on distributive lattiices. Discrete Math. **96**, 183–197 (1991)

[6] Cignoli, R., D'Ottaviano, I., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. Kluwer Academic Publishers, Dordrecht (2000)

[7] Georgescu, G., Iorgulescu, A., Leustean, I.: Monadic and closure MV-algebras. Mult. Val. Logic **3**, 235–257 (1998)

[8] Hájek, P.: Methamathematics of Fuzzy Logic. Kluwer Academic Publisher, Dordrecht (1998)

[9] Halmos, P.: Algebraic logic I. Compositio Mathematica **12**, 217–249 (1955)

[10] Halmos, P.: Algebraic Logic. Chelsea Publishing Company, White River Junction (1962)

[11] Iorgulescu, A.: Connections between MV_n algebras and n-valued Łukasiewicz-Moisil algebras Part I. Discrete Math. **181**, 155–177 (1998)

[12] Iturrioz, L.: Łukasiewicz and symmetrical Heyting algebras. Zeitschrift für Mathematische Logik und Grundlagen de Mathematik **23**, 131–136 (1977)

[13] Krongold, F.: Álgebra y lógica monádicas, Tesis de Licenciatura dirigida por el Dr. Roberto Cignoli (manuscript)

[14] Monteiro, L.: Algebras de Łukasiewicz trivalentes monádicas. Notas de Lógica Matemática, Vol. 32. Universidad Nacional del Sur (1974)

[15] Monteiro, A.: Sur les algèbres de Heyting symètriques. Portugaliae Mathematica **39**, 1–237 (1980)

[16] Petrovich, A.: An alternative definition of quantifier in three-valued Łukasiewicz algebra. XIV Simposio Latinoamericano de Lógica Matemática, Paraty, Rio De Janeiro (2008)

[17] Petrovich, A., Lattanzi, M.: An alternative notion of quantifiers on three-valued Łukasiewicz algebras. Mult.-Val. Logic Soft Comput. **28**(4–5), 335–360 (2017)

L. J. González and M. B. Lattanzi
Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
Avda. Uruguay 151
6300 Santa Rosa La Pampa
Argentina
e-mail: `lucianogonzalez@exactas.unlpam.edu.ar`;
`mblatt@exactas.unlpam.edu.ar`

A. G. Petrovich
Departamento de Matemática,
Facultad de Ciencias Exactas y Naturales
Universidad Nacional de Buenos Aires
Pabellón I - Ciudad Universitaria Buenos Aires
Argentina
e-mail: `apetrov@dm.uba.ar`

Received: July 4, 2016.
Accepted: October 7, 2017.