



Choice-Free Duality for Tarski Algebras

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Abstract

In [N. Bezhanishvili and W. H. Holliday. Choice-free Stone duality. *J. Symb. Log.*, 85(1):109–148, 2020.], the authors develop a choice-free topological duality for the algebraic category of Boolean algebras. We adapt the techniques and constructions given by Bezhanishvili and Holliday to develop a topological duality for the algebraic category of Tarski algebras without using the Axiom of Choice. Then, we show that the duality presented here for Tarski algebras is in fact a generalization of the duality given by Bezhanishvili and Holliday for Boolean algebras. We also obtain a choice-free topological duality for the algebraic category of generalized Boolean algebra.

Keywords Tarski algebra · Implicative filter · Regular open algebra · Boolean algebra · Generalized Boolean algebra

MSC classification: 03G25 · 06F35 · 06E75

1 Introduction

A Tarski algebra (also known as implication algebra) is an algebra $\langle A, \rightarrow, 1 \rangle$ of type (2,0) satisfying certain identities (see Definition 3.1). It is known that the variety of Tarski algebras provides an algebraic semantic of the implication fragment of classical logic (see for

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instance [11, 17]), and this variety corresponds to the $\{\rightarrow\}$ -subreducts of Boolean algebras. One of the pioneers in studying this variety was J. C. Abbott. Among other results, Abbott proved in [2] that there is a one-to-one correspondence between the variety of Tarski algebras (implication algebras as Abbott calls them) and the class of semi-boolean algebras (join-semilattices with a greatest element in which every principal upset is a Boolean algebra). Another important author who studied this variety was A. Monteiro [14–16]. Tarski algebras have been intensively studied since then by many authors. We mentioned some of them, but we do not pretend to be exhaustive: [3, 4, 6–9, 13, 16, 17].

There are in the literature at least four topological dualities for the algebraic category of Tarski algebras. In [3], the authors give a full description of a Tarski algebra as a union of a unique family of filters of a suitable Boolean algebra that they called Boolean closure. Then, they introduce the concept of implication space as follows. An implication space is a triple $\langle X, u, \mathcal{C} \rangle$ where X is a Boolean space, u is a fixed element of the space X , $\mathcal{C}$ is an antichain of closed sets of X such that $\bigcap \mathcal{C} = \{u\}$, and moreover it satisfies an additional condition. After that, they establish a categorical duality between the algebraic category of Tarski algebras and a certain category of implication spaces. This duality appeals to the Boolean duality for Boolean algebras. In [6], another topological duality is developed for the category of Tarski algebras. In order to obtain this duality, the authors use the maximal filters of Tarski algebras as the points of the dual spaces and use the Stone map to generate a topology on the set of all maximal filters. The dual objects of Tarski algebras are structures $\langle X, \mathcal{K}, \tau \rangle$ such that τ is a topology on X , $\mathcal{K}$ is a basis of compact subsets for τ , $A \cap B^c \in \mathcal{K}$, for all $A, B \in \mathcal{K}$, and satisfies two additional conditions (see Definition 4.12). Unlike the duality given in [3], this duality does not rely on Stone's duality for Boolean algebras, and the Stone duality for Boolean algebras can be obtained from the duality in [6]. The third and fourth topological dualities for Tarski algebras that we know are presented in [10]. Actually, [10] presents an abstract spectral-like duality and an abstract Priestley-like duality for the algebraic categories of the classes of algebras that are canonically associated with some filter distributive finitary congruential logic with theorems. The implication fragment of classical logic falls under this scope, and thus there are two dualities for the class of Tarski algebras (the class canonically associated with the implication fragment of classical logic): a spectral one and a Priestley one.

A common denominator of the four topological dualities previously mentioned is that they need some equivalent form of the Axiom of Choice. In the present article, we intend to develop a new topological duality for the variety of Tarski algebras which is free from the Axiom of Choice and all their equivalent formulations. In order to achieve this, we follow and adapt some of the techniques and constructions given by Bezhanishvili and Holliday in [5]. We refer the reader to [5] for details about the choice-free duality for Boolean algebras and compare them with the results developed here.

The paper is organized as follows. In Sect. 2, we present the basic definitions about partially ordered sets and topology. Section 3 is devoted to giving the necessary notions and results about Tarski algebras. We consider the notion of annihilator, and we present some of its properties, which will be important for what follows. In Sect. 4, we obtain a choice-free representation for Tarski algebras through certain topological spaces. Unlike the duality in [6], where the authors use the maximal implicative filters for the points of their dual spaces, we take the collection of all proper implicative filters as the points of a topological space. In our dual spaces, we identify a distinguished subbasis formed by subsets

which are both compact open subsets in the spaces and regular open in the upset topology of the specialization order of the spaces. Then we give an abstract characterization of these topological spaces, which we call CFT-spaces, and we establish the equivalence, at object level, between Tarski algebras and CFT-spaces. Section 5 presents the dual morphisms corresponding to the semi-homomorphisms and homomorphisms between Tarski algebras, and establishes the dual equivalence. In Sect. 6, we show some connections between Tarski algebras and generalized Boolean algebras. We introduce the notion of a free generalized Boolean algebra and we show two ways to obtain it. One of them is by means of the dual equivalence of the previous sections. We see also some relation between our CFT-spaces and the UV-spaces (the dual spaces to the Boolean algebras according to [5]). Section 7 applies the duality of Sects. 4 and 5 to the algebraic category of generalized Boolean algebras to obtain a choice-free duality. In Sect. 8, we conclude by showing that the choice-free duality for Boolean algebra can be derived from our choice-free duality for Tarski algebras.

2 Preliminaries

Let $\langle X, \leq \rangle$ be a poset. A subset Y of X is said to be an *upset* of X when for all $a, b \in X$, if $a \leq b$ and $a \in Y$, then $b \in Y$. Dually, we have the notion of *downset*. For every $Y \subseteq X$, let $[Y] := \{x \in X : \exists y \in Y (y \leq x)\}$ be the least upset of X containing Y . We write $[x]$ instead of $[\{x\}]$. Dually, we have the least downset $\downarrow Y$ of X containing Y , and the downset $\downarrow x$. We denote by $\text{Up}(X, \leq)$, or simply by $\text{Up}(X)$, the collection of all upsets of X . It is well-known that $(\text{Up}(X), \cap, \cup, \emptyset, X)$ is a completely distributive algebraic lattice. Thus, $\text{Up}(X)$ is a complete Heyting algebra, where the Heyting implication on $\text{Up}(X)$ is defined as follows: for all $U, V \in \text{Up}(X)$,

$$U \Rightarrow V := \{x \in X : [x] \cap U \subseteq V\}.$$

We assume that the reader is familiar with the basic notions of topology. Let $\langle X, \tau \rangle$ be a topological space. We denote respectively by $\text{cl}(\cdot)$ and $\text{int}(\cdot)$ the closure operator and the interior operator. Recall that a subset $U \subseteq X$ is called *regular open* if $U = \text{int}(\text{cl}(U))$. Let $\mathcal{RO}(X)$ be the collection of all regular open subsets of the space X . It is known that $(\mathcal{RO}(X), \cap, \vee_{\mathcal{RO}}, \neg_{\mathcal{RO}}, \emptyset, X)$ is a Boolean algebra, where

$$U \vee_{\mathcal{RO}} V = \text{int}(\text{cl}(U \cup V)) \quad \text{and} \quad \neg_{\mathcal{RO}} U = \text{cl}(U)^c = \text{int}(U^c).$$

for all $U, V \in \mathcal{RO}(X)$. Notice that $\neg_{\mathcal{RO}} \neg_{\mathcal{RO}} U = \text{int}(\text{cl}(U))$. Thus, U is regular open if and only if $\neg_{\mathcal{RO}} \neg_{\mathcal{RO}} U = U$.

Let $\langle X, \leq \rangle$ be a poset. Then, $\text{Up}(X)$ is a topology on X called the *upset topology* on X or the *Alexandroff topology* on X . Throughout the paper we write $\tau_{\leq}$ for the upset topology of the poset $\langle X, \leq \rangle$, that is, $\tau_{\leq} = \text{Up}(X)$. In the topological space $\langle X, \tau_{\leq} \rangle$, it follows that for every $Y \subseteq X$,

- (U1) $\text{cl}_{\leq}(Y) = \{x \in X : [x] \cap Y \neq \emptyset\} = [Y]$;
- (U2) $\text{int}_{\leq}(Y) = \{x \in X : [x] \subseteq Y\}$;
- (U3) $\text{int}_{\leq}(\text{cl}_{\leq}(Y)) = \{x \in X : \forall x' \geq x ([x'] \cap Y \neq \emptyset)\}$.

In particular, the Boolean implication in the Boolean algebra $\mathcal{RO}(X, \tau_{\leq})$ is given by

$$U \rightarrow_{\mathcal{RO}} V = \neg_{\mathcal{RO}} U \vee_{\mathcal{RO}} V = \{x \in X : [x] \cap U \subseteq V\} = U \Rightarrow V.$$

Moreover for all $U_1, \dots, U_n, V \in \mathcal{RO}(X, \tau_{\leq})$, we have that

$$U_1 \rightarrow_{\mathcal{RO}} (U_2 \rightarrow_{\mathcal{RO}} (\dots (U_n \rightarrow_{\mathcal{RO}} V) \dots)) = X \iff U_1 \cap \dots \cap U_n \subseteq V. \quad (2.1)$$

Let $\langle X, \tau \rangle$ be a T_0 -space. Let $\preccurlyeq_{\tau}$ be the *specialization order* of $\langle X, \tau \rangle$, that is, for all $x, y \in X$,

$$x \preccurlyeq_{\tau} y \iff \forall U \in \tau (x \in U \implies y \in U).$$

We drop the subscript of $\preccurlyeq_{\tau}$ when confusion is unlikely. Let us introduce some notations. Let $\langle X, \tau \rangle$ be a T_0 -space and $\preccurlyeq$ its specialization order. Thus, we obtain the upper topology $\tau_{\preccurlyeq} = \text{Up}(X, \preccurlyeq)$. Hence, we have two topologies τ and $\tau_{\preccurlyeq}$ defined on the same set X . We define the following collections of subsets:

- $\mathcal{O}(X)$ is the collection of all open subsets of $\langle X, \tau \rangle$.
- $\mathcal{KO}(X)$ is the collection of all compact and open subsets of $\langle X, \tau \rangle$.
- $\mathcal{RO}(X)$ and $\mathcal{RO}(X, \tau_{\preccurlyeq})$ denote the collection of all regular open subsets of $\langle X, \tau_{\preccurlyeq} \rangle$.

3 Tarski Algebras

In this section, we present the basics of Tarski algebras. For relevant background material concerning Tarski algebras, see [2, 6, 14, 16, 17].

Definition 3.1 An algebra $A = \langle A, \rightarrow, 1 \rangle$ of type (2,0) is a *Tarski algebra* if it satisfies the following identities:

- (T1) $1 \rightarrow a = a,$
- (T2) $a \rightarrow a = 1,$
- (T3) $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c),$
- (T4) $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a.$

Each Tarski algebra A has associated a partial order defined as follows: for all $a, b \in A$, $a \leq b \iff a \rightarrow b = 1$. In fact, Abbot proved in [1] that every Tarski algebra A is a join-semilattice, where the join is defined by $a \vee b = (a \rightarrow b) \rightarrow b$, such that every principal upset $[a] = \{b \in A : a \leq b\}$ is a Boolean algebra regarding the order $\leq$. When we write $x \wedge y$ in a Tarski algebra A we mean that the meet (infimum) of x and y exists in A and it is $x \wedge y$. Notice that if $a, b, c \in A$ are such that $c \leq a, b$, then $a \wedge b$ exists in A .

Proposition 3.2 Let A be a Tarski algebra. Let $a, b, c \in A$. Then:

1. $a \vee b = 1 \iff a \rightarrow b = b.$

2. $a \vee (a \rightarrow b) = 1$.
3. $a \vee (b \rightarrow c) = (a \vee b) \rightarrow (a \vee c)$.
4. $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$.

Let A be a Tarski algebra. A subset F of A is called an *implicative filter* (also called *deductive system*) of A if $1 \in F$, and if $a, a \rightarrow b \in F$, then $b \in F$. Let us denote by $\text{Fi}(A)$ the collection of all implicative filters of A . It is clear that $\text{Fi}(A)$ is an algebraic closure system. If $X \subseteq A$, then the least implicative filter containing X is denoted by $\text{Fig}(X)$, and if X is non-empty we have that

$$\text{Fig}(X) = \{a \in A : x_1 \rightarrow (x_2 \rightarrow (\cdots \rightarrow (x_n \rightarrow a) \cdots)) = 1, \text{ for some } x_1, \dots, x_n \in X\}.$$

Hence $\langle \text{Fi}(A), \cap, \cup, \{1\}, A \rangle$ is a complete distributive lattice, where $F \vee G = \text{Fig}(F \cup G)$, for all $F, G \in \text{Fi}(A)$. Moreover, it is a useful fact that for every implicative filter F and $a \in A$, $\text{Fig}(F, a) = \{b \in A : a \rightarrow b \in F\}$.

Proposition 3.3 *Let A be a Tarski algebra and $F \subseteq A$. Then, F is an implicative filter if and only if $1 \in F$, F is an upset and if $a, b \in F$ such that $a \wedge b$ exists in A , then $a \wedge b \in F$.*

Proof ($\Rightarrow$) Assume that F is an implicative filter. We already know that $1 \in F$ and F is an upset. Let $a, b \in F$ and assume that $a \wedge b$ exists in A . Let $c = a \wedge b$. Since $[c]$ is a Boolean algebra and $a, b \in [c]$, it follows that $b \leq a \rightarrow (a \wedge b)$. Then we have that $a, a \rightarrow (a \wedge b) \in F$. Hence $a \wedge b \in F$.

($\Leftarrow$) Let $a, a \rightarrow b \in F$. Since $a \leq a \vee b = (a \rightarrow b) \rightarrow b$, we have $(a \rightarrow b) \rightarrow b \in F$. Notice that $a \rightarrow b$ and $(a \rightarrow b) \rightarrow b$ are in $[b]$. Thus there exists $(a \rightarrow b) \wedge [(a \rightarrow b) \rightarrow b]$ in A . Hence $(a \rightarrow b) \wedge [(a \rightarrow b) \rightarrow b] \in F$. Since $(a \rightarrow b) \rightarrow b \leq (a \rightarrow b) \rightarrow b$, it follows by the residuation law in the Boolean algebra $[b]$ that $(a \rightarrow b) \wedge [(a \rightarrow b) \rightarrow b] \leq b$. Therefore $b \in F$. $\square$

Let A be a Tarski algebra. The *annihilator* of an element $a \in A$ is the set

$$a^\top = \{x \in A : x \vee a = 1\} = \{x \in A : x \rightarrow a = a\}$$

Proposition 3.4 *Let A be a Tarski algebra. Then, for all $a, b \in A$,*

1. $a^\top \in \text{Fi}(A)$
2. $(a \vee b)^\top = a^\top \vee b^\top$.
3. $a^\top \vee [a] = A$ and $a^\top \cap [a] = \{1\}$.

Proof (1) Clearly $1 \in a^\top$. Let $x, x \rightarrow y \in a^\top$. Thus $a \vee x = 1$ and $a \vee (x \rightarrow y) = 1$. Then, $1 = a \vee (x \rightarrow y) = (a \vee x) \rightarrow (a \vee y) = 1 \rightarrow (a \vee y) = a \vee y$. Hence, $y \in a^\top$. Therefore $a^\top \in \text{Fi}(A)$.

(2) It is clear that $a^\top, b^\top \subseteq (a \vee b)^\top$. Thus $a^\top \vee b^\top \subseteq (a \vee b)^\top$. Let now $x \in (a \vee b)^\top$. By Proposition 3.2.1, we have that $(a \vee b) \rightarrow x = x$. Then $(a \rightarrow x) \wedge (b \rightarrow x) = x$. Since $a \vee (a \rightarrow x) = 1$, it follows

- that $a \rightarrow x \in a^\top \subseteq a^\top \vee b^\top$. Similarly, $b \rightarrow x \in a^\top \vee b^\top$. Then, by Proposition 3.3, $x = (a \rightarrow x) \wedge (b \rightarrow x) \in a^\top \vee b^\top$. Hence $(a \vee b)^\top \subseteq a^\top \vee b^\top$.
- (3) On the one hand, it is clear that $a^\top \cap [a] = \{1\}$. On the other hand, notice that $a^\top \vee [a] = \text{Fig}(a^\top, a) = \{x \in A : a \rightarrow x \in a^\top\}$. Let $x \in A$. Since $a \vee (a \rightarrow x) = 1$, it follows that $a \rightarrow x \in a^\top$. Thus $x \in a^\top \vee [a]$. Hence $A = a^\top \vee [a]$. $\square$

Proposition 3.5 *Let A be a Tarski algebra, $a \in A$ and F a proper implicative filter of A .*

1. If $a \notin F$, then $a \notin F \vee a^\top$.
2. If $a^\top \subseteq F$, then $a \notin F$.
3. If $a^\top \not\subseteq F$, then the implicative filter $F \vee [a]$ is proper.

Proof (1) Assume that $a \notin F$. Suppose that $a \in F \vee a^\top$. Then,

$$[a] = (F \vee a^\top) \cap [a] = (F \cap [a]) \vee (a^\top \cap [a]) = F \cap [a].$$

Hence $a \in F$, which is a contradiction. Therefore, $a \notin F \vee a^\top$.

- (2) Assume $a^\top \subseteq F$. Suppose $a \in F$. Thus $[a] \subseteq F$. Then $A = a^\top \vee [a] \subseteq F$, a contradiction because F is proper. Hence $a \notin F$.
- (3) Assume $a^\top \not\subseteq F$. Suppose that $F \vee [a] = A$. Then

$$a^\top = (F \vee [a]) \cap a^\top = (F \cap a^\top) \vee ([a] \cap a^\top) = F \cap a^\top.$$

Which is a contradiction. Hence, $F \vee [a]$ is proper. $\square$

4 Representation

In this section, we develop a choice-free representation for Tarski algebras. We define a topology on the set of all proper implicative filters of a Tarski algebra. Then, we characterize topologically the spaces that arise as the spaces of the proper implicative filters of a Tarski algebras.

Let $\langle A, \rightarrow, 1 \rangle$ be a Tarski algebra. Let

$$X_A = \text{Fi}(A) \setminus \{A\},$$

that is, X_A is the collection of all proper implicative filters of A . Let us define the map $\alpha : A \rightarrow \mathcal{P}(X_A)$ as follows: for every $a \in A$,

$$\alpha(a) = \{F \in X_A : a \in F\}.$$

Let $\mathcal{L}_A = \{\alpha(a) : a \in A\}$. Let τ_A be the topology on X_A generated by the family $\mathcal{L}_A$. In other words, $\mathcal{L}_A$ is a subbasis for the topology τ_A on X_A . Notice that the specialization order $\preceq_{\tau_A}$ is the set-theoretical inclusion on X_A , that is, for all $x, y \in X_A$, $x \preceq_{\tau_A} y \iff x \subseteq y$. Hence the upset topology $\tau_{\preceq_{\tau_A}}$ on X_A is simply the upset topology $\tau_{\subseteq}$ of the poset $\langle X_A, \subseteq \rangle$.

Proposition 4.1 *For every $a \in A$, $\alpha(a)$ is a regular open of the space $\langle X_A, \tau_{\subseteq} \rangle$.*

Proof Let $a \in A$. It is clear that $\alpha(a) \subseteq \text{int}_{\subseteq}(\text{cl}_{\subseteq}(\alpha(a)))$. Let now $x \in X_A$ be such that $x \notin \alpha(a)$. So $a \notin x$. By (1) of Proposition 3.5, the implicative filter $y := x \vee a^\top$ is proper. Now we show that $[y]_{\subseteq} \cap \alpha(a) = \emptyset$. Suppose that there is $z \in [y]_{\subseteq} \cap \alpha(a)$. Thus $y \subseteq z$ and $a \in z$. Then $a^\top \subseteq z$ and $a \in z$, which is a contradiction by Proposition 3.5. Hence, since $x \subseteq y$ and $[y]_{\subseteq} \cap \alpha(a) = \emptyset$, it follows by (U3) that $x \notin \text{int}_{\subseteq}(\text{cl}_{\subseteq}(\alpha(a)))$. Therefore, $\alpha(a) = \text{int}_{\subseteq}(\text{cl}_{\subseteq}(\alpha(a)))$. That is, $\alpha(a) \in \mathcal{RO}(X_A, \tau_{\subseteq})$. $\square$

The previous proposition tells us that $\mathcal{L}_A \subseteq \mathcal{RO}(X_A, \tau_{\subseteq})$. The next proposition shows that $\mathcal{L}_A$ is an implicative subreduct of the Boolean algebra $\mathcal{RO}(X_A, \tau_{\subseteq})$. Let $\rightarrow_{\mathcal{RO}}$ be the implication of the Boolean algebra $\mathcal{RO}(X_A, \tau_{\subseteq})$.

Proposition 4.2 *For all $a, b \in A$, we have $\alpha(a \rightarrow b) = \alpha(a) \rightarrow_{\mathcal{RO}} \alpha(b)$. Moreover, $\alpha : \langle A, \rightarrow, 1 \rangle \rightarrow \langle \mathcal{L}_A, \rightarrow_{\mathcal{RO}}, X_A \rangle$ is an isomorphism.*

Proof Let $a, b \in A$. Recall that $\alpha(a) \rightarrow_{\mathcal{RO}} \alpha(b) = \{x \in X_A : [x]_{\subseteq} \cap \alpha(a) \subseteq \alpha(b)\}$. It is straightforward show directly that $\alpha(a \rightarrow b) \subseteq \alpha(a) \rightarrow_{\mathcal{RO}} \alpha(b)$. Let now $x \notin \alpha(a \rightarrow b)$. Thus $a \rightarrow b \notin x$. Then $b \notin \text{Fig}(x, a)$. We obtain that $[x]_{\subseteq} \cap \alpha(a) \not\subseteq \alpha(b)$. Hence $x \notin \alpha(a) \rightarrow_{\mathcal{RO}} \alpha(b)$. Therefore, $\alpha(a \rightarrow b) = \alpha(a) \rightarrow_{\mathcal{RO}} \alpha(b)$. $\square$

By the previous proposition, we obtain that $\langle \mathcal{L}_A, \rightarrow_{\mathcal{RO}}, X_A \rangle$ is a Tarski algebra, and thus $\langle A, \rightarrow, 1 \rangle$ is isomorphic to an implicative subreduct of the Boolean algebra $\langle \mathcal{RO}(X_A, \tau_{\subseteq}), \rightarrow_{\mathcal{RO}}, \neg_{\mathcal{RO}}, \emptyset, X_A \rangle$. Moreover, since the joins are definable in terms of the implication on Tarski and Boolean algebras, we have by the previous proposition that the embedding $\alpha : A \rightarrow \mathcal{RO}(X_A, \tau_{\subseteq})$ preserves joins: for all $a_1, \dots, a_n \in A$,

$$\alpha(a_1 \vee \dots \vee a_n) = \alpha(a_1) \vee_{\mathcal{RO}} \dots \vee_{\mathcal{RO}} \alpha(a_n) = \text{int}_{\subseteq}(\text{cl}_{\subseteq}(\alpha(a_1) \cup \dots \cup \alpha(a_n))).$$

Now let us characterize the structures of the form $\langle X_A, \tau_A, \mathcal{L}_A \rangle$ for all Tarski algebras A . For the following definition, we need to consider the following property. Let X be a non-empty set and $\mathcal{L} \subseteq \mathcal{P}(X)$. Let $\nabla \subseteq \mathcal{L}$. We shall say that ∇ is $\mathcal{L}$ -proper is satisfies:

$$\exists U \in \mathcal{L} \text{ such that } \forall U_1, \dots, U_n \in \nabla, U_1 \cap \dots \cap U_n \not\subseteq U \quad (\text{P})$$

Definition 4.3 A structure $\langle X, \tau, \mathcal{L} \rangle$ is called a *CFT-space*.¹ if satisfies the following:

- (Ts1) $\langle X, \tau \rangle$ is a T_0 -space where $\mathcal{L}$ is a subbasis for τ .
- (Ts2) $\mathcal{L} \subseteq \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\leq})$ and $X \in \mathcal{L}$.
- (Ts3) For all $U, V \in \mathcal{L}$, $U \rightarrow_{\mathcal{RO}} V \in \mathcal{L}$.
- (Ts4) For all $x \in X$, there is $U \in \mathcal{L}$ such that $x \notin U$.
- (Ts5) For all $\nabla \cup \Delta \subseteq \mathcal{L}$, if ∇ satisfies condition (P) and $\bigcap \nabla \subseteq \bigcup \Delta$, then there are $U_1, \dots, U_n \in \nabla$ and $V \in \Delta$ such that $U_1 \cap \dots \cap U_n \subseteq V$.

¹“CFT” is for *choice-free Tarski*.

Given a CFT-space $\langle X, \tau, \mathcal{L} \rangle$, let

$$T_{\mathcal{L}}(X) := \langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle.$$

Proposition 4.4 *If $\langle X, \tau, \mathcal{L} \rangle$ is a structure satisfying conditions (Ts1)–(Ts3), then $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a Tarski algebra.*

Proof We know that $\langle \mathcal{RO}(X, \tau_{\leq}), \rightarrow_{\mathcal{RO}}, \emptyset, X \rangle$ is a Boolean algebra. From (Ts1)–(Ts3) we have that $\langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a subalgebra of $\langle \mathcal{RO}(X, \tau_{\leq}), \rightarrow_{\mathcal{RO}}, X \rangle$. Hence $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a Tarski algebra. $\square$

We say that $T_{\mathcal{L}}(X)$ is the *dual Tarski algebra* of the CFT-space $\langle X, \tau, \mathcal{L} \rangle$.

Remark 4.5 Let $\langle X, \tau, \mathcal{L} \rangle$ be a structure satisfying (Ts1)–(Ts3). Since $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a subalgebra of $\langle \mathcal{RO}(X, \tau_{\leq}), \rightarrow_{\mathcal{RO}}, X \rangle$, it follows that $U \vee_{\mathcal{RO}} V = (U \rightarrow_{\mathcal{RO}} V) \rightarrow_{\mathcal{RO}} V \in \mathcal{L}$, for all $U, V \in \mathcal{L}$. Hence $U \vee_{\mathcal{RO}} V$ is the supremum of U and V in the Tarski algebra $T_{\mathcal{L}}(X)$, for all $U, V \in \mathcal{L}$.

Recall that in the Boolean algebra $\langle \mathcal{RO}(X, \tau_{\leq}), \rightarrow_{\mathcal{RO}}, \emptyset, X \rangle$ the infimum is given by intersection of sets.

Proposition 4.6 *Let $\langle X, \tau, \mathcal{L} \rangle$ be a structure satisfying (Ts1)–(Ts3). Let $U, V \in \mathcal{L}$. If the infimum $U \wedge V$ exists in the Tarski algebra $T_{\mathcal{L}}(X)$, then $U \wedge V = U \cap V$.*

Proof Let $U, V \in \mathcal{L}$ be such that $U \wedge V$ exists in the Tarski algebra $T_{\mathcal{L}}(X)$. Let $W := U \wedge V$. Then

$$\begin{aligned} U \wedge V &= (V \rightarrow_{\mathcal{RO}} (U \rightarrow_{\mathcal{RO}} W)) \rightarrow_{\mathcal{RO}} W \\ &= \neg_{\mathcal{RO}}(\neg_{\mathcal{RO}}V \vee_{\mathcal{RO}} (\neg_{\mathcal{RO}}U \vee_{\mathcal{RO}} W)) \vee_{\mathcal{RO}} W \\ &= (V \cap (U \cap \neg_{\mathcal{RO}}W)) \vee_{\mathcal{RO}} W \\ &= (V \vee_{\mathcal{RO}} W) \cap (U \vee_{\mathcal{RO}} W) \cap (\neg_{\mathcal{RO}}W \vee_{\mathcal{RO}} W) \\ &= V \cap U. \end{aligned}$$

$\square$

Let $\langle X, \tau, \mathcal{L} \rangle$ be a structure satisfying conditions (Ts1)–(Ts3). For every $x \in X$, let

$$\epsilon(x) = \{U \in \mathcal{L} : x \in U\}. \quad (4.1)$$

Recall that $X_{T_{\mathcal{L}}(X)}$ denotes the collection of all proper implicative filters of the Tarski algebra $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{RO}}, X \rangle$, for each structure $\langle X, \tau, \mathcal{L} \rangle$ satisfying conditions (Ts1)–(Ts3).

Proposition 4.7 *Let $\langle X, \tau, \mathcal{L} \rangle$ be a structure satisfying conditions (Ts1)–(Ts4). Then, the following conditions are equivalent:*

1. $\epsilon : X \rightarrow X_{T_{\mathcal{L}}(X)}$ as defined in (4.1) is an onto map.
2. For all $\nabla \cup \Delta \subseteq \mathcal{L}$, if ∇ satisfies condition (P) and $\bigcap \nabla \subseteq \bigcup \Delta$, then there are $U_1, \dots, U_n \in \nabla$ and $V \in \Delta$ such that $U_1 \cap \dots \cap U_n \subseteq V$.

Proof (1) $\Rightarrow$ (2) Let $\nabla \cup \Delta \subseteq \mathcal{L}$ be such that ∇ satisfies condition (P) and $\bigcap \nabla \subseteq \bigcup \Delta$. Let $F := \text{Fig}_{T_{\mathcal{L}}(X)}(\nabla)$. By condition (P) and (2.1), it follows that F is proper. Thus $F \in X_{T_{\mathcal{L}}(X)}$. Then, by (1), there is $x \in X$ such that $\epsilon(x) = F$. It follows that $x \in \bigcap \nabla$, and thus $x \in \bigcup \Delta$. Then, there is $V \in \Delta$ such that $x \in V$. Since $V \in F = \text{Fig}_{T_{\mathcal{L}}(X)}(\nabla)$, we have that there are $U_1, \dots, U_n \in \nabla$ such that $U_1 \rightarrow_{\mathcal{R}\mathcal{O}} (U_2 \rightarrow_{\mathcal{R}\mathcal{O}} (\dots (U_n \rightarrow_{\mathcal{R}\mathcal{O}} V) \dots)) = X_{T_{\mathcal{L}}(X)}$. Hence $U_1 \cap \dots \cap U_n \subseteq V$.

(2) $\Rightarrow$ (1) It is straightforward that $\epsilon(x)$ is an implicative filter of $T_{\mathcal{L}}(X)$, and by (Ts4), it follows that $\epsilon(x)$ is proper, for all $x \in X$. Hence $\epsilon(x) \in X_{T_{\mathcal{L}}(X)}$, for all $x \in X$. Let $F \in X_{T_{\mathcal{L}}(X)}$ and let $\Delta = \mathcal{L} \setminus F$. Since F is a proper implicative filter, it follows that F satisfies condition (P). Moreover, since $F \cap \Delta = \emptyset$, it follows that for all $U_1, \dots, U_n \in F$ and $V \in \Delta$, $U_1 \cap \dots \cap U_n \not\subseteq V$. Hence, by (2), we have that $\bigcap F \not\subseteq \bigcup \Delta$. Let $x \in \bigcap F \setminus \bigcup \Delta$. Therefore $\epsilon(x) = F$. $\square$

Let A be a Tarski algebra. Recall that τ_A is the topology on X_A generated by the family $\mathcal{L}_A = \{\alpha(a) : a \in A\}$, and the specialization order $\preceq_{\tau_A}$ of $\langle X_A, \tau_A \rangle$ is the inclusion order, that is, $\preceq_{\tau_A} = \subseteq$. Let

$$X(A) := \langle X_A, \tau_A, \mathcal{L}_A \rangle.$$

Proposition 4.8 *For every $a \in A$, $\alpha(a)$ is a compact subset of X_A .*

Proof Let $a \in A$ and suppose that $\alpha(a) \subseteq \bigcup_{i \in I} U_i$ with $U_i \in \tau_A$ for all $i \in I$. If $[a] = A$, then $\alpha(a)$ is empty, and thus it is compact. If $[a] \neq A$, then $[a] \in \alpha(a)$. Thus there is $i \in I$ such that $[a] \in U_i$. Since $\mathcal{L}_A$ is subbasis, there are $a_1, \dots, a_n \in A$ such that $[a] \in \alpha(a_1) \cap \dots \cap \alpha(a_n) \subseteq U_i$. Then $\alpha(a) \subseteq U_i$. Hence $\alpha(a)$ is compact. $\square$

Theorem 4.9 *Let A be a Tarski algebra. Then, the topological space $X(A) = \langle X_A, \tau_A, \mathcal{L}_A \rangle$ is a CFT-space.*

Proof It is straightforward that the space $X(A)$ is T_0 . We have $X_A = \alpha(1) \in \mathcal{L}_A$. Thus $X(A)$ satisfies (Ts1). By Proposition 4.1, it follows that $\mathcal{L}_A \subseteq \mathcal{R}\mathcal{O}(X_A, \tau_{\subseteq})$. By Proposition 4.8, it follows that $\mathcal{L}_A \subseteq \text{KO}(X_A)$. Hence, $\mathcal{L}_A \subseteq \text{KO}(X_A) \cap \mathcal{R}\mathcal{O}(X_A, \tau_{\subseteq})$, that is, (Ts2) holds. By Proposition 4.2, it follows that condition (Ts3) holds. Let $x \in X_A$. Since x is a proper implicative filter of A , there is $a \in A$ such that $a \notin x$. Hence $x \notin \alpha(a) \in \mathcal{L}_A$. Then, (Ts4) holds. Since $\alpha : A \rightarrow \langle \mathcal{L}_A, \rightarrow_{\mathcal{R}\mathcal{O}}, X_A \rangle$ is an isomorphism, we have that every proper implicative filter $\mathcal{F}$ of $\langle \mathcal{L}_A, \rightarrow_{\mathcal{R}\mathcal{O}}, X_A \rangle$ is the form $\alpha[x]$, for some $x \in X_A$. Then $\mathcal{F} = \alpha[x] = \{\alpha(a) \in \mathcal{L}_A : a \in x\} = \{\alpha(a) \in \mathcal{L}_A : x \in \alpha(a)\}$. Hence, $\epsilon : X_A \rightarrow X_{T_{\mathcal{L}_A}(X_A)}$ is an onto map. Therefore, $X(A)$ satisfies (Ts5). $\square$

Given a Tarski algebra A , we say that $X(A)$ is the *dual CFT-space* of A . Notice that the dual Tarski algebra of the CFT-space $X(A) = \langle X_A, \tau_A, \mathcal{L}_A \rangle$ is $T_{\mathcal{L}_A}(X(A)) = \langle \mathcal{L}_A, \rightarrow_{\mathcal{R}\mathcal{O}}, X_A \rangle$. Now we are ready to present a choice-free topological representation for Tarski algebras.

Corollary 4.10 (Representation) *Every Tarski algebra A is isomorphic to the dual Tarski algebra $T_{\mathcal{L}_A}(X(A))$ of the CFT-space $X(A)$.*

Proof It is a consequence of the previous theorem and Proposition 4.2. $\square$

The following theorem shows that the CFT-spaces are precisely the CFT-spaces $X(A)$, with A a Tarski algebra.

Theorem 4.11 *Let $\langle X, \tau, \mathcal{L} \rangle$ be a CFT-space. Then, the function $\epsilon : X \rightarrow X(T_{\mathcal{L}}(X))$ defined by $\epsilon(x) = \{U \in \mathcal{L} : x \in U\}$ is a homeomorphism. Moreover, $\mathcal{L}_{T_{\mathcal{L}}(X)} = \{\epsilon[U] : U \in \mathcal{L}\}$.*

Proof Recall that $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{K}\mathcal{O}}, X \rangle$ and $X(T_{\mathcal{L}}(X)) = \langle X_{T_{\mathcal{L}}(X)}, \tau_{T_{\mathcal{L}}(X)}, \mathcal{L}_{T_{\mathcal{L}}(X)} \rangle$, where $\mathcal{L}_{T_{\mathcal{L}}(X)} = \{\alpha(U) : U \in \mathcal{L}\}$ is the corresponding subbasis of $X(T_{\mathcal{L}}(X))$.

From conditions (Ts4) and (Ts5) (see also Proposition 4.7), it follows that ϵ is a well-defined onto map. Since $\mathcal{L}$ is a subbasis, it follows that ϵ is injective. Now we prove that ϵ is continuous. Let $U \in \mathcal{L}$ and $x \in X$. Then

$$x \in \epsilon^{-1}[\alpha(U)] \iff \epsilon(x) \in \alpha(U) \iff U \in \epsilon(x) \iff x \in U.$$

Thus $\epsilon^{-1}[\alpha(U)] = U$ is an open subset of X . Hence ϵ is continuous. By the above, and since ϵ is a one-to-one correspondence, it follows that for all $U \in \mathcal{L}$, $\epsilon[U] = \alpha(U)$. Hence, ϵ is a homeomorphism and $\mathcal{L}_{T_{\mathcal{L}}(X)} = \{\epsilon[U] : U \in \mathcal{L}\}$. $\square$

We end this section by establishing a connection between our choice-free representation for Tarski algebras and the representation given by [6]. In [5], the authors show that the dual UV-space of a Boolean algebra B (this is the choice-free duality for Boolean algebra) can be realized as the *upper Vietoris topology* ([5, 18]) on the hyperspace of nonempty closed sets of the dual Stone space of B . In the context of Tarski algebras we have a similar result using the dual Tarski spaces for Tarski algebras given in [6].

Definition 4.12 A *Tarski space* is a structure $\langle X, \mathcal{K}, \tau \rangle$ such that

1. $\langle X, \tau \rangle$ is a topological space where $\mathcal{K}$ is a basis of compact subsets for τ .
2. For every $A, B \in \mathcal{K}$, $A \cap B^c \in \mathcal{K}$.
3. For every $x, y \in X$, if $x \neq y$, then there exists $U \in \mathcal{K}$ such that $x \notin U$ and $y \in U$.
4. If F is a closed subset and $K = \{U_i : i \in I\}$ is a dually directed subfamily of $\mathcal{K}$ such that $F \cap U_i \neq \emptyset$ for all $i \in I$, then $F \cap \bigcap \{U_i : i \in I\} \neq \emptyset$.

Let $\langle X, \mathcal{K}, \tau \rangle$ be a Tarski space. Let $D_{\mathcal{K}}(X) = \{U^c : U \in \mathcal{K}\}$. Then $\langle D_{\mathcal{K}}(X), \Rightarrow, X \rangle$, where $A \Rightarrow B = A^c \cup B$, is a Tarski algebra. Following a similar argument to the given in [5], we can obtain, with slight variation, the dual CFT-space a Tarski algebra A through of an upper Vietories topology on the hyperspace of nonempty closed sets of the dual Tarski space of A . Let $C(X)^*$ be the collection of all nonempty closed subsets of $\langle X, \mathcal{K}, \tau \rangle$. For every $U \in D_{\mathcal{K}}(X)$, let

$$\square U = \{C \in C(X)^* : C \subseteq U\}.$$

Let $\mathcal{UV}(X)$ be the space of nonempty closed subsets of X with the topology generated by the family $\{\Box U : U \in D_{\mathcal{K}}(X)\}$. Notice that $\Box U \cap \Box V = \Box(U \cap V)$. Hence, $\{\Box U : U \in D_{\mathcal{K}}(X)\}$ form a basis for the space $\mathcal{UV}(X)$.

Proposition 4.13 *Let $\langle X, \mathcal{K}, \tau \rangle$ be a Tarski space. Let $A = D_{\mathcal{K}}(X)$ be the dual Tarski algebra of X . Then, the hyperspace $\mathcal{UV}(X)$ is homeomorphic to the dual CFT-space $\langle X_A, \tau_A \rangle$ of A .*

Proof Notice that X_A is the collection of proper filters of $D_{\mathcal{K}}(X)$. Let $f : C(X)^* \rightarrow X_A$ be defined by $f(C) = \{U \in D_{\mathcal{K}}(X) : C \subseteq U\}$. Since $D_{\mathcal{K}}(X)$ is a basis for the closed subsets of X , it follows that $C \neq \emptyset$ if and only if $f(C) \neq D_{\mathcal{K}}(X)$. For this and by [6, Lemma 3.9], it follows that f is a one-to-one map. Now we show that f is continuous and open. Recall that the basic opens of the CFT-space X_A are of the form $\alpha(U) = \{\mathcal{F} \in X_A : U \in \mathcal{F}\}$, with $U \in A = D_{\mathcal{K}}(X)$. Let $U \in A = D_{\mathcal{K}}(X)$ and $C \in C(X)^*$. Then

$$C \in f^{-1}[\alpha(U)] \iff f(C) \in \alpha(U) \iff U \in f(C) \iff C \subseteq U.$$

Thus, we obtain that $f^{-1}[\alpha(U)] = \Box U$. Hence f is continuous. We now show that f is open. Let $U \in D_{\mathcal{K}}(X)$. Then, using the fact that f is onto, we have

$$f[\Box U] = \{f(C) : C \subseteq U\} = \{f(C) : u \in f(C)\} = \alpha(u).$$

Hence, f is open. Therefore, f is a homeomorphism. $\square$

5 Morphisms and Choice-Free Duality for Tarski Algebras

In this section, we extend the correspondence between Tarski algebras and CFT-spaces to a full dual equivalence between certain categories of Tarski algebras and morphisms and certain categories of CFT-spaces and morphisms.

Definition 5.1 Let A and B be Tarski algebras. A mapping $h : A \rightarrow B$ is said to be:

- a *semi-homomorphism* if $h(a \rightarrow b) \leq h(a) \rightarrow h(b)$, for every $a, b \in A$, and $h(1) = 1$.
- a *homomorphism* if $h(a \rightarrow b) = h(a) \rightarrow h(b)$, for all $a, b \in A$.

Notice that each homomorphism is a semi-homomorphism, and every semi-homomorphism $h : A \rightarrow B$ is order-preserving, i.e., if $a \leq b$, then $h(a) \leq h(b)$, for every $a, b \in A$. We present some characterizations of semi-homomorphism and homomorphism.

Let A and B be Tarski algebras. Let $h : A \rightarrow B$ be an order-preserving map. For each $F \in \text{Fi}(B)$, let $h_+(F) = h^{-1}[F]$.

Proposition 5.2 *Let A and B be Tarski algebras. Let $h : A \rightarrow B$ be an order-preserving map. Then, h is a semi-homomorphism if and only if $h_+(F) \in \text{Fi}(A)$, for every $F \in \text{Fi}(B)$. Hence, $h_+ : \text{Fi}(B) \rightarrow \text{Fi}(A)$ is a well-defined map, for all semi-homomorphism $h : A \rightarrow B$.*

Proof The implication from left to right is given in [6, Prop. 4.9]. Assume that $h_+(F) \in \text{Fi}(A)$, for every $F \in \text{Fi}(B)$. We suppose that there exist $a, b \in A$ such that $h(a \rightarrow b) \not\leq h(a) \rightarrow h(b)$. Then $h(a) \rightarrow h(b) \notin F = \text{Fig}_B(h(a \rightarrow b))$. Let $H = \text{Fig}_B(F, h(a))$. Since $h(a) \rightarrow h(b) \notin F$, it follows that $h(b) \notin H$. Thus $b \notin h^{-1}[H] = h_+(H)$. On the other hand, since $F \subseteq H$ and $h(a) \in H$, we have that $a \rightarrow b, a \in h^{-1}[H] = h_+(H)$. By hypothesis, $h_+(H)$ is an implicative filter of A , then we have $b \in h_+(H)$, which is an absurd. Hence $h(a \rightarrow b) \leq h(a) \rightarrow h(b)$, for all $a, b \in A$. $\square$

It is known that if A and B are Boolean algebras, then a mapping $h : A \rightarrow B$ is a semi-homomorphism if and only if h is a meet-homomorphism, i.e., $h(a \wedge b) = h(a) \wedge h(b)$, for all $a, b \in A$, and $h(1) = 1$. We have a similar result for Tarski algebras.

Proposition 5.3 *Let A and B be Tarski algebras and $h : A \rightarrow B$ be an order-preserving map. Then, the following are equivalent:*

1. h is a semi-homomorphism.
2. h preserves existing finite meets.

Proof By the previous proposition, (1) is equivalent to the condition that $h^{-1}[G] \in \text{Fi}(A)$, for all $G \in \text{Fi}(B)$. In [12, Prop. 5.10] it is proved that a map $f : P \rightarrow Q$, between posets P and Q , preserves existing finite meets if and only if for all upset G of Q closed under existing finite meets in Q , $f^{-1}[G]$ is an upset of P closed under existing finite meets in P . Now by Proposition 3.3, we have $G \in \text{Fi}(A)$ if and only if G is an upset closed under existing finite meets and $1 \in G$. Hence, (1) is equivalent to (2). $\square$

Let $\langle X_1, \leq_1 \rangle$ and $\langle X_2, \leq_2 \rangle$ be posets. A map $f : X_1 \rightarrow X_2$ is said to be a *p-morphism* if it is order-preserving and satisfies the following:

$$\text{if } f(x) \leq_2 y, \text{ then there is } x' \in X_1 \text{ such that } x \leq x' \text{ and } f(x') = y. \quad (\text{pM})$$

Proposition 5.4 *Let A and B be Tarski algebras. Let $h : A \rightarrow B$ be an order-preserving map. Then, h is a homomorphism if and only if $h_+ : \langle \text{Fi}(B), \subseteq \rangle \rightarrow \langle \text{Fi}(A), \subseteq \rangle$ is a p-morphism.*

Proof Assume that $h : A \rightarrow B$ is a homomorphism. By Proposition 5.2, $h_+ : \text{Fi}(B) \rightarrow \text{Fi}(A)$ is a map. It is clear that h_+ is order-preserving. Now let $G \in \text{Fi}(B)$ and $F \in \text{Fi}(A)$ be such that $h_+[G] \subseteq F$. Let $H = \text{Fig}(h[F] \cup G)$. Then $G \subseteq H$ and $F \subseteq h^{-1}[H]$. We prove that $h^{-1}[H] \subseteq F$. Let $h(b) \in H$. Then there exist $a_1, \dots, a_n \in F$ such that $h(a_1) \rightarrow (h(a_2) \rightarrow (\dots (h(a_n) \rightarrow h(b)) \dots)) \in G$, and since h is homomorphism, it follows that $a_1 \rightarrow (a_2 \rightarrow (\dots (a_n \rightarrow b) \dots)) \in h^{-1}[G] \subseteq F$. By modus ponens, $b \in F$. Hence, we have proved that there exists $H \in \text{Fi}(B)$ such that $G \subseteq H$ and $h_+(H) = F$. Therefore, h_+ is a p-morphism.

Assume that $h_+ : \text{Fi}(B) \rightarrow \text{Fi}(A)$ is a p-morphism. Thus, by Proposition 5.2, h is a semi-homomorphism. Now let $a, b \in A$ and suppose that $h(a) \rightarrow h(b) \not\leq h(a \rightarrow b)$. Then $h(a \rightarrow b) \notin \text{Fig}_B(h(a) \rightarrow h(b)) = G$, i.e., $a \rightarrow b \notin h^{-1}[G]$. Since $h^{-1}[G] = h_+(G) \in \text{Fi}(A)$, it follows that $b \notin \text{Fig}_A(h^{-1}[G] \cup \{a\}) = H$. As $h^{-1}[G] \subseteq H$ and h_+ is a p-morphism,

there exists $K \in \text{Fi}(B)$ such that $G \subseteq K$ and $h^{-1}[K] = H$. We have that $h(a) \rightarrow h(b) \in K$. As $a \in H = h^{-1}[K]$, we get $h(a) \in K$. Thus, $h(b) \in K$, i.e., $b \in h^{-1}[K] = H$, which is a contradiction. Thus, $h(a) \rightarrow h(b) \leq h(a \rightarrow b)$. Therefore, h is a homomorphism. $\square$

Let us consider the following two algebraic categories:

$$\begin{aligned} \mathcal{STA} &= \text{Tarski algebras} + \text{semi-homomorphisms} \\ \mathcal{HTA} &= \text{Tarski algebras} + \text{homomorphisms.} \end{aligned}$$

Our next task is to define the corresponding morphisms between CFT-spaces will be the duals of the semi-homomorphisms and homomorphisms. First, an observation. Given a homomorphism $h : A \rightarrow B$ between Tarski algebras, we know that $h_+ : \text{Fi}(B) \rightarrow \text{Fi}(A)$ is a map (actually, a p-morphism). However, it is not necessarily true that h_+ be a map from X_B into X_A , because there might be homomorphisms $h : A \rightarrow B$ and $G \in X_B$ such that $h_+(G) = h^{-1}[G] = A \notin X_A$. Hence, we cannot take the h_+ as the dual of the homomorphisms between Tarski algebras.

We start with the following definitions. Let $R \subseteq X_1 \times X_2$ be a relation. For every $x \in X_1$, let $R(x) = \{y \in X_2 : (x, y) \in R\}$. For every $Z \subseteq X_1$, let $R[Z] = \{y \in X_2 : \exists z \in Z \text{ s.t. } (z, y) \in R\}$. We define the map $h_R : \mathcal{P}(X_2) \rightarrow \mathcal{P}(X_1)$ as follows: for every $Y \subseteq X_2$,

$$h_R(Y) = \{x \in X_1 : R(x) \subseteq Y\}. \quad (5.1)$$

Given two relations $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, we set the composition $S \circ R \subseteq X_1 \times X_3$ as follows:

$$(x, z) \in S \circ R \iff \exists y \in X_2 \text{ such that } (x, y) \in R \text{ and } (y, z) \in S.$$

Definition 5.5 Let $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ be CFT-spaces. A relation $R \subseteq X_1 \times X_2$ is called a *semi-CFT-morphism* if satisfies the following:

- (R1) For every $v \in \mathcal{L}_2$, $h_R(v) \in \mathcal{L}_1$.
- (R2) If $x \preceq_1 x'$, then $R(x') \subseteq R(x)$.
- (R3) $R(x) = \bigcap \{v \in \mathcal{L}_2 : R(x) \subseteq v\}$.

Notice that for every semi-CFT-morphism $R \subseteq X_1 \times X_2$ and every $x \in X_1$, $R(x)$ is an upset of $\langle X_2, \preceq_2 \rangle$. The next proposition aims to show that it is possible to consider a category whose objects are the CFT-spaces and whose morphisms are the semi-CFT-morphisms. First, we need the following technical result.

Lemma 5.6 Let $R \subseteq X_1 \times X_2$ be a semi-CFT-morphism. Let $\nabla \subseteq \mathcal{L}_1$ be such that satisfies condition (P) and let $C = \bigcap \nabla$. Then $R[C] = \bigcap \{v \in \mathcal{L}_2 : R[C] \subseteq v\}$.

Proof It is clear that $R[C] \subseteq \bigcap \{v \in \mathcal{L}_2 : R[C] \subseteq v\}$. Let $y \in \bigcap \{v \in \mathcal{L}_2 : R[C] \subseteq v\}$. Suppose that $y \notin R[C]$. Then, for every $x \in C$, $y \notin R(x)$. By (R3), it follows that for every $x \in C$, there is $v_x \in \mathcal{L}_2$ such that $R(x) \subseteq v_x$ and $y \notin v_x$. Notice that for every $x \in C$, $x \in h_R(v_x)$. Let us consider the family $\Delta = \{h_R(v_x) : x \in C\} \subseteq \mathcal{L}_1$. Then $\bigcap \nabla = C \subseteq \bigcup \Delta$. By (Ts5),

there are $u_1, \dots, u_n \in \nabla$ and $x \in C$ such that $u_1 \cap \dots \cap u_n \subseteq h_R(v_x)$. Then $C \subseteq h_R(v_x)$. It follows that $R[C] \subseteq v_x$, and this is a contradiction because $y \notin v_x$. $\square$

Proposition 5.7 *Let $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$, $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ and $\langle X_3, \tau_3, \mathcal{L}_3 \rangle$ be CFT-spaces. Let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be semi-CFT-morphism. Then:*

1. The relation $I_{X_1} \subseteq X_1 \times X_1$ defined by $I_{X_1}(x) = [x]_{\preccurlyeq_1}$ is a semi-CFT-morphism.
2. $I_{X_2} \circ R = R$ and $S \circ I_{X_2} = S$.
3. $h_{S \circ R}(w) = (h_R \circ h_S)(w)$, for all $w \in \mathcal{L}_3$.
4. $S \circ R$ is a semi-CFT-morphism.

Proof (1) Since $\preccurlyeq_1$ is the specialization order on X_1 and $\mathcal{L}_1$ is a subbasis, it follows that $h_{I_{X_1}}(u) = \{x \in X_1 : I_{X_1}(x) \subseteq u\} = \{x \in X_1 : [x]_{\preccurlyeq_1} \subseteq u\} = u$, for all $u \in \mathcal{L}_1$. Hence I_{X_1} holds condition (R1). It is straightforward that I_{X_1} satisfies (R2). We also have that $\bigcap \{u \in \mathcal{L}_1 : I_{X_1}(x) \subseteq u\} = \bigcap \{u \in \mathcal{L}_1 : [x]_{\preccurlyeq_1} \subseteq u\} = [x]_{\preccurlyeq_1} = I_{X_1}(x)$, for all $x \in X_1$. Hence, I_{X_1} satisfies condition (R3).

(2) Since $R(x)$ is an $\preccurlyeq_2$ -upset for all $x \in X_1$, we have for $x \in X_1$ and $y \in X_2$ that

$$\begin{aligned} (x, y) \in I_{X_2} \circ R &\iff \exists y' \in X_2 \text{ s.t. } (x, y') \in R \text{ and } (y', y) \in I_{X_2} \\ &\implies y' \in R(x) \text{ and } y' \preccurlyeq_2 y \\ &\implies y \in R(x) \implies (x, y) \in R. \end{aligned}$$

On the other hand, taking $y' = y$, we obtain that $(x, y) \in R$ implies that $(x, y) \in I_{X_2} \circ R$. Hence $I_{X_2} \circ R = R$. Analogously, we have $S \circ I_{X_2} = S$.

(3) Let $w \in \mathcal{L}_3$. Let $x \in X_1$. Notice that

$$x \in (h_R \circ h_S)(w) \iff R(x) \subseteq h_S(w).$$

Assume that $x \in h_{S \circ R}(w)$. Thus $(S \circ R)(x) \subseteq w$. Let $y \in R(x)$ and $z \in S(y)$. Then $(x, z) \in S \circ R$, that is, $z \in (S \circ R)(x)$. Thus $z \in w$. We have proved that $y \in h_S(w)$ for all $y \in R(x)$, and thus $R(x) \subseteq h_S(w)$. Hence $x \in (h_R \circ h_S)(w)$.

Now let $x \in (h_R \circ h_S)(w)$. Then $R(x) \subseteq h_S(w)$. Let $z \in (S \circ R)(x)$. Thus there is $y \in X_2$ such that $(x, y) \in R$ and $(y, z) \in S$. Then $y \in h_S(w)$. It follows that $z \in S(y) \subseteq w$. Hence $z \in w$. We have proved that $(S \circ R)(x) \subseteq w$. Therefore, $x \in h_{S \circ R}(w)$.

(4) By (3), we have that $h_{S \circ R}(w) = h_R(h_S(w)) \in \mathcal{L}_1$, for all $w \in \mathcal{L}_3$. Hence $S \circ R$ satisfies condition (R1). It is straightforward to show that $x \preccurlyeq_1 x'$ implies that $(S \circ R)(x') \subseteq (S \circ R)(x)$. Hence (R2) holds. In order to prove (R3), let $x \in X_1$. Notice that $(S \circ R)(x) = S[R(x)]$. If $R(x) = \emptyset$, then $(S \circ R)(x) = \emptyset$. By condition (Ts4), we obtain that

$$\bigcap \{w \in \mathcal{L}_3 : (S \circ R)(x) \subseteq w\} = \bigcap \mathcal{L}_3 = \emptyset = (S \circ R)(x).$$

If $R(x) \neq \emptyset$, then there is $y \in R(x)$. By (Ts4), there is $v_0 \in \mathcal{L}_2$ such that $y \notin v_0$. Thus $R(x) \not\subseteq v_0$. This implies that $\nabla = \{v \in \mathcal{L}_2 : R(x) \subseteq v\}$ satisfies condition (P). Notice that, by (R3), $R(x) = \bigcap \nabla$. Hence, by Lemma 5.6, we obtain that

$$(S \circ R)(x) = S[R(x)] = \bigcap \{w \in \mathcal{L}_3 : S[R(x)] \subseteq w\}.$$

Hence, condition (R3) holds for $S \circ R$. Therefore, we conclude that $S \circ R$ is a semi-CFT-morphism. $\square$

Now we can define the following category:

$SCFT = \text{CFT-spaces} + \text{semi-CFT-morphisms}$

where the composition is the usual composition of relations (as defined above), and for every CFT-space X , I_X is the identity morphism.

Let $h : A \rightarrow B$ be a semi-homomorphism between Tarski algebras A and B . We define the relation $R_h \subseteq X_B \times X_A$ as follows: let $(y, x) \in X_B \times X_A$,

$$(y, x) \in R_h \iff h^{-1}[y] \subseteq x. \quad (5.2)$$

Proposition 5.8 1. If $h : A \rightarrow B$ is a semi-homomorphism between the Tarski algebras A and B , then $R_h \subseteq X_B \times X_A$ is a semi-CFT-morphism between the duals CFT-spaces $X(B)$ and $X(A)$.

2. If $R \subseteq X_1 \times X_2$ is a semi-CFT-morphism between the CFT-spaces $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$, then $h_R : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ is a semi-homomorphism between the dual Tarski algebras $T_{\mathcal{L}_2}(X_2)$ and $T_{\mathcal{L}_1}(X_1)$.
3. If $h : A \rightarrow B$ and $g : B \rightarrow C$ are semi-homomorphisms, then $R_{g \circ h} = R_h \circ R_g$.
4. If $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ are semi-CFT-morphisms, then $h_{S \circ R} = h_R \circ h_S$.
5. If A is a Tarski algebra and $Id_A : A \rightarrow A$ is the identity map, then $R_{Id_A} = I_{X_A}$.
6. If $\langle X, \tau, \mathcal{L} \rangle$ is a CFT-space, then $h_{I_X} = Id_{\mathcal{L}}$.

Proof (1) Let $h : A \rightarrow B$ be a semi-homomorphism. We need to prove that R_h satisfies conditions (R1)–(R3). Recall that $\mathcal{L}_A = \{\alpha_A(a) : a \in A\}$ and $\mathcal{L}_B = \{\alpha_B(b) : b \in B\}$. In order to prove (R1), we show that $h_{R_h}(\alpha_A(a)) = \alpha_B(h(a))$, for all $a \in A$. Let $a \in A$. Let $y \in h_{R_h}(\alpha_A(a))$. Thus $R_h(y) \subseteq \alpha_A(a)$. We have that $h^{-1}[y] = A$ or $h^{-1}[y] \in X_A$. On the one hand, if $h^{-1}[y] = A$, then $h(a) \in y$. Hence $y \in \alpha_B(h(a))$. On the other hand, if $x := h^{-1}[y] \in X_A$, then $x \in R_h(y)$. Thus $h^{-1}[y] = x \in \alpha_A(a)$. It follows that $h(a) \in y$, and thus $y \in \alpha_B(h(a))$. Hence $h_{R_h}(\alpha_A(a)) \subseteq \alpha_B(h(a))$. Let now $y \in \alpha_B(h(a))$. Then $a \in h^{-1}[y]$. Hence, for every $x \in R_h(y)$, we have that $x \in \alpha_A(a)$. Thus $R_h(y) \subseteq \alpha_A(a)$. Then $y \in h_{R_h}(\alpha_A(a))$. Hence $\alpha_B(h(a)) \subseteq h_{R_h}(\alpha_A(a))$. Therefore, (R1) holds. To prove (R2), let $y_1, y_2 \in X_B$ be such that $y_1 \subseteq y_2$. We need to show that $R_h(y_2) \subseteq R_h(y_1)$. Let $x \in R_h(y_2)$. Thus $h^{-1}[y_2] \subseteq x$. Then $h^{-1}[y_1] \subseteq h^{-1}[y_2] \subseteq x$. Hence $x \in R_h(y_1)$. Therefore, (R2) holds. To prove (R3), $y \in X_B$. Let $x \in \bigcap \{\alpha_A(a) : R_h(y) \subseteq \alpha_A(a)\}$. We need to show that $x \in R_h(y)$. Let $a \in h^{-1}[y]$. Then $R_h(y) \subseteq \alpha_A(a)$. Thus $x \in \alpha_A(a)$. Hence $a \in x$. We have proved that $h^{-1}[y] \subseteq x$. Hence $x \in R_h(y)$. Therefore, (R3) holds.

- (2) Let $R \subseteq X_1 \times X_2$ be a semi-CFT-morphism. Condition (R1) implies that $h_R : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ is well-defined. By Proposition 4.6, we know that the existing finite meets in $T_{\mathcal{L}_i}(X_i) = \langle \mathcal{L}_i, \rightarrow_{\mathcal{R}\mathcal{O}}, X_i \rangle$ are the intersections. Moreover, it is straightforward that the map h_R preserves finite intersections. Hence, by Proposition 5.3, h_R is a semi-homomorphism.

- (3) Let $h : A \rightarrow B$ and $g : B \rightarrow C$ be semi-homomorphisms. Thus $R_g \subseteq X_C \times X_B$ and $R_h \subseteq X_B \times X_A$. Let $z \in X_C$. If $g^{-1}[z] = B$, then $h^{-1}[g^{-1}[z]] = A$. Thus $R_g(z) = \emptyset$ and $R_{g \circ h}(z) = \emptyset$. Hence $R_{g \circ h}(z) = \emptyset = (R_h \circ R_g)(z)$. Now suppose that $g^{-1}[z] \neq B$, and thus by Proposition 5.2, we have $g^{-1}[z] \in X_B$. Let $y := g^{-1}[z]$. Let $x \in X_A$. Then,

$$\begin{aligned} x \in R_{g \circ h}(z) &\implies h^{-1}[g^{-1}[z]] \subseteq x \implies h^{-1}[y] \subseteq x \\ &\implies (z, y) \in R_g \text{ and } (y, x) \in R_h \implies x \in (R_h \circ R_g)(z). \end{aligned}$$

Hence $R_{g \circ h}(z) \subseteq (R_h \circ R_g)(z)$. The opposite inclusion is straightforward. Therefore, $R_{g \circ h} = R_h \circ R_g$.

- (4) It was proved in (3) of Proposition 5.7.
 (5) Let A be a Tarski algebra and $Id_A : A \rightarrow A$ the identity map. Let $x, x' \in X_A$. Then

$$(x', x) \in R_{Id_A} \iff Id_A^{-1}(x') \subseteq x \iff x' \subseteq x \iff x \in [x']_{\subseteq} = I_{X_A}.$$

Hence $R_{Id_A} = I_{X_A}$.

- (6) It follows from the proof of (1) of Proposition 5.7. □

The previous proposition justifies that the following are functors between the corresponding categories:

- $X : \mathcal{STA} \rightarrow \mathcal{SCFT}$ defined as follows:

- $X(A) = \langle X_A, \tau_A, \mathcal{L}_A \rangle$, for every $A \in \mathcal{STA}$.
- $X(h) = R_h \subseteq X_B \times X_A$, for every semi-homomorphism $h : A \rightarrow B$.

- $T : \mathcal{SCFT} \rightarrow \mathcal{STA}$ defined as follows:

- $T(\langle X, \tau, \mathcal{L} \rangle) = T_{\mathcal{L}}(X) = \langle \mathcal{L}, \rightarrow_{\mathcal{RCO}}, X \rangle$, for every $\langle X, \tau, \mathcal{L} \rangle \in \mathcal{SCFT}$.
- $T(R) = h_R : T_{\mathcal{L}_2}(X_2) \rightarrow T_{\mathcal{L}_1}(X_1)$, for every semi-CFT-morphism $R \subseteq X_1 \times X_2$.

Before proving that the functors X and T are dual equivalences, we need the following result.

Proposition 5.9 *Let $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ be CFT-spaces. Let $\epsilon : X_1 \rightarrow X_2$ be a homeomorphism such that $\mathcal{L}_2 = \{\epsilon[U] : U \in \mathcal{L}_1\}$. Then, the relation $R_\epsilon \subseteq X_1 \times X_2$ defined by*

$$(x, y) \in R_\epsilon \iff \epsilon(x) \preccurlyeq_2 y, \quad (5.3)$$

is an isomorphism of the category $\mathcal{SCFT}$.

Proof First, we need to show that R_ϵ is a semi-CFT-morphism.

- (R1) Let $V = \epsilon[U]$ with $U \in \mathcal{L}_1$. Let us show that $h_{R_\epsilon}(\epsilon[U]) = U$. Let $x \in h_{R_\epsilon}(\epsilon[U])$. Thus $R_\epsilon(x) \subseteq \epsilon[U]$. Since $\epsilon(x) \in X_2$ and $\epsilon(x) \preccurlyeq_2 \epsilon(x)$, it follows

that $\epsilon(x) \in R_\epsilon(x)$. Then $\epsilon(x) \in \epsilon[U]$. Given that ϵ is a homeomorphism, we have that $x \in U$. Hence $h_{R_\epsilon}(\epsilon[U]) \subseteq U$. Let now $x \in U$. Let $y \in R_\epsilon(x)$. Thus $\epsilon(x) \preceq_2 y$. Let $x' \in X_1$ be such that $\epsilon(x') = y$. Since $\epsilon(x) \preceq_2 \epsilon(x')$, it follows that $x \preceq_1 x'$. Then, since U is an $\preceq_1$ -upset, we have that $x' \in U$. Thus $y = \epsilon(x') \in \epsilon[U]$. We have proved that $R_\epsilon(x) \subseteq \epsilon[U]$. Hence $x \in h_{R_\epsilon}(\epsilon[U])$. That is, $U \subseteq h_{R_\epsilon}(\epsilon[U])$. Therefore, $h_{R_\epsilon}(V) = h_{R_\epsilon}(\epsilon[U]) = U \in \mathcal{L}_1$, for every $V \in \mathcal{L}_2$.

(R2) It is straightforward from the definition of R_ϵ .

(R3) Let $x \in X_1$. Let $y \in \bigcap \{\epsilon[U] : U \in \mathcal{L}_1 \text{ and } R_\epsilon(x) \subseteq \epsilon[U]\}$. We need to show that $\epsilon(x) \preceq_2 y$. Let $U \in \mathcal{L}_1$ be such that $\epsilon(x) \in \epsilon[U]$. Since $\epsilon[U] \in \mathcal{L}_2$ is an $\preceq_2$ -upset, it follows that $R_\epsilon(x) \subseteq \epsilon[U]$. Then $y \in \epsilon[U]$. Hence $\epsilon(x) \preceq_2 y$, that is, $y \in R_\epsilon(x)$. Therefore, $R_\epsilon(x) = \bigcap \{\epsilon[U] : U \in \mathcal{L}_1 \text{ and } R_\epsilon(x) \subseteq \epsilon[U]\}$.

Now we show that R_ϵ is an isomorphism in $SCFT$. Since $\epsilon^{-1} : X_2 \rightarrow X_1$ is also a homeomorphism such that $\mathcal{L}_1 = \{\epsilon^{-1}[V] : V \in \mathcal{L}_2\}$, we have that $R_{\epsilon^{-1}} \subseteq X_2 \times X_1$, defined by $(y, x) \in R_{\epsilon^{-1}} \iff \epsilon^{-1}(y) \preceq_1 x$, is a semi-CFT-morphism. We show that $R_{\epsilon^{-1}} \circ R_\epsilon = I_{X_1}$ and $R_\epsilon \circ R_{\epsilon^{-1}} = I_{X_2}$. Let $x, x' \in X_1$. Notice that

$$(x, x') \in R_{\epsilon^{-1}} \circ R_\epsilon \iff \exists y \in X_2 (\epsilon(x) \preceq_2 y \quad \epsilon^{-1}(y) \preceq_1 x').$$

Let $(x, x') \in R_{\epsilon^{-1}} \circ R_\epsilon$. Then, there is $y \in X_2$ such that $\epsilon(x) \preceq_2 y$ and $\epsilon^{-1}(y) \preceq_1 x'$. Since ϵ is onto, there is $x'' \in X_1$ such that $y = \epsilon(x'')$. Thus $\epsilon(x) \preceq_2 y = \epsilon(x'')$. Then $x \preceq_1 x'' = \epsilon^{-1}(y) \preceq_1 x'$. Hence $(x, x') \in I_{X_1}$. We have proved that $R_{\epsilon^{-1}} \circ R_\epsilon \subseteq I_{X_1}$. Let now $(x, x') \in I_{X_1}$. Thus $x \preceq_1 x'$. Taking $y = \epsilon(x) \in X_2$, we obtain that $\epsilon(x) \preceq_2 y$ and $\epsilon^{-1}(y) \preceq_1 x'$. Hence $(x, x') \in R_{\epsilon^{-1}} \circ R_\epsilon$. By an analogous argument, we have that $R_\epsilon \circ R_{\epsilon^{-1}} = I_{X_2}$. Therefore, R_ϵ is an isomorphism of the category $SCFT$. $\square$

Now we are ready to establish one of the main results of this section.

Theorem 5.10 *The categories STA and $SCFT$ are dually equivalent under the functors $X : STA \rightarrow SCFT$ and $T : SCFT \rightarrow STA$.*

Proof Let us define $\eta : Id_{STA} \rightarrow T \circ X$ and $\mu : Id_{SCFT} \rightarrow X \circ T$ as follows:

- For every Tarski algebra A , $\eta(A) = \alpha_A : A \rightarrow T_{\mathcal{L}_A}(X(A))$.
- For every CFT-space $\langle X, \tau, \mathcal{L} \rangle$, $\mu(X) = R_\epsilon \subseteq X \times X(T_{\mathcal{L}}(X))$.

By Corollary 4.10 and Proposition 5.9, we obtain respectively that α_A and R_ϵ are isomorphisms in the categories STA and $SCFT$. It only remains to show that η and μ are natural transformations. Let $h : A \rightarrow B$ be a morphism of the category STA and let $R \subseteq X_1 \times X_2$ be a morphism in the category $SCFT$. We need to show that the following diagrams

$$\begin{array}{ccccc} A & \xrightarrow{h} & B & & X_1 & \xrightarrow{R} & X_2 \\ \alpha_A \downarrow & & \downarrow \alpha_B & & \downarrow R_{\epsilon_1} & & \downarrow R_{\epsilon_2} \\ T_{\mathcal{L}_A}(X(A)) & \xrightarrow{h_{R_h}} & T_{\mathcal{L}_B}(X(B)) & & X(T_{\mathcal{L}_1}(X_1)) & \xrightarrow{R_{h_R}} & X(T_{\mathcal{L}_2}(X_2)) \end{array}$$

commute. By the proof of (1) in Proposition 5.8, it follows that $h_{R_h} \circ \alpha_A = \alpha_B \circ h$. Hence the first diagram commutes. In order to prove that the second diagram commutes, let $x \in X_1$. We need to show that $(R_{h_R} \circ R_{\epsilon_1})(x) = (R_{\epsilon_2} \circ R)(x)$. Let $\bar{z} \in X(T_{\mathcal{L}_2}(X_2))$. Recall that $\bar{z}$ is a proper implicative filter of $T_{\mathcal{L}_2}(X_2)$. Notice that

$$\bar{z} \in (R_{h_R} \circ R_{\epsilon_1})(x) \iff \exists \bar{y} \in X(T_{\mathcal{L}_1}(X_1)) \text{ s.t. } \epsilon_1(x) \subseteq \bar{y} \text{ and } h_R^{-1}[\bar{y}] \subseteq \bar{z}$$

and

$$\bar{z} \in (R_{\epsilon_2} \circ R)(x) \iff \exists y \in X_2 \text{ s.t. } y \in R(x) \text{ and } \epsilon_2(y) \subseteq \bar{z}.$$

Let $\bar{z} \in (R_{h_R} \circ R_{\epsilon_1})(x)$. Thus, there is $\bar{y} \in X(T_{\mathcal{L}_1}(X_1))$ such that $\epsilon_1(x) \subseteq \bar{y}$ and $h_R^{-1}[\bar{y}] \subseteq \bar{z}$. Since ϵ_2 is a homeomorphism, there is $y \in X_2$ such that $\epsilon_2(y) = \bar{z}$. We show that $y \in R(x)$. Let $V \in \mathcal{L}_2$ be such that $R(x) \subseteq V$. Suppose that $y \notin V$. Thus $V \notin \epsilon_2(y)$. Then $V \notin \bar{z}$. This implies that $V \notin h_R^{-1}[\bar{y}]$, and thus $h_R(V) \notin \bar{y}$. Then $x \notin h_R(V)$. Thus $R(x) \not\subseteq V$, a contradiction. Then $y \in V$. Hence, by (R3), $y \in R(x)$. Therefore $\bar{z} \in (R_{\epsilon_2} \circ R)(x)$.

Let now $\bar{z} \in (R_{\epsilon_2} \circ R)(x)$. Thus there is $y \in X_2$ such that $y \in R(x)$ and $\epsilon_2(y) \subseteq \bar{z}$. Let $\bar{y} = \epsilon_1(x) \in X(T_{\mathcal{L}_1}(X_1))$. We show that $h_R^{-1}[\bar{y}] \subseteq \bar{z}$. Let $V \in h_R^{-1}[\bar{y}]$. Thus $h_R(V) \in \bar{y} = \epsilon_1(x)$. Then $R(x) \subseteq V$. It follows that $y \in V$, and thus $V \in \epsilon_2(y)$. Then $V \in \bar{z}$. Hence $h_R^{-1}[\bar{y}] \subseteq \bar{z}$. Therefore, $\bar{z} \in (R_{h_R} \circ R_{\epsilon_1})(x)$.

Therefore, we have that the functors $X : \mathcal{STA} \rightarrow \mathcal{SCFT}$ and $T : \mathcal{SCFT} \rightarrow \mathcal{STA}$ are dual equivalences. $\square$

Our next aim is to establish a full dual equivalence between the category $\mathcal{HTA}$ (Tarski algebras + homomorphisms) and a category of CFT-spaces. To this end, we need to define the dual morphisms of homomorphisms.

Definition 5.11 Let $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ be CFT-spaces. A relation $R \subseteq X_1 \times X_2$ is called a *CFT-morphism* if it is a semi-CFT-morphisms and satisfies the following:

$$(R4) \quad \text{If } y \in R(x), \text{ then there is } x' \in X_1 \text{ such that } x \preccurlyeq_1 x' \text{ and } R(x') = [y]_{\preccurlyeq_2}.$$

In the following proposition, we give all the needed results to establish the announced duality.

Proposition 5.12 1. For every CFT-space $\langle X, \tau, \mathcal{L} \rangle$, the relation $I_X \subseteq X \times X$ is a CFT-morphism.

2. If $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ are CFT-morphisms between CFT-spaces, then $S \circ R \subseteq X_1 \times X_3$ is also a CFT-morphism.

3. If $h : A \rightarrow B$ is a homomorphism, then $R_h \subseteq X_B \times X_A$ is a CFT-morphism between the duals CFT-spaces $X(B)$ and $X(A)$.

4. If $R \subseteq X_1 \times X_2$ is a CFT-morphism, then $h_R : T_{\mathcal{L}_2}(X_2) \rightarrow T_{\mathcal{L}_1}(X_1)$ is a homomorphism between the duals Tarski algebras.

5. If $\epsilon : X_1 \rightarrow X_2$ is a homeomorphism from the CFT-space $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ to $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ such that $\mathcal{L}_2 = \{\epsilon[U] : U \in \mathcal{L}_1\}$, then the relation $R_\epsilon \subseteq X_1 \times X_2$ as defined in (5.3) is a CFT-morphism.

- Proof** (1) From Proposition 5.7, we know that I_X is a semi-CFT-morphism. Condition (R4) follows from the very definition of I_X .
- (2) From Proposition 5.7, we have that $S \circ R \subseteq X_1 \times X_3$ is a semi-CFT-morphism. Now we prove condition (R4). Let $x \in X_1$ and $z \in X_3$ be such that $z \in (S \circ R)(x)$. Thus there is $y \in X_2$ such that $y \in R(x)$ and $z \in S(y)$. By (R4), there is $x' \in X_1$ such that $x \preccurlyeq_1 x'$ and $R(x') = [y]_{\preccurlyeq_2}$. There is also $y' \in X_2$ such that $y \preccurlyeq_2 y'$ and $S(y') = [z]_{\preccurlyeq_3}$. We have that $y' \in R(x')$. Then, by (R4), there is $x'' \in X_1$ such that $x' \preccurlyeq_1 x''$ and $R(x'') = [y']_{\preccurlyeq_2}$. Hence $x \preccurlyeq_1 x''$. Now we show that $(S \circ R)(x'') = [z]_{\preccurlyeq_3}$. Let $z' \in (S \circ R)(x'')$. Thus there is $y'' \in X_2$ such that $y'' \in R(x'')$ and $z' \in S(y'')$. Then $y' \preccurlyeq_2 y''$. Thus by (R2), we have $S(y'') \subseteq S(y')$. It follows that $z' \in S(y') = [z]_{\preccurlyeq_3}$. Hence $(S \circ R)(x'') \subseteq [z]_{\preccurlyeq_3}$. Let now $z' \in [z]_{\preccurlyeq_3} = S(y')$. Then $z' \in S(y')$ and $y' \in R(x'')$. Thus $z' \in (S \circ R)(x'')$. Hence $[z]_{\preccurlyeq_3} \subseteq (S \circ R)(x'')$. Therefore $S \circ R$ satisfies condition (R4).
- (3) By Proposition 5.8, we have that $R_h \subseteq X_B \times X_A$ is a semi-CFT-morphism. Let $x \in X_B$ and $y \in X_A$ be such that $y \in R_h(x)$. By definition of R_h , we have that $h^{-1}[x] \subseteq y$. Then, by Proposition 5.4, there is $x' \in X_B$ such that $x \subseteq x'$ and $h^{-1}[x'] = y$. It is straightforward that $R_h(x') = [y]_{\subseteq}$. Hence R_h satisfies condition (R4).
- (4) By Proposition 5.8, we have that $h_R : T_{\mathcal{L}_2}(X_2) \rightarrow T_{\mathcal{L}_1}(X_1)$ is a semi-homomorphism. In order to prove that h_R is a homomorphism, we show that the map $(h_R)_+ : \text{Fi}(T_{\mathcal{L}_1}(X_1)) \rightarrow \text{Fi}(T_{\mathcal{L}_2}(X_2))$ is a p-morphism (see Proposition 5.4). Let $F \in \text{Fi}(T_{\mathcal{L}_1}(X_1))$ and $G \in \text{Fi}(T_{\mathcal{L}_2}(X_2))$. Without loss of generality we can assume that F and G are proper. Then, by Theorem 4.11, $F = \epsilon_1(x)$ and $G = \epsilon_2(y)$ for some $x \in X_1$ and $y \in X_2$. Suppose that $(h_R)_+(\epsilon_1(x)) \subseteq \epsilon_2(y)$. That is, $h_R^{-1}[\epsilon_1(x)] \subseteq \epsilon_2(y)$. By condition (R3), it follows that $y \in R(x)$. Thus, by (R4), there is $x' \in X_1$ such that $x \preccurlyeq_1 x'$ and $R(x') = [y]_{\preccurlyeq_2}$. Then $\epsilon_1(x) \subseteq \epsilon_1(x')$ and $h_R^{-1}[\epsilon_1(x)] = \epsilon_2(y)$. We proved that there exists $\epsilon_1(x') \in \text{Fi}(T_{\mathcal{L}_1}(X_1))$ such that $\epsilon_1(x) \subseteq \epsilon_1(x')$ and $(h_R)_+(\epsilon_1(x')) = \epsilon_2(y)$. Hence $(h_R)_+$ is a p-morphism. Therefore, h_R is a homomorphism.
- (5) By Proposition 5.9, we know that $R_\epsilon \subseteq X_1 \times X_2$ is a semi-CFT-morphism. Let now $y \in R_\epsilon(x)$. Then $\epsilon(x) \preccurlyeq_2 y$. Since ϵ is onto, it follows that there is $x' \in X_1$ such that $\epsilon(x') = y$. Hence, it is straightforward that $x \preccurlyeq_1 x'$ and $R_\epsilon(x') = [y]_{\preccurlyeq_2}$. $\square$

From (1) and (2) of the previous proposition, we can define the following category:

$\mathcal{CFT} = \text{CFT-spaces} + \text{CFT-morphisms}$

Now we can establish the second main aim of this section.

Theorem 5.13 *The categories $\mathcal{HTA}$ and $\mathcal{CFT}$ are dually equivalent under the functors $X : \mathcal{HTA} \rightarrow \mathcal{CFT}$ and $T : \mathcal{CFT} \rightarrow \mathcal{HTA}$.*

Proof It follows by (3)–(5) of Proposition 5.12 and by Theorem 5.10. We left the details to the reader. $\square$

6 Generalized Boolean algebra, Boolean algebra and UV-space

In this section, we consider the concept of generalized Boolean algebra, and we relate it with Tarski algebras by means of CFT-spaces. We show that the finite CFT-spaces are UV-spaces (the dual spaces under the choice-free duality in [5]).

Definition 6.1 A distributive lattice L is said to be *relatively complemented* if each element $x \in L$ is complemented in every interval $[a, b]$ of L containing x .

Definition 6.2 A *generalized Boolean algebra* is a relatively complemented distributive lattice with upper bound 1.

As we can notice, generalized Boolean algebras can be presented in several ways. We leave the proofs of the following three propositions to the reader. They are straightforward.

Proposition 6.3 A distributive lattice L with upper bound 1 is relatively complemented (generalized Boolean algebra) if and only if for all $a \in L$, $[a]$ is a Boolean algebra.

Proposition 6.4 1. If $\langle A, \rightarrow, 1 \rangle$ is a Tarski algebra such that there exist the meets of all elements $a, b \in A$ (regarding the natural order on A), then $A_* = \langle A, \wedge, \vee, 1 \rangle$ is a generalized Boolean algebra.
 2. If $\langle G, \wedge, \vee, 1 \rangle$ is a generalized Boolean algebra, then $G^* = \langle G, \rightarrow, 1 \rangle$ is a Tarski algebra such that there exist the meets of all elements $a, b \in G$, where $a \rightarrow b$ is the complement of a in $[a \wedge b, 1]$, for all $a, b \in G$.
 3. Let A be a Tarski algebra such that there exists the meet of all elements $a, b \in A$, and let G be a generalized Boolean algebra. Then $(A_*)^* = A$ and $(G^*)_* = G$.

Proposition 6.5 Let $\langle G, \wedge, \vee, \rightarrow, 1 \rangle$ be an algebra of type $(2, 2, 2, 0)$. Then, $\langle G, \wedge, \vee, 1 \rangle$ is a generalized Boolean algebra where $a \rightarrow b$ is the complement of a in $[a \wedge b, 1]$ if and only if $\langle G, \wedge, \vee, \rightarrow, 1 \rangle$ satisfies the following:

1. $\langle G, \rightarrow, 1 \rangle$ is a Tarski algebra;
2. $x \vee y = (x \rightarrow y) \rightarrow y$;
3. $x \wedge y = (x \rightarrow (y \rightarrow (x \wedge y))) \rightarrow (x \wedge y)$.

Throughout this article generalized Boolean algebras will be understood as algebras on the algebraic language $\{\wedge, \vee, \rightarrow, 1\}$ of type $(2, 2, 2, 0)$ satisfying the conditions of Proposition 6.5. Notice that isn't necessary to include the join operation on the signature of a generalized Boolean algebra because it can be defined by the implication operation. We include it only to point out that $\langle G, \wedge, \vee \rangle$ is a distributive lattice.

Notice that every $\{\wedge, \vee, \rightarrow, 1\}$ -subreduct of a Boolean algebra $\langle B, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a generalized Boolean algebra. Let B be a Boolean algebra and $L \subseteq B$. Let

$$G_L = \{a_1 \wedge \cdots \wedge a_n : a_1, \dots, a_n \in L\}.$$

Proposition 6.6 Let $\langle B, \wedge, \vee, \rightarrow, 0, 1 \rangle$ be a Boolean algebra. Let L be a $\{\rightarrow\}$ -subreduct of B . Then, $\langle G_L, \wedge, \vee, \rightarrow, 1 \rangle$ is a generalized Boolean algebra.

Proof It is clear that G_L is closed under $\wedge$. Because the join operation $\vee$ can be defined by the implication operation $\rightarrow$ in B , we only need to prove that G_L is closed under $\rightarrow$. Let $u, v \in G_L$. Thus $u = a_1 \wedge \cdots \wedge a_n$ and $v = b_1 \wedge \cdots \wedge b_m$ with $a_i, b_j \in L$. Then

$$u \rightarrow v = u \rightarrow (b_1 \wedge \cdots \wedge b_m) = (u \rightarrow b_1) \wedge \cdots \wedge (u \rightarrow b_m).$$

For every $j \in \{1, \dots, m\}$, we have

$$u \rightarrow b_j = (a_1 \wedge \cdots \wedge a_n) \rightarrow b_j = (a_1 \rightarrow b_j) \vee \cdots \vee (a_n \rightarrow b_j)$$

Since L is closed under $\rightarrow$, and thus under $\vee$, we obtain that $c_j := u \rightarrow b_j \in L$, for all j . Hence, $u \rightarrow v = c_1 \wedge \cdots \wedge c_k \in G_{\mathcal{L}}$. $\square$

Proposition 6.7 *Let $\langle B, \wedge, \vee, \rightarrow, 0, 1 \rangle$ be a Boolean algebra. Let $L \subseteq B$ be a $\{\rightarrow\}$ -subreduct of B . Then, for every generalized Boolean algebra G and every $\{\rightarrow\}$ -homomorphism $h : L \rightarrow G$, there exists a unique $\{\wedge, \vee, \rightarrow\}$ -homomorphism $\hat{h} : G_L \rightarrow G$ such that $\hat{h} \upharpoonright_L = h$. Moreover, if h is an embedding, then $\hat{h}$ is an embedding.*

Proof We define $\hat{h} : G_L \rightarrow G$ as follows: for every $u = a_1 \wedge \cdots \wedge a_n \in G_L$, let $\hat{h}(u) = h(a_1) \wedge \cdots \wedge h(a_n)$. It is straightforward to show that $\hat{h}$ is the desired $\{\wedge, \vee, \rightarrow\}$ -homomorphism. $\square$

Definition 6.8 Let A be a Tarski algebra. A generalized Boolean algebra G is said to be a *free generalized Boolean extension* of A if there is an $\{\rightarrow\}$ -embedding $e : A \rightarrow G$ such that for every generalized Boolean algebra H and every $\{\rightarrow\}$ -embedding $h : A \rightarrow H$, there exists a unique generalized Boolean embedding $\hat{h} : G \rightarrow H$ such that $\hat{h} \circ e = h$.

By a standard categorical argument, if a Tarski algebra has a free generalized Boolean algebra extension, it is unique up to isomorphism. By [2, Theorem 17], we know that each Tarski algebra A can be embedded into a Boolean algebra B . Then, by taking into account Proposition 6.7, we obtain that $G_A = \{a_1 \wedge \cdots \wedge a_n : a_1, \dots, a_n \in A\}$ is, up to isomorphism, the free generalized Boolean algebra extension of A . Now we provide an alternative way to obtain the free generalized Boolean algebra extension of a Tarski algebra.

Let $\langle X, \tau, \mathcal{L} \rangle$ be a CFT-space. Recall that $T_{\mathcal{L}}(X) = \langle \mathcal{L}, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a subreduct of the Boolean algebra $\langle \mathcal{RO}(X, \tau_{\leq}), \cap, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, \emptyset, X \rangle$. Let

$$G_{\mathcal{L}} := \{u_1 \cap \cdots \cap u_n : u_1, \dots, u_n \in \mathcal{L}\}.$$

Notice that $G_{\mathcal{L}}$ is a basis for τ .

Proposition 6.9 *Let $\langle X, \tau, \mathcal{L} \rangle$ be a CFT-space.*

1. $G_{\mathcal{L}} \subseteq \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\leq})$.
2. $\langle G_{\mathcal{L}}, \cap, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a generalized Boolean algebra.
3. $G_{\mathcal{L}} \cup \{\emptyset\} = \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\leq})$.
4. $G_{\mathcal{L}}$ is the free generalized Boolean algebra extension of the Tarski algebra $T_{\mathcal{L}}(X)$.

Proof (1) It is clear that $G_{\mathcal{L}} \subseteq \text{O}(X) \cap \mathcal{RO}(X, \tau_{\leq})$. It only remains to show that every element of $G_{\mathcal{L}}$ is a compact subset of X . In order to prove this, we assume,

by Theorem 4.11, that $\langle X, \tau, \mathcal{L} \rangle$ is the dual CFT-space of a Tarski algebra A . That is, we assume that $X = X_A$ and $\mathcal{L} = \mathcal{L}_A = \{\alpha(a) : a \in A\}$. Let $U \in G_{\mathcal{L}}$. Thus $U = \alpha(a_1) \cap \cdots \cap \alpha(a_n)$ for some $a_1, \dots, a_n \in A$. Suppose that

$$\alpha(a_1) \cap \cdots \cap \alpha(a_n) \subseteq \bigcup_{i \in I} \left(\bigcap \mathcal{A}_i \right)$$

with each $\mathcal{A}_i \subseteq \mathcal{L}$ finite. Let $F := \text{Fig}(a_1, \dots, a_n)$. If $F = A$, then $U = \alpha(a_1) \cap \cdots \cap \alpha(a_n) = \emptyset$. Thus $U = \emptyset$ is compact. Suppose that $F \neq A$. Then $F \in \alpha(a_1) \cap \cdots \cap \alpha(a_n)$. Let $i \in I$ be such that $F \in \bigcap \mathcal{A}_i$. Hence $U = \alpha(a_1) \cap \cdots \cap \alpha(a_n) \subseteq \bigcap \mathcal{A}_i$. Therefore, U is a compact subset of X .

- (2) Since $\mathcal{L}$ is a $\{\rightarrow_{\mathcal{R}\mathcal{O}} \mathcal{O}\}$ -subreduct of the Boolean algebra $\langle \mathcal{R}\mathcal{O}(X, \tau_{\leq}), \cap, \vee_{\mathcal{R}\mathcal{O}}, \rightarrow_{\mathcal{R}\mathcal{O}}, \emptyset, X \rangle$, it follows by Proposition 6.6 that $\langle G_{\mathcal{L}}, \cap, \vee_{\mathcal{R}\mathcal{O}}, \rightarrow_{\mathcal{R}\mathcal{O}}, X \rangle$ is a generalized Boolean algebra.
- (3) Let $U \in \text{KO}(X) \cap \mathcal{R}\mathcal{O}(X, \tau_{\leq}) - \{\emptyset\}$. Since $G_{\mathcal{L}}$ is a basis for X and $U \neq \emptyset$ is a compact open subset, there are $\mathcal{A}_1, \dots, \mathcal{A}_n$ finite subsets of $\mathcal{L}$ such that $U = \bigcup_{1 \leq i \leq n} \left(\bigcap \mathcal{A}_i \right)$. Because $U \in \mathcal{R}\mathcal{O}(X, \tau_{\leq})$ and each $\bigcap \mathcal{A}_i \in G_{\mathcal{L}} \subseteq \mathcal{R}\mathcal{O}(X, \tau_{\leq})$, we have that $U = \bigvee_{\mathcal{R}\mathcal{O}} \{\bigcap \mathcal{A}_i : 1 \leq i \leq n\}$. Hence, by (2), it follows that $U \in G_{\mathcal{L}}$.

- (4) It is consequence of Proposition 6.7. $\square$

Let A be a Tarski algebra and $\langle X_A, \tau_A, \mathcal{L}_A \rangle$ its dual CFT-space. Notice that

$$G_{\mathcal{L}_A} = \{\alpha(a_1) \cap \cdots \cap \alpha(a_n) : a_1, \dots, a_n \in A\}.$$

Then, by Corollary 4.10 and Proposition 6.9, we obtain that $\langle G_{\mathcal{L}_A}, \cap, \vee_{\mathcal{R}\mathcal{O}}, \rightarrow_{\mathcal{R}\mathcal{O}}, X_A \rangle$ is the free generalized Boolean algebra extension of A . Moreover, if A is finite, then $\emptyset = \bigcap \alpha[A] \in G_{\mathcal{L}_A}$. Hence $\langle G_{\mathcal{L}_A}, \cap, \vee_{\mathcal{R}\mathcal{O}}, \rightarrow_{\mathcal{R}\mathcal{O}}, \emptyset, X_A \rangle$ is a Boolean subalgebra of $\mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$. By Proposition 6.9, it follows that $G_{\mathcal{L}_A} = \text{KO}(X_A) \cap \mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$, and thus $\text{KO}(X_A) \cap \mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$ is a Boolean subalgebra of $\mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$, where $\mathcal{L}_A$ is finitely meet-dense on $\text{KO}(X_A) \cap \mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$. We summarize this in the following.

Proposition 6.10 *If A is a finite Tarski algebra, then $\text{KO}(X_A) \cap \mathcal{R}\mathcal{O}(X_A, \tau_{\leq})$ is the free generalized Boolean algebra extension of A , and it is a Boolean algebra.*

Now we present the definition of the dual topological spaces (called *UV-spaces*) of Boolean algebras used in the choice-free duality given in [5], and we show the first connection between our CFT-spaces and UV-spaces. In fact, we prove that every finite CFT-space is a UV-space.

Definition 6.11 A T_0 -space $\langle X, \tau \rangle$ is said to be a *UV-space* if:

- (UV1) $\text{KO}(X) \cap \mathcal{R}\mathcal{O}(X, \tau_{\leq})$ is closed under $\cap$ and $\neg_{\mathcal{R}\mathcal{O}} = \text{int}_{\tau_{\leq}}(X \setminus \cdot)$;
 (UV2) If $x \not\leq y$, then there is a $U \in \text{KO}(X) \cap \mathcal{R}\mathcal{O}(X, \tau_{\leq})$ such that $x \in U$ and $y \notin U$;
 (UV3) Every proper filter in $\text{KO}(X) \cap \mathcal{R}\mathcal{O}(X, \tau_{\leq})$ is of the form $\{U \in \text{KO}(X) \cap \mathcal{R}\mathcal{O}(X, \tau_{\leq}) : x \in U\}$.

Theorem 6.12 *Let A be a finite Tarski algebra and $\langle X_A, \tau_A, \mathcal{L}_A \rangle$ its dual CFT-space. Then $\langle X_A, \tau_A \rangle$ is a UV-space.*

Proof Recall that the specialization order $\preceq$ of $\langle X_A, \tau_A \rangle$ is $\subseteq$.

(UV1) By Proposition 6.10 (or by the paragraph above it), we have that $\text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$ is a Boolean subalgebra of $\mathcal{RO}(X_A, \tau_A)$. Hence $\text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$ is closed under $\cap$ and $\neg_{\mathcal{RO}} = \text{int}_{\preceq}(X_A \setminus \cdot)$.

(UV2) It holds because $\mathcal{L}_A$ is a subbasis of X_A and $\mathcal{L}_A \subseteq \text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$.

(UV3) Let $\mathcal{F}$ be a proper filter of $\text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$. Since $\alpha : A \rightarrow \text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$ is an $\{\rightarrow\}$ -embedding, it follows that $x := \alpha^{-1}[\mathcal{F}]$ is an implicative filter of A . If $\alpha^{-1}[\mathcal{F}] = A$, then $\alpha[A] \subseteq \mathcal{F}$. Since A is finite, it follows that $\emptyset = \bigcap \alpha[A] \in \mathcal{F}$, which is a contradiction. Hence $x = \alpha^{-1}[\mathcal{F}]$ is proper. Let us prove that $\mathcal{F} = \{U \in \text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq}) : x \in U\}$. Let $U \in \text{KO}(X_A) \cap \mathcal{RO}(X_A, \tau_{\preceq})$. We have that $U = \alpha(a_1) \cap \dots \cap \alpha(a_n)$ for some $a_1, \dots, a_n \in A$. Then,

$$\begin{aligned} U \in \mathcal{F} &\iff \alpha(a_1) \cap \dots \cap \alpha(a_n) \in \mathcal{F} \\ &\iff \alpha(a_1), \dots, \alpha(a_n) \in \mathcal{F} \\ &\iff a_1, \dots, a_n \in \alpha^{-1}[\mathcal{F}] = x \\ &\iff x \in \alpha(a_1) \cap \dots \cap \alpha(a_n) \\ &\iff x \in U. \end{aligned}$$

Therefore, (UV3) holds.

Corollary 6.13 *If $\langle X, \tau, \mathcal{L} \rangle$ is a finite CFT-space, then $\langle X, \tau \rangle$ is a UV-space.*

7 Choice-free duality for generalized Boolean algebras

The aim of this section is to apply the choice-free duality for Tarski algebras given in Sect. 5 to obtain a choice-free duality for the algebraic category of generalized Boolean algebras and homomorphisms. We start presenting the dual spaces.

Definition 7.1 A structure $\langle X, \tau, \mathcal{L} \rangle$ is called a *GUV-space*² if satisfies the following:

- (G1) $\langle X, \tau \rangle$ is a T_0 -space where $\mathcal{L}$ is a basis for τ .
- (G2) $\mathcal{L} \cup \{\emptyset\} = \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\preceq})$ and $X \in \mathcal{L}$.
- (G3) $\mathcal{L}$ is closed under $\rightarrow_{\mathcal{RO}}$ and $\cap$.
- (G4) For all $x \in X$, there is $U \in \mathcal{L}$ such that $x \notin U$.
- (G5) For all $\nabla \cup \Delta \subseteq \mathcal{L}$, if ∇ satisfies condition (P) and $\bigcap \nabla \subseteq \bigcup \Delta$, then there are $U_1, \dots, U_n \in \nabla$ and $V \in \Delta$ such that $U_1 \cap \dots \cap U_n \subseteq V$.

Let $\langle X, \tau, \mathcal{L} \rangle$ be a GUV-space. By (G1)–(G3), it follows that

$$\text{G}_{\mathcal{L}}(X) = \langle \mathcal{L}, \cap, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, X \rangle$$

² GUV-space means generalized UV-space.

is a subreduct of the Boolean algebra $\langle \mathcal{RO}(X, \tau_{\preceq}), \cap, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, \emptyset, X \rangle$. Hence $G_{\mathcal{L}}(X) = \langle \mathcal{L}, \cap, \vee_{\mathcal{RO}}, \rightarrow_{\mathcal{RO}}, X \rangle$ is a generalized Boolean algebra.

Let $\langle G, \wedge, \vee, \rightarrow, 1 \rangle$ be a generalized Boolean algebra. Recall that X_G is the collection of all proper implicative filters of G , and $\alpha : G \rightarrow \mathcal{P}(X_G)$ is $\alpha(a) = \{x \in X_G : a \in x\}$, for all $a \in G$. Let

$$X(G) = \langle X_G, \tau_G, \mathcal{L}_G \rangle$$

be the dual CFT-space of the Tarski algebra $\langle G, \rightarrow, 1 \rangle$. Recall that $\mathcal{L}_G = \{\alpha(a) : a \in G\}$ is a subbasis for X_G . Moreover, recall that the specialization order of the space X_G is $\preceq_{\tau_G} = \subseteq$. We drop the subscript of $\preceq_{\tau_G}$ when confusion is unlikely.

Proposition 7.2 *For all $a, b \in G$, $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b)$. Moreover, $\mathcal{L}_G$ is a basis for X_G .*

Proof By Proposition 3.3, it follows that the implicative filters of G are exactly the lattice filters. Hence, it is straightforward that $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b)$, for all $a, b \in G$. Moreover, since $\mathcal{L}_G$ is a subbasis for X_G and is closed under finite intersection, it follows that $\mathcal{L}_G$ is a basis. $\square$

Proposition 7.3 *Let G be a generalized Boolean algebra. Then, $\mathcal{L}_G \cup \{\emptyset\} = KO(X) \cap \mathcal{RO}(X, \tau_{\preceq})$.*

Proof By Propositions 4.1 and 4.8, we have that $\mathcal{L}_G \cup \{\emptyset\} \subseteq KO(X) \cap \mathcal{RO}(X, \tau_{\preceq})$. Let $U \in KO(X_G) \cap \mathcal{RO}(X_G, \tau_{\preceq})$ be non-empty. Since $\mathcal{L}_G$ is a basis for τ_G , it follows that $U = \bigcup_{1 \leq i \leq n} \alpha(a_i)$ for some $a_1, \dots, a_n \in G$. Since $U \in \mathcal{RO}(X_G, \tau_{\preceq})$ and every $\alpha(a_i) \in \mathcal{RO}(X_G, \tau_{\preceq})$, we obtain that $U = \bigvee_{RO} \{\alpha(a_i) : 1 \leq i \leq n\}$. Hence, by Proposition ??, $U = \alpha(a_1 \vee \dots \vee a_n) \in \mathcal{L}_G$. $\square$

Theorem 7.4 (Representation) *Let G be a generalized Boolean algebra. Then, $X(G) = \langle X_G, \tau_G, \mathcal{L}_G \rangle$ is a GUV-space and $\alpha : G \rightarrow G_{\mathcal{L}}(X_G)$ is an isomorphism of generalized Boolean algebras.*

Proof It follows by Theorem 4.9 and Propositions 7.2 and 7.3. $\square$

Let us consider the following two categories:

$\mathcal{GBA}$ = generalized Boolean algebras + homomorphisms
and

$\mathcal{GUV}$ = GUV-spaces + CFT-morphisms.

We define the following:

- $X : \mathcal{GBA} \rightarrow \mathcal{GUV}$ defined as follows:
 - $X(G) = \langle X_G, \tau_G, \mathcal{L}_G \rangle$, for every $G \in \mathcal{GBA}$.
 - $X(h) = R_h \subseteq X_{G_2} \times X_{G_1}$ as defined in (5.2), for every homomorphism $h : G_1 \rightarrow G_2$.

• $G : \mathcal{GUV} \rightarrow \mathcal{GBA}$ defined as follows:

- $G(\langle X, \tau, \mathcal{L} \rangle) = G_{\mathcal{L}}(X) = \langle \mathcal{L}, \cap, \vee_{\mathcal{RO}}, \neg_{\mathcal{RO}}, X \rangle$, for every $\langle X, \tau, \mathcal{L} \rangle \in \mathcal{GUV}$.
- $G(R) = h_R : G_{\mathcal{L}_2}(X_2) \rightarrow G_{\mathcal{L}_1}(X_1)$ as defined in (5.1), for every CFT-morphism $R \subseteq X_1 \times X_2$.

Theorem 7.5 *The categories $\mathcal{GBA}$ and $\mathcal{GUV}$ are dually equivalent under the functors $X : \mathcal{GBA} \rightarrow \mathcal{GUV}$ and $G : \mathcal{GUV} \rightarrow \mathcal{GBA}$.*

Proof It follows straightforwardly by the results in Sect. 5, in particular, see Propositions 5.12 and 5.8. $\square$

Proposition 7.6 *Let $\langle X, \tau, \mathcal{L} \rangle$ be a GUV-space. Then, $\langle X, \tau \rangle$ is a UV-space if and only if $\emptyset \in \mathcal{L}$.*

Proof ($\Rightarrow$) Assume that $\langle X, \tau \rangle$ is a UV-space. Since $\langle X, \tau, \mathcal{L} \rangle$ is a GUV-space, we have that $\mathcal{L} \cup \{\emptyset\} = \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\approx})$. Let $u \in \mathcal{L}$ be such that $u \neq X$ (this follows by (G4)). Since $\text{KO}(X) \cap \mathcal{RO}(X, \tau_{\approx})$ is closed under $\neg_{\mathcal{RO}}$ and $u \in \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\approx})$, it follows that $\neg_{\mathcal{RO}} u \in \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\approx})$. Since $\neg_{\mathcal{RO}} u \neq \emptyset$, we have that $\neg_{\mathcal{RO}} u \in \mathcal{L}$. Now since $\mathcal{L}$ is closed under $\cap$, we obtain that $\emptyset = u \cap \neg_{\mathcal{RO}} u \in \mathcal{L}$.

($\Leftarrow$) Assume that $\emptyset \in \mathcal{L}$. Thus $\mathcal{L} = \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\approx})$. Then, by (G2) and (G3), it follows that $\langle X, \tau \rangle$ satisfies conditions (UV1) and (UV2). Condition (UV3) follows from Theorem 4.11. $\square$

Corollary 7.7 *Let B be a Boolean algebra and $\langle X_B, \tau_B, \mathcal{L}_B \rangle$ its dual GUV-space. Then $\langle X_B, \tau_B \rangle$ is an UV-space where $\mathcal{L}_B = \text{KO}(X_B) \cap \mathcal{RO}(X_B, \tau_{\subseteq})$.*

8 Choice-Free Duality for Boolean Algebras

In the previous sections, we have developed choice-free topological dualities for the categories of Tarski algebras + homomorphisms (semi-homomorphisms) and generalized Boolean algebras + homomorphisms. In this section, we show how these dualities reduce to the Bezhanishvili and Holliday duality when applied to Boolean algebras and homomorphisms, [5]. For this, recall Definition 6.11. The dual morphisms of the Boolean homomorphisms are defining as follows. A map $f : X_1 \rightarrow X_2$ is said to be a *UV-map* if $f^{-1}[V] \in \text{KO}(X_1) \cap \mathcal{RO}(X_1, \tau_{\leq_1})$ for all $V \in \text{KO}(X_2) \cap \mathcal{RO}(X_2, \tau_{\leq_2})$. Consider the categories

$\mathcal{BA}$ = Boolean algebras + homomorphisms

and

$\mathcal{UV}$ = UV-spaces + UV-maps.

We say that $\langle X, \tau, \mathcal{L} \rangle$ is a UV-space if $\langle X, \tau \rangle$ is a UV-space (Def. 6.11) and $\mathcal{L} = \text{KO}(X) \cap \mathcal{RO}(X, \tau_{\leq})$. Now a CFT-morphism $R \subseteq X_1 \times X_2$ (see Def. 5.11) between UV-spaces $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ is said to be *functional* if for all $x \in X_1$, there exists $y \in X_2$ such that $R(x) = [y]_{\leq_2}$. We consider the following category

$\mathcal{FUV} = \text{UV-spaces} + \text{functional CFT-morphisms.}$

Theorem 8.1 *The categories $\mathcal{BA}$ and $\mathcal{FUV}$ are dually equivalent.*

Proof It follows from the results in Sect. 7, and by the fact that for all Boolean homomorphism $h : B_1 \rightarrow B_2$, $h^{-1}[G]$ is a proper filter of B_1 , for all proper filter G of B_2 . $\square$

Proposition 8.2 *Let $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ and $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ be UV-spaces.*

1. If $R \subseteq X_1 \times X_2$ is a functional CFT-morphism, then the map $f_R : X_1 \rightarrow X_2$, defining by $f_R(x) = y \iff R(x) = [y]_{\preceq_2}$, is a UV-map.
2. If $f : X_1 \rightarrow X_2$ is a UV-map, then the relation $R_f \subseteq X_1 \times X_2$, defining by $R_f(x) = [f(x)]_{\preceq_2}$, is a functional CFT-morphism.
3. Let $R \subseteq X_1 \times X_2$ be a functional CFT-morphism and $f : X_1 \rightarrow X_2$ a UV-map. Then, $R_{f_R} = R$ and $f_{R_f} = f$.

Proof (1) It follows from the fact that

$$h_R(v) = \{x \in X_1 : R(x) \subseteq v\} = \{x \in X_1 : [f_R(x)]_{\preceq_2} \subseteq v\} = f_R^{-1}[v]$$

for all $v \in \mathcal{L}_2 = \text{KO}(X_2) \cap \mathcal{RO}(X_2)$.

(2) (R1) follows from the following fact:

$$h_{R_f}(v) = \{x \in X_1 : R_f(x) \subseteq v\} = \{x \in X_1 : [f(x)]_{\preceq_2} \subseteq v\} = \{x \in X_1 : f(x) \in v\} = f^{-1}[v]$$

for all $v \in \mathcal{L}_2 = \text{KO}(X_2) \cap \mathcal{RO}(X_2, \tau_{\preceq_2})$. Condition (R2) follows from the fact that f is order-preserving. (R3) is consequence from the fact that $\mathcal{L}_2 = \text{KO}(X_2) \cap \mathcal{RO}(X_2, \tau_{\preceq_2})$ is a basis. Condition (R4) is a consequence that the map $f : \langle X_1, \preceq_1 \rangle \rightarrow \langle X_2, \preceq_2 \rangle$ is a p-morphism (see [5, Def. 6.1]). Hence R_f is a CFT-morphism, and moreover it is clear that R_f is functional.

(3) It is straightforward from the definitions. $\square$

The above proposition allows us to obtain that

Theorem 8.3 *The categories $\mathcal{FUV}$ and $\mathcal{UV}$ are isomorphic.*

Therefore, by Theorem 8.1, we arrive at the result presented in [5].

Theorem 8.4 *The categories $\mathcal{BA}$ and $\mathcal{UV}$ are dually equivalent.*

9 Final Remarks

In this final section, we add some interesting observations and questions we have received from the referee, which may be addressed in future work.

- (1) In Sect. 5, we proposed the categories of Tarski algebras with semi-homomorphisms and with homomorphisms between Tarski algebras, and we found their dual morphisms between CFT-spaces, which are binary relations satisfying some conditions. A natural question could arise. Which are the duals, between Tarski algebras, of the maps f from a CFT-space $\langle X_1, \tau_1, \mathcal{L}_1 \rangle$ into a CFT-space $\langle X_2, \tau_2, \mathcal{L}_2 \rangle$ such that $f^{-1}[V] \in \mathcal{L}_1$, for all $V \in \mathcal{L}_2$? We have not yet addressed this question.
- (2) Given a Tarski algebra A , we have shown that $G_{\mathcal{L}_A}$ is the “smallest” generalized Boolean algebra in which A can be embedded, see Definition 6.8 and Proposition 6.9. Let $\mathcal{B}_{\mathcal{L}_A}$ and $\mathcal{B}_{G_{\mathcal{L}_A}}$ be the Boolean subalgebras of $\mathcal{RO}(X_A, \tau_{\subseteq})$ generated by $\mathcal{L}_A$ and $G_{\mathcal{L}_A}$, respectively. It can be proved that $\mathcal{B}_{\mathcal{L}_A} = \mathcal{B}_{G_{\mathcal{L}_A}}$. We have tried to show that $\mathcal{B}_{\mathcal{L}_A}$ is the “smaller” Boolean algebra, in any sense, in which A can be embedded. However, this remains open.
- (3) Let A be a Tarski algebra and let $\mathcal{B}_{\mathcal{L}_A}$ be the Boolean subalgebra of $\mathcal{RO}(X_A, \tau_{\subseteq})$ generated by $\mathcal{L}_A$. Let σ_A be the topology on X_A generated by the collection $\mathcal{B}_{\mathcal{L}_A}$. Since $\mathcal{B}_{\mathcal{L}_A}$ is closed under finite intersections, $\mathcal{B}_{\mathcal{L}_A}$ is a basis for σ_A . We have shown that the specialization order given by σ_A is the inclusion order. Thus $\preceq_{\tau_A} = \subseteq = \preceq_{\sigma_A}$. Hence $\mathcal{RO}(X_A, \tau_{\preceq_{\tau_A}}) = \mathcal{RO}(X_A, \tau_{\subseteq}) = \mathcal{RO}(X_A, \tau_{\preceq_{\sigma_A}})$. We also proved that $\mathcal{B}_{\mathcal{L}_A} = \text{KO}(X_A, \sigma_A) \cap \mathcal{RO}(X_A, \tau_{\subseteq})$. Thus, we can ask: is the space $\langle X_A, \sigma_A \rangle$ a UV-space? It remains to show that satisfies condition 3. of Definition 5.1 in [5]. In the affirmative case, is it the dual UV-space of the Boolean algebra $\mathcal{B}_{\mathcal{L}_A}$? These remain open.

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