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COMPLETELY DISTRIBUTIVE COMPLETIONS OF POSETS

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Abstract. A Δ_1-completion of a poset is a completion for which, simultaneously, every element is reachable as a join of meets and a meet of joins from the original poset. We focus our attention on Δ_1-completions that can be obtained from polarities $\langle \mathcal{F}, \mathcal{I}, R \rangle$ where $\mathcal{F}$ is a collection of upsets containing the principal upsets, $\mathcal{I}$ is a collection of downsets containing the principal downsets of the original poset, and $R \subseteq \mathcal{F} \times \mathcal{I}$ is the relation of nonempty intersection. These Δ_1-completions are called $\langle \mathcal{F}, \mathcal{I} \rangle$-completions, and they satisfy a compactness property. In this paper, we show that if a pair $\langle \mathcal{F}, \mathcal{I} \rangle$ satisfies a separating condition (similar to the Prime Filter Theorem for distributive lattices), then the $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of the original poset is a completely distributive algebraic lattice. Given a poset P and an algebraic closure system $\mathcal{F}$ of upsets of P satisfying a distributivity condition, we show how to choose a collection of downsets $\mathcal{I}$ of P such that the $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P is a completely distributive algebraic lattice. Then, we study the extensions of additional operations on posets to their corresponding $\langle \mathcal{F}, \mathcal{I} \rangle$-completions. Finally, we use the previous results to obtain adequate $\langle \mathcal{F}, \mathcal{I} \rangle$-completions for the classes of Tarski algebras and Hilbert algebras, and for the classes of algebras that are canonically associated (in the sense of abstract algebraic logic) with some propositional logics.

1. Introduction

The problem of how to obtain a completion of a partially ordered set, that is, how to embedding a partially ordered set in a complete lattice was probably first addressed by MacNeille [31]. Then, other completions for ordered algebraic structures richer than partially ordered sets were developed. For instance, the canonical extension for Boolean algebras with operators [29,30], and the canonical extension for bounded (distributive) lattices

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with additional operations [18,21–23]. The notion of canonical extension for bounded lattices was generalised to partially ordered sets by Dunn et al. [12]. Dunn et al. [12] show how to extended n-ary operations on partially ordered sets to their canonical extensions, and they used this to prove a relational completeness of some substructural logics.

However, not every completion of a partially ordered set is adequate. For instance, the MacNeille completion preserves existing finite meets and joins, but it does not have good algebraic preservation properties, e.g., it does not preserve homomorphisms, and it does not necessarily retain the distributivity condition when is applied to distributive lattices. The theory of canonical extensions for partially ordered sets due to Dunn et al. [12] was employed to obtain some complete relational semantics for several substructural logics, but it is not appropriated, for instance, for obtaining an adequate completion for Hilbert algebras as was shown in [13, Example B.16]. Gehrke et al. [19] introduced a new and general concept of canonical extension for algebras corresponding to the algebraic counterpart of a finitary congruential logic. This completion, called $\mathcal{S}$-canonical extension, can be developed as a logical construct rather than just as a purely order-theoretical construct. Then, Gehrke et al. [19] obtained the IPC-canonical extension for Hilbert algebras, which correspond to the algebraic counterpart of the intuitionistic propositional calculus (IPC), proving that the IPC-canonical extension of a Hilbert algebra is a complete Heyting algebra.

A Δ_1-completion of a poset is a completion for which, simultaneously, each element is reachable as a join of meets of elements from the original poset and as a meet of joins of elements from the original poset. In particular, Δ_1-completions include the MacNeille completion and the canonical extension of a poset. Gehrke et al. [20] studied in detailed Δ_1-completions for posets. The main result in [20] is a full classification of the Δ_1-completions of a poset P in terms of certain polarities $\langle \mathcal{F}, \mathcal{I}, R\rangle$ where $\mathcal{F}$ is a collection of upsets of P, $\mathcal{I}$ is a collection of downsets of P, and R is a relation from $\mathcal{F}$ into $\mathcal{I}$ satisfying several properties. Gehrke et al. [20] show that for each polarity $\langle \mathcal{F}, \mathcal{I}, R\rangle$, there is (up to isomorphism) a unique completion of P, called the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P, which satisfies a compactness property and a density property.

In this paper, we study $\langle \mathcal{F}, \mathcal{I}\rangle$-completions that are completely distributive algebraic lattices. We prove that under a certain natural condition on a polarity $\langle \mathcal{F}, \mathcal{I}, R\rangle$, the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion is a completely distributive algebraic lattice (Section 3). It is worth mentioning the article [34], where the author study some completions through polarities satisfying certain conditions. Then, given a poset P and an algebraic closure system of upsets $\mathcal{F}$ of P containing the principal upsets and satisfying a distributivity condition, we show how to choose a collection of downsets $\mathcal{I}$ of P such that the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P is a completely distributive algebraic lattice. This allows us to show that the canonical extension ([12]) of a distributive meet-semilattice is a completely distributive algebraic lattice (Section 4). In Section 5, we study the extensions of additional operations defined on posets to their corresponding $\langle \mathcal{F}, \mathcal{I} \rangle$-completions. Finally, in Section 6, we apply the results of the previous sections to obtain appropriated $\langle \mathcal{F}, \mathcal{I} \rangle$-completions for several ordered algebraic structures associated with some propositional logics. We obtain completions for Tarski algebras, for Hilbert algebras, and for the algebras that are canonically associated (in the sense of abstract algebraic logic) with some finitary congruential logics.

2. Preliminaries

In this section, we review the main results about Δ_1-completions, and we establish several properties needed for what follows. The definitions and results on Δ_1-completions can be found in [20]. The main references for Order and Lattice theory are [9,27].

Let X be a set. Let $\mathcal{P}(X)$ be the power set of X. For $A \subseteq X$, we denote the complement of A with respect to X by A^c. By $A \subseteq_\omega X$ we mean that A is a (possibly empty) finite subset of X.

Let P be a poset. A subset $F \subseteq P$ is said to be an *upset* of P if for every $x \in F$ and $y \in P$ we have that $x \leq y$ implies $y \in F$. For $x \in P$, the *principal upset* of x is the set $\uparrow x = \{y \in P : x \leq y\}$. We denote by $\mathsf{Up}(P)$ the collection of all upsets of P. Dually, we have the notion of *downset*, and the *principal downset* of $x \in P$, $\downarrow x = \{y \in P : y \leq x\}$. Notice that for every poset P, $\langle \mathsf{Up}(P), \cap, \cup, \emptyset, P \rangle$ is a completely distributive algebraic lattice.

Let P be a poset. A subset $U \subseteq P$ is said to be *up-directed* if for all $a, b \in U$, there is $c \in U$ such that $a, b \leq c$.

A map $h\colon P \to Q$ from a poset P into a poset Q is said to be an *order embedding* if for all $a, b \in P$, $a \leq_P b \iff e(a) \leq_Q e(b)$. As usual, we drop the subscript when confusion is unlikely.

Let $\langle L, \wedge, \vee \rangle$ be a lattice. A nonempty subset F of L is said to be a *filter* of L if it is an upset, and if $a, b \in F$, then $a \wedge b \in F$. Dually, we have the notion of *ideal*. Let us denote the collection of filters of L by $\mathsf{Fi}(L)$, and the lattice of ideals of L by $\mathsf{Id}(L)$. For every nonempty $X \subseteq L$, $\mathrm{Fig}_L(X)$ denotes the filter of L generated by X, and $\mathrm{Idg}_L(X)$ denotes the ideal of L generated by X.

A complete lattice L is said to be a *completion* of P if there is an order embedding $e\colon P \to L$. We also say that $\langle L, e \rangle$ is a completion of P. For every $u \in L$, we consider the following sets:

$$\uparrow_P u := \{a \in P : u \leq e(a)\} \quad \text{and} \quad \downarrow_P u := \{a \in P : e(a) \leq u\}.$$

2.1. $\mathbf{\Delta_1}$-completions based on standard $\mathbf{\Delta_1}$-polarities. Let P be a poset. A collection $\mathcal{F}$ of upsets of P is called *standard* provided that $\{\uparrow a : a \in P\} \subseteq \mathcal{F}$. Dually, a collection $\mathcal{I}$ of downsets of P is called *standard* if $\{\downarrow a : a \in P\} \subseteq \mathcal{I}$. For every standard collection of upsets $\mathcal{F}$ and every standard collection of downsets $\mathcal{I}$, we consider the polarity $\langle \mathcal{F}, \mathcal{I}, R\rangle$ where $R \subseteq \mathcal{F} \times \mathcal{I}$ is defined as follows: for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$,

$$FRI \iff F \cap I \neq \emptyset.$$

For short, we will say that $\langle \mathcal{F}, \mathcal{I}\rangle$ is a *standard* Δ_1*-polarity over* P when $\mathcal{F}$ is a standard collection of upsets of P, $\mathcal{I}$ is a standard collection of downsets P, and R is defined just before.

Let P be a poset and $\langle \mathcal{F}, \mathcal{I}\rangle$ a standard Δ_1-polarity over P. Then, the polarity $\langle \mathcal{F}, \mathcal{I}, R\rangle$ gives rise to the following Galois connection (Φ_R, Ψ_R): given by

$$\Phi_R \colon \mathcal{P}(\mathcal{F}) \to \mathcal{P}(\mathcal{I}), \quad X \mapsto \Phi_R(X) = \big\{ I \in \mathcal{I} : (\forall F \in \mathcal{F})(F \in X \Rightarrow FRI)\big\},$$

$$\Psi_R \colon \mathcal{P}(\mathcal{I}) \to \mathcal{P}(\mathcal{F}), \quad Y \mapsto \Psi_R(Y) = \big\{ F \in \mathcal{F} : (\forall I \in \mathcal{I})(I \in Y \Rightarrow FRI)\big\}.$$

The *Galois closed subsets* of $\mathcal{F}$ and $\mathcal{I}$ are, respectively,

$$\mathcal{G}(\mathcal{F}) = \big\{ X \in \mathcal{P}(\mathcal{F}) : (\Psi_R \circ \Phi_R)(X) = X\big\} = \big\{ \Psi(Y) : Y \in \mathcal{P}(\mathcal{I})\big\},$$

$$\mathcal{G}^d(\mathcal{I}) = \big\{ Y \in \mathcal{P}(\mathcal{I}) : (\Phi_R \circ \Psi_R)(Y) = Y\big\} = \big\{ \Phi(X) : X \in \mathcal{P}(\mathcal{F})\big\}.$$

The Galois closed subsets of $\mathcal{F}$ form a closure system. Thus, $\langle \mathcal{G}(\mathcal{F}), \subseteq\rangle$ is a complete lattice. The map $e_P \colon P \to \mathcal{G}(\mathcal{F})$ defined as follows: $e_P(a) = \{F \in \mathcal{F} : a \in F\}$ is an order embedding. Hence $\langle \mathcal{G}(\mathcal{F}), e_P\rangle$ is a completion of P.

Let P be a poset and let $\langle \mathcal{F}, \mathcal{I}\rangle$ be a standard Δ_1-polarity over P. Let $\langle L, e\rangle$ be a completion of P. We consider the following sets:

- $\mathsf{K}_{\mathcal{F}}(L) := \{x \in L : x = \bigwedge e[F] \text{ for some } F \in \mathcal{F}\}$;
- $\mathsf{O}_{\mathcal{I}}(L) := \{y \in L : y = \bigvee e[I] \text{ for some } I \in \mathcal{I}\}$.

We drop the subscript when confusion is unlikely. The elements of the set $\mathsf{K}_{\mathcal{F}}(L)$ are called $\mathcal{F}$*-closed* (or simply *closed*), and the elements of $\mathsf{O}_{\mathcal{I}}(L)$ are called $\mathcal{I}$*-open* (or simply *open*).

Definition 2.1 [20]. Let P be a poset and let $\langle \mathcal{F}, \mathcal{I}\rangle$ be a standard Δ_1-polarity of P. A completion $\langle L, e\rangle$ of P is said to be $\langle \mathcal{F}, \mathcal{I}\rangle$*-compact* provide the following condition holds:

(C) for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$, if $\bigwedge e[F] \leq \bigvee e[I]$, then $F \cap I \neq \emptyset$.

A completion $\langle L, e\rangle$ of P is said to be $\langle \mathcal{F}, \mathcal{I}\rangle$*-dense* provide the following condition holds:

(D) for each $u \in L$, $u = \bigwedge\{y \in \mathsf{O}(L) : u \leq y\}$ and $u = \bigvee\{x \in \mathsf{K}(L) : x \leq u\}$.

We will refer to condition (C) as the $\langle \mathcal{F}, \mathcal{I} \rangle$-*compactness* of L, and condition (D) as the $\langle \mathcal{F}, \mathcal{I} \rangle$-*density* of L.

DEFINITION 2.2 [20, Definition 5.9]. Let P be a poset and let $\langle \mathcal{F}, \mathcal{I} \rangle$ be a standard Δ_1-polarity of P. We say that a completion $\langle L, e \rangle$ of P is an $\langle \mathcal{F}, \mathcal{I} \rangle$-*completion* if it is $\langle \mathcal{F}, \mathcal{I} \rangle$-compact and $\langle \mathcal{F}, \mathcal{I} \rangle$-dense.

THEOREM 2.3 [20, Theorem 5.10]. *Let P be a poset and $\langle \mathcal{F}, \mathcal{I} \rangle$ a standard Δ_1-polarity. Then, the completion $\langle \mathcal{G}(\mathcal{F}), e_P \rangle$ is, up to isomorphism, the unique $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P.*

For what follows, we need to introduce some concepts. Let X be a nonempty set. Given a collection $\mathcal{F}$ of subsets of X, we denote by $\mathcal{C}_{\mathcal{F}}(X)$ the closure system on X generated by $\mathcal{F}$, and $C_{\mathcal{F}}$ denotes the closure operator associated with $\mathcal{C}_{\mathcal{F}}(X)$. That is, for every $A \subseteq X$,

$$\mathcal{C}_{\mathcal{F}}(X) = \Big\{ \bigcap \mathcal{F}_0 : \mathcal{F}_0 \subseteq \mathcal{F} \Big\} \quad \text{and} \quad C_{\mathcal{F}}(A) = \bigcap \big\{ F \in \mathcal{F} : A \subseteq F \big\}.$$

Let C and C' be two closure operators on the same set X. We define $C \leq C'$ as follows: $C \leq C'$ if and only if $C(A) \subseteq C'(A)$ for all $A \subseteq X$. If $C \leq C'$, then the closure operator C' is said to be *stronger* than C.

Recall that a closure operator C on a set X is said to be *finitary* if for all $A \subseteq X$, $C(A) = \bigcup \{ C(B) : B \subseteq_\omega A \}$. A closure system $\mathcal{C}$ on X is said to be *algebraic* if it is closed under unions of chains. Let C be a closure operator and let $\mathcal{C}$ be the closure system associated with C. Then, C is finitary if and only if $\mathcal{C}$ is algebraic.

A finitary closure operator can be naturally associated with every closure operator. In other words, for each closure operator C there exists a finitary closure operator C^{f}, which is the strongest of all finitary closure operators C' such that $C' \leq C$. More precisely,

PROPOSITION 2.4. *Let X be a nonempty subset and let $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator on X. Then, the operator $C^{\mathrm{f}} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $C^{\mathrm{f}}(A) = \bigcup \{ C(A_0) : A_0 \subseteq_\omega A \}$ for every $A \in \mathcal{P}(X)$ satisfy the following*:

(1) $C^{\mathrm{f}}(A) = C(A)$ *for all* $A \subseteq_\omega X$;

(2) $C = C^{\mathrm{f}}$ *if and only if* C *is finitary*;

(3) C^{f} *is a finitary closure operator on* X *and* $C^{\mathrm{f}} \leq C$;

(4) C^{f} *is the strongest of all finitary closure operators* C' *on* X *such that* $C' \leq C$.

The operator C^{f} is called *the finitary companion of* C. We denote by $\mathcal{C}^{\mathrm{f}}(X)$ the closure system associated with C^{f}.

Thus, given a collection $\mathcal{F}$ of subsets of a set X, $C^{\mathrm{f}}_{\mathcal{F}}$ is the finitary companion of $C_{\mathcal{F}}$, and its associated closure system is denoted by $\mathcal{C}^{\mathrm{f}}_{\mathcal{F}}(X)$. Notice that if $\mathcal{F}$ is a closure system, then $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{F}$, and if $\mathcal{F}$ is a finitary closure system, we have $\mathcal{C}^{\mathrm{f}}_{\mathcal{F}}(X) = \mathcal{C}_{\mathcal{F}}(X) = \mathcal{F}$.

REMARK 2.5. Let P be a poset and $\mathcal{F}$ a standard collection of upsets of P. Then, we have $\mathcal{C}_{\mathcal{F}}(P) \subseteq \mathcal{C}^{\mathrm{f}}_{\mathcal{F}}(P) \subseteq \mathsf{Up}(P)$, and $C^{\mathrm{f}}_{\mathcal{F}}(a) = C_{\mathcal{F}}(a) = {\uparrow} a$, for all $a \in P$. Dually, we have similar facts for standard collections of downsets of P.

2.2. Some properties of $\langle \mathcal{F}, \mathcal{I} \rangle$-completions. Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P. If we consider $\mathcal{F}$ and $\mathcal{I}$ ordered by inclusion, then by [20, Proposition 5.4] is straightforward to show that $\mathsf{K}(L)$ is dually order isomorphic to $\mathcal{F}$, and $\mathsf{O}(L)$ is order isomorphic to $\mathcal{I}$ by the following maps:

$$\begin{array}{rrcl} \bigwedge: & \mathcal{F} & \rightleftarrows & \mathsf{K}(L) : {\uparrow}_P(.) \\ & F & \mapsto & \bigwedge e[F] \\ & {\uparrow}_P x & \leftarrow & x \end{array} \qquad \begin{array}{rrcl} \bigvee: & \mathcal{I} & \rightleftarrows & \mathsf{O}(L) : {\downarrow}_P(.) \\ & I & \mapsto & \bigvee e[I] \\ & {\downarrow}_P y & \leftarrow & y \end{array}$$

Since $\{{\uparrow} a : a \in P\} \subseteq \mathcal{F}$ and $\{{\downarrow} a : a \in P\} \subseteq \mathcal{I}$, it follows that $e[P] \subseteq \mathsf{K}(L) \cap \mathsf{O}(L)$.

PROPOSITION 2.6. *Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. Then, the order embedding e satisfies the following conditions: for every $A \subseteq P$ and $b \in P$,*

(1) *$b \in C_{\mathcal{F}}(A)$ if and only if $e(b) \in {\uparrow}(\bigwedge e[A])$;*
(2) *$b \in C_{\mathcal{I}}(A)$ if and only if $e(b) \in {\downarrow}(\bigvee e[A])$;*
(3) *$b \in C^{\mathrm{f}}_{\mathcal{F}}(A)$ if and only if $e(b) \in \mathrm{Fig}_L(e[A])$;*
(4) *$b \in C^{\mathrm{f}}_{\mathcal{I}}(A)$ if and only if $e(b) \in \mathrm{Idg}_L(e[A])$.*

PROOF. Let $A \subseteq P$ and $b \in P$.

(1) Taking into account that $\mathcal{F}$ is dually order isomorphic to $\mathsf{K}(L)$, and from the $\langle \mathcal{F}, \mathcal{I}\rangle$-density, we have that

$$\begin{aligned} b \in C_{\mathcal{F}}(A) &\iff (\forall x \in \mathsf{K}(L))(A \subseteq {\uparrow}_P x \implies b \in {\uparrow}_P x) \\ &\iff (\forall x \in \mathsf{K}(L))\Big(x \leq \bigwedge e[A] \implies x \leq e(b)\Big) \\ &\iff \bigwedge e[A] \leq e(b) \iff e(b) \in {\uparrow}\Big(\bigwedge e[A]\Big). \end{aligned}$$

(2) It can be proved by a dual argument to (1).

(3) Assume that $b \in C^{\mathrm{f}}_{\mathcal{F}}(A)$. By definition of $C^{\mathrm{f}}_{\mathcal{F}}$, there exists $A_0 \subseteq_\omega A$ such that $b \in C_{\mathcal{F}}(A_0)$. Then, by (1), we have $\bigwedge e[A_0] \leq e(b)$. Hence $e(b) \in \mathrm{Fig}_L(e[A])$. Now assume that $e(b) \in \mathrm{Fig}_L(e[A])$. So, there exists $A_0 \subseteq_\omega A$ such that $\bigwedge e[A_0] \leq e(b)$. Then, by (1), $b \in C_{\mathcal{F}}(A_0)$. Thus $b \in C^{\mathrm{f}}_{\mathcal{F}}(A)$.

(4) It can be proved by a dual argument to (3). □

Remark 2.7. Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. Let $b \in P$. If $b \in C_{\mathcal{F}}(\emptyset)$, then b is the greatest element of P, and $b \in F$ for all $F \in \mathcal{F}$. Thus $e(b)$ is the greatest element of L. But, in general, the order embedding e not necessarily preserves the greatest element of P, when it exists.

Corollary 2.8. *Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. Then, conditions* (3) *and* (4) *of Proposition* 2.6 *are equivalent to the following conditions, respectively*:

(3′) $e^{-1}[F] \in \mathcal{C}^{\mathrm{f}}_{\mathcal{F}}(P)$ *for all* $F \in \mathsf{Fi}(L)$;

(4′) $e^{-1}[I] \in \mathcal{C}^{\mathrm{f}}_{\mathcal{I}}(P)$ *for all* $I \in \mathsf{Id}(L)$.

Proposition 2.9. *Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. Then, for every $A \subseteq P$,*

$$\bigwedge e[A] = \bigwedge e[C^{\mathrm{f}}_{\mathcal{F}}(A)] = \bigwedge e[C_{\mathcal{F}}(A)]$$

and

$$\bigvee e[A] = \bigvee e[C^{\mathrm{f}}_{\mathcal{I}}(A)] = \bigvee e[C_{\mathcal{I}}(A)].$$

Proof. Let us prove that $\bigwedge e[A]$ is the infimum of $e[C^{\mathrm{f}}_{\mathcal{F}}(A)]$. Let $b \in C^{\mathrm{f}}_{\mathcal{F}}(A)$. So, there is $A_0 \subseteq_\omega A$ such that $b \in C_{\mathcal{F}}(A_0)$. Then, by Proposition 2.6, $\bigwedge e[A_0] \le e(b)$, and thus we obtain that $\bigwedge e[A] \le \bigwedge e[A_0] \le e(b)$. Hence $\bigwedge e[A]$ is a lower bound of $e[C^{\mathrm{f}}_{\mathcal{F}}(A)]$. Let $u \in L$ be such that $u \le e(b)$ for all $b \in C^{\mathrm{f}}_{\mathcal{F}}(A)$. So, in particular, $u \le e(a)$ for all $a \in A$. Then $u \le \bigwedge e[A]$. Hence $\bigwedge e[A] = \bigwedge e[C^{\mathrm{f}}_{\mathcal{F}}(A)]$. With a similar argument it can be proved that $\bigwedge e[A]$ is the infimum of $e[C_{\mathcal{F}}(A)]$. The second set of equalities are proved by a dual argumentation. □

Corollary 2.10. *Let P be a poset and $\langle L, e\rangle$ an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. If $\mathcal{F}$ is a closure system, then $\mathsf{K}(L)$ is closed under arbitrary meets. Dually, if $\mathcal{I}$ is a closure system, then $\mathsf{O}(L)$ is closed under arbitrary joins.*

Proof. Let $X \subseteq \mathsf{K}(L)$. By Proposition 2.9, we obtain that

$$\begin{aligned}\bigwedge X &= \bigwedge_{x\in X}\Big(\bigwedge e[{\uparrow_P} x]\Big) = \bigwedge\Big(\bigcup\{e[{\uparrow_P} x] : x \in X\}\Big)\\ &= \bigwedge\Big(e\Big[\bigcup\{{\uparrow_P} x : x \in X\}\Big]\Big) = \bigwedge e\Big[C_{\mathcal{F}}\Big(\bigcup\{{\uparrow_P} x : x \in X\}\Big)\Big].\end{aligned}$$

Hence, since $\mathcal{F}$ is a closure system, $\bigwedge X \in \mathsf{K}(L)$. □

3. $\langle\mathcal{F},\mathcal{I}\rangle$-completions through separating collections

Throughout this section, P will be a poset and $\langle\mathcal{F},\mathcal{I}\rangle$ will be a standard Δ_1-polarity of P. We consider the following set:

$$\Omega_{\langle\mathcal{F},\mathcal{I}\rangle}(P) := \left\{ F \in \mathcal{F} : F^c \in \mathcal{I} \right\}.$$

For brevity, and when there is not danger of confusion, we sometimes omit the subscripts. Let us consider the following property:

$$\text{(P)} \qquad \begin{cases} \text{For every } F \in \mathcal{F} \text{ and every } I \in \mathcal{I}, \text{ if } F \cap I = \emptyset, \text{ then} \\ \text{there exists } H \in \Omega(P) \text{ such that } F \subseteq H \text{ and } H \cap I = \emptyset. \end{cases}$$

Property (P) is a generalisation of the Prime Filter Theorem of several ordered algebraic structures; for instance, of the Prime Filter Theorem for Boolean algebras and distributive lattices. Now, we show that a standard Δ_1-polarity satisfying property (P) separates points.

LEMMA 3.1. *Let P be a poset and $\langle\mathcal{F},\mathcal{I}\rangle$ a standard Δ_1-polarity satisfying property* (P). *If $a, b \in P$ such that $a \nleq b$, then there is $H \in \Omega(P)$ such that $a \in H$ and $b \notin H$.*

PROOF. It follows straightforward by property (P), and from the fact that $\mathcal{F}$ and $\mathcal{I}$ are standard collections of upsets and downsets, respectively. □

Considering the poset $\langle\Omega(P), \subseteq\rangle$, we define the map $\alpha\colon P \to \mathsf{Up}(\Omega(P))$ as follows:

$$\alpha(a) = \left\{ H \in \Omega(P) : a \in H \right\}.$$

It is clear that α is well defined, and since $\mathcal{F} \subseteq \mathsf{Up}(P)$, it follows that α is order preserving.

THEOREM 3.2. *Let P be a poset and let $\langle\mathcal{F},\mathcal{I}\rangle$ be a standard Δ_1-polarity of P such that satisfies property* (P). *Then $\langle\mathsf{Up}(\Omega(P)), \alpha\rangle$ is, up to isomorphism, the $\langle\mathcal{F},\mathcal{I}\rangle$-completion of P.*

PROOF. From the previous lemma, it is straightforward to check that α is an order embedding, and thus $\langle\mathsf{Up}(\Omega(P)), \alpha\rangle$ is a completion of P.

First we show that $\langle\mathsf{Up}(\Omega(P)), \alpha\rangle$ is an $\langle\mathcal{F},\mathcal{I}\rangle$-compact completion of P. Let $F \in \mathcal{F}$ and $I \in \mathcal{I}$ be such that $\bigcap \alpha[F] \subseteq \bigcup \alpha[I]$. We need to show that $F \cap I \neq \emptyset$. Suppose, towards a contradiction, that $F \cap I = \emptyset$. By property (P), there is $H \in \Omega(P)$ such that $F \subseteq H$ and $H \cap I = \emptyset$. Then $H \in \bigcap \alpha[F]$ and $H \notin \bigcup \alpha[I]$, which is a contradiction. Hence $F \cap I \neq \emptyset$, and therefore $\mathsf{Up}(\Omega(P))$ is an $\langle\mathcal{F},\mathcal{I}\rangle$-compact completion of P.

Now we prove that $\langle\mathsf{Up}(\Omega(P)), \alpha\rangle$ is an $\langle\mathcal{F},\mathcal{I}\rangle$-dense completion of P. Let $U \in \mathsf{Up}(\Omega(P))$. We need to prove that:

(1) $U = \bigcap\{\bigcup \alpha[I] : I \in \mathcal{I},\ U \subseteq \bigcup \alpha[I]\}$ and
(2) $U = \bigcup\{\bigcap \alpha[F] : F \in \mathcal{F},\ \bigcap \alpha[F] \subseteq U\}$.

(1) It is clear that $U \subseteq \bigcap\{\bigcup \alpha[I] : I \in \mathcal{I},\ U \subseteq \bigcup \alpha[I]\}$. Let $H \in \Omega(P)$ be such that $H \in \bigcup \alpha[I]$ for all $I \in \mathcal{I}$ such that $U \subseteq \bigcup \alpha[I]$. Suppose that $H \notin U$. Since U is an upset of $\Omega(P)$, it follows that for every $G \in U$ there exists $a_G \in G \setminus H$. Let $I := H^c$. So, $I \in \mathcal{I}$. It follows straightforward that $U \subseteq \bigcup \alpha[I]$ and $H \notin \bigcup \alpha[I]$, a contradiction. Hence $H \in U$. Therefore, (1) holds.

(2) It is clear that $\bigcup\{\bigcap \alpha[F] : F \in \mathcal{F},\ \bigcap \alpha[F] \subseteq U\} \subseteq U$. Let $H \in U$. So $H \in \Omega(P) \subseteq \mathcal{F}$. Since U is an upset, it follows that $H \in \bigcap \alpha[H] \subseteq U$. Then $H \in \bigcup\{\bigcap \alpha[F] : F \in \mathcal{F},\ \bigcap \alpha[F] \subseteq U\}$, and hence (2) holds. □

Corollary 3.3. *Let P be a poset. If $\langle \mathcal{F}, \mathcal{I}\rangle$ is a standard Δ_1-polarity of P satisfying property* (P), *then the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P is a completely distributive algebraic lattice.*

For every poset P, we can always choose a standard Δ_1-polarity $\langle \mathcal{F}, \mathcal{I}\rangle$ satisfying property (P). Indeed, let $\mathcal{F}$ be the collection of all upsets of P and $\mathcal{I}$ the collection of all downsets of P. So, it is obvious that $\langle \mathcal{F}, \mathcal{I}\rangle$ is a standard Δ_1-polarity. Notice that $\Omega(P) = \{F \in \mathcal{F} : F^c \in \mathcal{I}\} = \mathcal{F}$. Thus, the standard Δ_1-polarity $\langle \mathcal{F}, \mathcal{I}\rangle$ satisfies trivially property (P). Hence, by Theorem 3.2, we obtain that $\langle \mathsf{Up}(\mathcal{F}), \alpha\rangle$ is the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. Then, we have shown the following proposition.

Proposition 3.4. *Every poset P has an $\langle \mathcal{F}, \mathcal{I}\rangle$-completion, for some $\langle \mathcal{F}, \mathcal{I}\rangle$ standard Δ_1-polarity, being a completely distributive algebraic lattice.*

Example 3.5. Consider the poset P in Fig. 1. Let $\mathcal{F}_u$ be the collection of all upsets and $\mathcal{I}_d$ the collection of all downsets of P. We know that $\langle \mathcal{F}_u, \mathcal{I}_d\rangle$ is a standard Δ_1-polarity satisfying property (P). Thus, we have that $\langle \mathsf{Up}\,(\mathcal{F}_u), \alpha\rangle$ is the $\langle \mathcal{F}_u, \mathcal{I}_d\rangle$-completion of P. The lattice $L := \mathsf{Up}\,(\mathcal{F}_u)$ is depicted in Fig. 1. Recall that $\alpha[P] \subseteq \mathsf{K}_{\mathcal{F}_u}(L) \cap \mathsf{O}_{\mathcal{I}_d}(L)$. Hence, the $\mathcal{F}_u$-closed elements of L are the elements of $\alpha[P]$ and the nodes with a square, and the $\mathcal{I}_d$-open elements of L are the elements of $\alpha[P]$ and the nodes with a circle. Then, it is straightforward to check that the completion L is $\langle \mathcal{F}_u, \mathcal{I}_d\rangle$-compact and $\langle \mathcal{F}_u, \mathcal{I}_d\rangle$-dense.

4. $\langle \mathcal{F}, \mathcal{I}\rangle$-completions for $\mathcal{F}$-distributive posets

Given a poset P and a standard algebraic closure system $\mathcal{F}$ of upsets of P satisfying a distributivity condition, we show how to choose a standard collection $\mathcal{I}$ of downsets of P such that the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P is a completely distributive algebraic lattice.

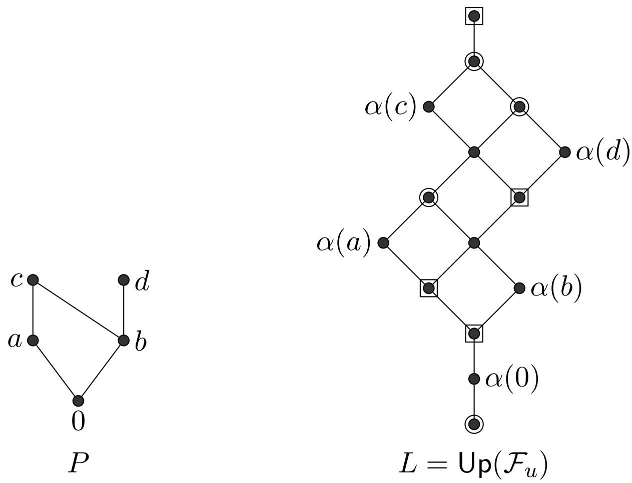

Fig. 1: The poset P and its $\langle \mathcal{F}_u, \mathcal{I}_d \rangle$-completion

REMARK 4.1. In [13] Esteban studied finitary closure operators C on posets P that satisfy the following conditions

$$\{A \subseteq P : C(A) = A\} \subseteq \mathsf{Up}(P) \quad \text{and} \quad C(a) = \uparrow a \text{ for all } a \in P. \tag{4.1}$$

As was noted in Remark 2.5, given a poset P and a standard collection of upsets $\mathcal{F}$ of P, the finitary closure operator $C^{\mathrm{f}}_{\mathcal{F}}$ satisfies Esteban's conditions. In this section, we continue studying finitary closure operators satisfying (4.1), and their associated algebraic closure systems. We refer the reader to [13, Sections 2.1, 2.2], where some of the concepts considered here are studied in a uniform and general way.

Throughout this section, unless otherwise stated, P will be a poset and $\mathcal{F}$ will be a standard algebraic closure system of upsets of P. Recall that $C_{\mathcal{F}}$ denotes the closure operator associated with $\mathcal{F}$, and since $\mathcal{F}$ is algebraic, we have that $C_{\mathcal{F}}$ is finitary. Thus, $C_{\mathcal{F}}$ satisfies the conditions in (4.1). Moreover, notice that $\mathcal{F}$ is a (complete) lattice where $F_1 \cap F_2$ is the meet of F_1 and F_2, and the join operation $\vee$ is given by $F_1 \vee F_2 = C_{\mathcal{F}}(F_1 \cup F_2)$, for all $F_1, F_2 \in \mathcal{F}$.

We denote by $\mathcal{F}_{\mathrm{pr}}$ the collection of all meet-prime (*prime* for short) elements of the lattice $\mathcal{F}$, and $\mathcal{F}_{\mathrm{irr}}$ denotes the collection of all meet-irreducible (*irreducible* for short) elements of the lattice $\mathcal{F}$.

PROPOSITION 4.2. *For every $F \in \mathcal{F}$, F is prime if and only if F^c is a nonempty up-directed downset of P.*

PROOF. It follows straightforwardly. $\square$

Proposition 4.3 [13, Lemma 2.1.1]. *For every $F \in \mathcal{F}$ and every nonempty up-directed downset $I \subseteq P$, if $F \cap I = \emptyset$, then there exists $H \in \mathcal{F}_{\mathrm{irr}}$ such that $F \subseteq H$ and $H \cap I = \emptyset$.*

Proposition 4.4. *The lattice $\mathcal{F}$ is distributive if and only if $\mathcal{F}_{\mathrm{pr}} = \mathcal{F}_{\mathrm{irr}}$.*

Proof. It is known that in all distributive lattices the collections of all meet-irreducible and meet-prime elements coincide. Conversely, assume now that $\mathcal{F}_{\mathrm{pr}} = \mathcal{F}_{\mathrm{irr}}$. Let $X_1, X_2, X_3 \in \mathcal{F}$. We need to show that $X_1 \cap (X_2 \vee X_3) \subseteq (X_1 \cap X_2) \vee (X_1 \cap X_3)$. Suppose towards a contradiction that the previous inclusion does not hold. So, there is $a \in X_1 \cap (X_2 \vee X_3)$ such that $a \notin (X_1 \cap X_2) \vee (X_1 \cap X_3)$. By Proposition 4.3, there exists $Y \in \mathcal{F}_{\mathrm{irr}}$ such that $(X_1 \cap X_2) \vee (X_1 \cap X_3) \subseteq Y$ and $a \notin Y$. Then $X_1 \cap X_2 \subseteq Y$ and $X_1 \cap X_3 \subseteq Y$. Since $Y \in \mathcal{F}_{\mathrm{irr}} = \mathcal{F}_{\mathrm{pr}}$, it follows that

$$(X_1 \subseteq Y \text{ or } X_2 \subseteq Y) \quad \text{and} \quad (X_1 \subseteq Y \text{ or } X_3 \subseteq Y).$$

As $a \in X_1$ and $a \notin Y$, we obtain that $X_2 \subseteq Y$ and $X_3 \subseteq Y$. Hence $X_2 \vee X_3 \subseteq Y$. This implies that $a \in Y$, which is a contradiction. This completes the proof. $\square$

Definition 4.5. We say that a poset P is *$\mathcal{F}$-distributive* if $\mathcal{F}$ is a standard algebraic closure system of upsets of P such that the lattice $\langle \mathcal{F}, \cap, \vee \rangle$ is distributive.

Corollary 4.6. *Let P be an $\mathcal{F}$-distributive poset. For every $F \in \mathcal{F}$ and every nonempty up-directed downset $I \subseteq P$, if $F \cap I = \emptyset$, then there exists $H \in \mathcal{F}_{\mathrm{pr}}$ such that $F \subseteq H$ and $H \cap I = \emptyset$.*

The next three propositions are some technical results needed for what follows. The proofs of the following two propositions are straightforward, and thus they are omitted.

Proposition 4.7. *Let $A \subseteq P$ and $x \in P$. If $C_{\mathcal{F}}(x) = C_{\mathcal{F}}(A)$, then $x = \bigwedge A$*[1].

Proposition 4.8. *The following conditions are equivalent*:

(1) *every $F \in \mathcal{F}$ is closed under existing* (*finite*) *meets*;

(2) *for every $A \subseteq P$* ($A \subseteq_\omega P$) *and $x \in P$, $x = \bigwedge A$ implies $C_{\mathcal{F}}(x) = C_{\mathcal{F}}(A)$.*

Now we establish (in a general setting) a characterization of the distributivity of the lattice of closed subsets of a finitary closure operator.

[1] For every $A \subseteq P$, $x = \bigwedge A$ means that the infimum of A in P exists and equals x. Moreover, if $A = \emptyset$, then x is the greatest element of P.

Proposition 4.9. *Let X be a nonempty set and let C be a finitary closure operator on X. Let $\mathcal{C}$ be the closure system associated with C. Then, the lattice $\mathcal{C}$ is distributive if and only if the following condition holds*:

$$\text{(E)} \qquad \begin{cases} \text{for every } A \subseteq_\omega X \text{ and every } x \in X, \text{ if } x \in C(A), \text{ then} \\ \text{there exists } B \subseteq_\omega \bigcup_{a\in A} C(a) \text{ such that } C(x) = C(B). \end{cases}$$

Proof. Assume that the lattice $\mathcal{C}$ is distributive. Let $A \subseteq_\omega X$ and $x \in X$ be such that $x \in C(A)$. So, by distributivity of $\mathcal{C}$ we have that

$$C(x) = C(x) \cap C(A) = C(x) \cap \Big(\bigvee_{a\in A} C(a) \Big) = \bigvee_{a\in A} (C(x) \cap C(a)).$$

Then $x \in \bigvee_{a\in A}(C(x) \cap C(a)) = C\big(\bigcup_{a\in A}(C(x)\cap C(a))\big)$. Since C is finitary, it follows that there exists $B \subseteq_\omega \bigcup_{a\in A}(C(x) \cap C(a))$ such that $x \in C(B)$. Thus $C(x) \subseteq C(B)$. Moreover, notice that $B \subseteq C(x)$. Hence $C(x) = C(B)$. Since $B \subseteq_\omega C(x) \cap (\bigcup_{a\in A} C(a))$, we obtain that $B \subseteq_\omega \bigcup_{a\in A} C(a)$. Therefore, condition (E) holds.

Now assume that condition (E) holds. Let $A_1, A_2, A_3 \in \mathcal{C}$. We need only prove that $A_1 \cap (A_2 \vee A_3) \subseteq (A_1 \cap A_2) \vee (A_1 \cap A_3)$. Let $x \in A_1 \cap (A_2 \vee A_3)$. So $x \in A_1$ and $x \in A_2 \vee A_3$. Since C is finitary, it follows that there is $B \subseteq_\omega A_2 \cup A_3$ such that $x \in C(B)$. By (E), there exists $A \subseteq_\omega \bigcup_{b\in B} C(b)$ such that $C(x) = C(A)$. As $x \in A_1 \in \mathcal{C}$, we have $A \subseteq C(A) \subseteq A_1$. Given that $B \subseteq_\omega A_2 \cup A_3$, we have $\bigcup_{b\in B} C(b) \subseteq A_2 \cup A_3$. Thus, we obtain that $A \subseteq A_2 \cup A_3$. Then, $A \subseteq A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$, and hence $C(A) \subseteq (A_1 \cap A_2) \vee (A_1 \cap A_3)$. Thus, $x \in (A_1 \cap A_2) \vee (A_1 \cap A_3)$. Therefore $\mathcal{C}$ is distributive. □

The following corollary is a consequence of Propositions 4.9 and 4.7.

Corollary 4.10. *Let P be an $\mathcal{F}$-distributive poset. For every $A \subseteq_\omega P$ and every $x \in P$, if $x \in C_{\mathcal{F}}(A)$, then there exists $B \subseteq_\omega \bigcup_{a\in A} C_{\mathcal{F}}(a)$ such that $x = \bigwedge B$.*

The following result tells us which standard collection of downsets $\mathcal{I}$ we have to choose for an $\mathcal{F}$-distributive poset so that the corresponding $\langle \mathcal{F}, \mathcal{I}\rangle$-completion be a completely distributive algebraic lattice.

Theorem 4.11. *Let P be an $\mathcal{F}$-distributive poset and let $\mathcal{I}_u$ be the collection of all nonempty up-directed downsets of P. Then, the $\langle \mathcal{F}, \mathcal{I}_u\rangle$-completion of P is $\mathsf{Up}(\mathcal{F}_{\mathrm{pr}}(P))$ (up to isomorphism), and thus it is a completely distributive algebraic lattice.*

Proof. Firstly, it is clear that $\langle \mathcal{F}, \mathcal{I}_u\rangle$ is a standard Δ_1-polarity of P. By Proposition 4.2, we have $\Omega_{\langle \mathcal{F}, \mathcal{I}_u\rangle}(P) = \mathcal{F}_{\mathrm{pr}}$. Then, by Corollary 4.6, we obtain that the Δ_1-polarity $\langle \mathcal{F}, \mathcal{I}_u \rangle$ satisfies property (P). Hence, by Theorem 3.2, $\mathsf{Up}(\mathcal{F}_{\mathrm{pr}})$ is the $\langle \mathcal{F}, \mathcal{I}_u \rangle$-completion of P. $\square$

Now we will show that for an $\mathcal{F}$-distributive poset P is possible to choose another standard collection of downsets $\mathcal{I}$ such that the $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P is a completely distributive algebraic lattice.

DEFINITION 4.12 [13, Definition 2.1.4]. Let P be a poset and $\mathcal{F}$ a standard algebraic closure system of upsets of P. A downset I of P is said to be *$\mathcal{F}$-strong* when for all $A \subseteq_\omega I$ and all $B \subseteq_\omega P$,

$$\bigcap_{a \in A} C_{\mathcal{F}}(a) \subseteq C_{\mathcal{F}}(B) \quad \text{implies} \quad C_{\mathcal{F}}(B) \cap I \neq \emptyset.$$

Let us denote by $\mathcal{I}_s$ the collection of all $\mathcal{F}$-strong subsets of P.

PROPOSITION 4.13. *If I is an $\mathcal{F}$-strong downset of P, then the following condition holds: for any $A \subseteq_\omega I$ and any $x \in P$,*

$$\bigcap_{a \in A} C_{\mathcal{F}}(a) \subseteq C_{\mathcal{F}}(x) \quad \textit{implies} \quad x \in I.$$

PROOF. Assume that I is an $\mathcal{F}$-strong downset of P, and let $A \subseteq_\omega I$ and $x \in P$ be such that $\bigcap_{a \in A} C_{\mathcal{F}}(a) \subseteq C_{\mathcal{F}}(x)$. Since I is an $\mathcal{F}$-strong, it follows that $C_{\mathcal{F}}(x) \cap I \neq \emptyset$. Thus, there is $b \in C_{\mathcal{F}}(x) = {\uparrow} x$ and $b \in I$. Then, since I is a downset, we have that $x \in I$. $\square$

In [13] the subsets I of P that satisfies the condition of the previous proposition are called *dual closed sets* of $C_{\mathcal{F}}$ (see [13, Definition 2.1.3]).

Notice that the principal downsets ${\downarrow} a$ of P are $\mathcal{F}$-strong. Hence, $\langle \mathcal{F}, \mathcal{I}_s \rangle$ is a standard Δ_1-polarity of P.

DEFINITION 4.14 [13, Definition 2.1.6]. Let P be a poset and $\mathcal{F}$ a standard algebraic closure system of upsets of P. An element $F \in \mathcal{F}$ is called a *strong $\mathcal{F}$-optimal* if there exists an $\mathcal{F}$-strong downset I of P such that F is a maximal element of the collection $\{G \in \mathcal{F} : G \cap I = \emptyset\}$ and I is a maximal element of the collection $\{J \in \mathcal{I}_{\mathrm{s}} : F \cap J = \emptyset\}$. Let us denote by $\mathsf{Opt}_{\mathrm{s}\mathcal{F}}(P)$ the collection of all strong $\mathcal{F}$-optimal.

The definition above is slightly different from [13, Definition 2.1.6]. Here, in Definition 4.14, we require that I be maximal concerning the $\mathcal{F}$-strong downsets disjoint with I, and in [13] I is maximal concerning the dual closed sets of $C_{\mathcal{F}}$ disjoint with I. Despite this slight difference, the corresponding proofs in [13] remain valid here. Moreover, notice by Proposition 4.13 that every $\mathcal{F}$-strong downset is a dual closed set of $C_{\mathcal{F}}$.

Recall that

$$\Omega_{\langle \mathcal{F}, \mathcal{I}_{\mathrm{s}} \rangle}(P) = \{F \in \mathcal{F} : F^{\mathrm{c}} \in \mathcal{I}_{\mathrm{s}}\}.$$

PROPOSITION 4.15 [13, Lemma 2.2.1]. *Let P be an $\mathcal{F}$-distributive poset. Then* $\mathsf{Opt}_{s\mathcal{F}}(P) = \Omega_{\langle \mathcal{F}, \mathcal{I}_s \rangle}(P)$.

COROLLARY 4.16. *Let P be an $\mathcal{F}$-distributive poset. For any $F \in \mathcal{F}$ and any $I \in \mathcal{I}_s$, if $F \cap I = \emptyset$, then there exists $H \in \Omega_{\langle \mathcal{F}, \mathcal{I}_s \rangle}(P)$ such that $F \subseteq H$ and $H \cap I = \emptyset$.*

PROOF. It is a consequence of the previous proposition and [13, Lemma 2.1.7]. □

THEOREM 4.17. *Let P be an $\mathcal{F}$-distributive poset. Then, the $\langle \mathcal{F}, \mathcal{I}_s \rangle$-completion of P is a completely distributive algebraic lattice.*

PROOF. It is a consequence of the previous corollary and Theorem 3.2. □

Now, as an application of the above results, we obtain a completion for certain partially ordered sets. In particular, we will show that the canonical extension (in the sense of [12]) of a distributive meet-semilattice is a completely distributive algebraic lattice; this was an open problem posed in [13, pp. 69] and proved in [26].

DEFINITION 4.18 [17]. Let P be a poset. A subset $F \subseteq P$ is said to be a *Frink filter* of P if for all $A \subseteq_\omega F$ and $x \in P$,

$$\bigcap_{a \in A} {\downarrow} a \subseteq {\downarrow} x \quad \text{implies} \quad x \in F.$$

Let us denote by $\mathsf{Fi}_\mathsf{F}(P)$ the collection of all Frink filters of P.

Let P be a poset. It is easy to check that $\mathsf{Fi}_\mathsf{F}(P)$ is a standard algebraic closure system on P. The posets for which the lattice $\mathsf{Fi}_\mathsf{F}(P)$ is distributive were studied by several authors [5,6,10,25,28]. In [10] the $\mathsf{Fi}_\mathsf{F}(P)$-distributive posets are called filter-distributive posets, and in [25] the $\mathsf{Fi}_\mathsf{F}(P)$-distributive posets are called meet-order distributive posets. Recall that $\mathcal{I}_u$ denotes the collection of all nonempty up-directed downsets of P.

COROLLARY 4.19. *If P is an $\mathsf{Fi}_\mathsf{F}(P)$-distributive poset, then the $\langle \mathsf{Fi}_\mathsf{F}(P), \mathcal{I}_u \rangle$-completion of P is a completely distributive algebraic lattice.*

PROOF. It is a direct consequence of Theorem 4.11. □

The $\langle \mathsf{Fi}_\mathsf{F}(P), \mathcal{I}_u \rangle$-completion of an $\mathsf{Fi}_\mathsf{F}(P)$-distributive poset was obtain in [26] through a topological duality.

EXAMPLE 4.20. Consider the poset P depicted in Fig. 1. Thus $\mathsf{Fi}_\mathsf{F}(P) = \{\emptyset, {\uparrow}a, {\uparrow}b, {\uparrow}c, {\uparrow}d, {\uparrow}0\}$. It is clear that the lattice $\langle \mathsf{Fi}_\mathsf{F}(P), \subseteq \rangle$ is distributive. Thus P is $\mathsf{Fi}_\mathsf{F}(P)$-distributive. If $\mathsf{Fi}^{\mathrm{pr}}_\mathsf{F}(P)$ denotes the meet-prime elements of the lattice $\mathsf{Fi}_\mathsf{F}(P)$, it follows by Theorem 4.11 that $\langle \mathsf{Up}\left(\mathsf{Fi}^{\mathrm{pr}}_\mathsf{F}(P)\right), \alpha \rangle$ is the

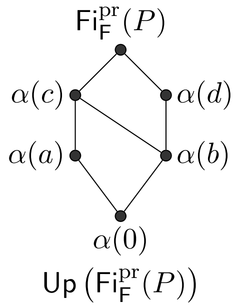

Fig. 2: The $\langle \mathsf{Fi}_{\mathsf{F}}(P), \mathcal{I}_u\rangle$-completion of the poset P in Fig. 1

$\langle \mathsf{Fi}_{\mathsf{F}}(P), \mathcal{I}_u\rangle$-completion of P. The lattice $\mathsf{Up}\left(\mathsf{Fi}^{\mathrm{pr}}_{\mathsf{F}}(P)\right)$ is shown in Fig. 2. Compare with the $\langle \mathcal{F}_u, \mathcal{I}_d\rangle$-completion of P depicted in Fig. 1.

Let $\langle M, \wedge, 1\rangle$ be a meet-semilattice with a greatest element 1. A nonempty subset F of M is a *filter* if (i) it is an upset and (ii) $a, b \in F$ implies $a \wedge b \in F$. We denote by $\mathsf{Fi}(M)$ the collection of all filters of M. It is straightforward that $\mathsf{Fi}(M)$ is a standard algebraic closure system on M.

A meet-semilattice $\langle M, \wedge, 1\rangle$ is *distributive* if for each $a, b_1, b_2 \in M$ with $b_1 \wedge b_2 \leq a$, there exist $a_1, a_2 \in M$ such that $b_1 \leq a_1$, $b_2 \leq a_2$ and $a = a_1 \wedge a_2$. As follows from [27, Sec. II 5.1, Lem. 184], a meet-semilattice M is distributive if and only if the lattice $\mathsf{Fi}(M)$ is distributive. Thus, a meet-semilattice M is distributive if and only if it is $\mathsf{Fi}(M)$-distributive.

The *canonical extension* (see [12]) of a meet-semilattice is, up to isomorphism, the $\langle \mathsf{Fi}(M), \mathcal{I}_u\rangle$-completion of M. It worth notice that the canonical extension of a semilattice is a generalisation of the concept of canonical extension for Boolean algebras [29] and distributive lattices [21]. Now, by Theorem 4.11, it is straightforward to prove directly the following corollary.

Corollary 4.21. *The canonical extension of a distributive meet-semilattice is a completely distributive algebraic lattice.*

5. Extensions of maps

In this section, we study extensions of maps. We will describe how can be extended n-ary operations on posets to their corresponding $\langle \mathcal{F}, \mathcal{I}\rangle$-completions.

Let P be a poset and $\langle \mathcal{F}, \mathcal{I}\rangle$ a standard Δ_1-polarity of P. Let $\langle L, e\rangle$ be the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of P. In the present section, for the sake of simplicity, we will consider that P is a sub-poset of L and $e\colon P \to L$ is the identity map. Thus, for instance, we have

- $\mathsf{K}(L) = \{x \in L : x = \bigwedge F \text{ for some } F \in \mathcal{F}\}$ and
- $\mathsf{O}(L) = \{y \in L : y = \bigvee I \text{ for some } I \in \mathcal{I}\}$.

Recall, from the $\langle\mathcal{F},\mathcal{I}\rangle$-density (see Definition 2.1) of L, that $\mathsf{K}(L)$ is join-dense in L and $\mathsf{O}(L)$ is meet-dense in L.

5.1. Extensions of unary maps. Throughout this subsection, for $i \in \{1,2\}$, P_i will denote a poset, $\mathcal{F}_i$ will be a standard algebraic closure system of upsets of P_i, and $\mathcal{I}_u^i$ will be the collection of all nonempty up-directed downsets of P_i. For every $i \in \{1,2\}$, $C_{\mathcal{F}_i}$ will be the finitary closure operator associated with $\mathcal{F}_i$, and $C_{\mathcal{I}_u^i}$ will denote the closure operator associated with the closure system generated by $\mathcal{I}_u^i$. Moreover, for $i \in \{1,2\}$, L_i will denote the $\langle\mathcal{F}_i,\mathcal{I}_u^i\rangle$-completion of P_i.

The following two definitions are generalisations of [12, Definition 3.2] (see also [32]).

Definition 5.1. Let $f\colon P_1 \to P_2$ be an order preserving map. We define the maps $f^\sigma\colon L_1 \to L_2$ and $f^\pi\colon L_1 \to L_2$ as follow: let $u \in L_1$,

$$f^\sigma(u) = \bigvee\Big\{\bigwedge\{f(a) : x \le a \in P_1\} : u \ge x \in \mathsf{K}(L_1)\Big\}$$

and

$$f^\pi(u) = \bigwedge\Big\{\bigvee\{f(a) : y \ge a \in P_1\} : u \le y \in \mathsf{O}(L_1)\Big\}.$$

Definition 5.2. Let $g\colon P_1 \to P_2$ be an order reversing map. We define the maps $g^\sigma\colon L_1 \to L_2$ and $g^\pi\colon L_1 \to L_2$ as follow: let $u \in L_1$,

$$g^\sigma(u) = \bigvee\Big\{\bigwedge\{g(a) : y \ge a \in P_1\} : u \le y \in \mathsf{O}(L_1)\Big\}$$

and

$$g^\pi(u) = \bigwedge\Big\{\bigvee\{g(a) : x \le a \in P_1\} : u \ge x \in \mathsf{K}(L_1)\Big\}.$$

In the following two propositions we collect the main properties of the functions f^σ, f^π, g^σ and g^π. Their proofs are very similar to those given in the framework of [12].

Proposition 5.3. *Let $f\colon P_1 \to P_2$ be an order preserving map. Then:*
(1) *for all $x \in \mathsf{K}(L_1)$, $f^\sigma(x) = \bigwedge\{f(a) : x \le a \in P_1\}$;*
(2) *for all $y \in \mathsf{O}(L_1)$, $f^\pi(y) = \bigvee\{f(a) : y \ge a \in P_1\}$;*
(3) *for all $x \in \mathsf{K}(L_1)$, $f^\sigma(x) \in \mathsf{K}(L_2)$;*
(4) *for all $y \in \mathsf{O}(L_1)$, $f^\pi(y) \in \mathsf{O}(L_2)$;*
(5) *for all $u \in L_1$, $f^\sigma(u) = \bigvee\{f^\sigma(x) : u \ge x \in \mathsf{K}(L_1)\}$;*
(6) *for all $u \in L_1$, $f^\pi(u) = \bigwedge\{f^\pi(y) : u \le y \in \mathsf{O}(L_1)\}$;*
(7) *f^σ and f^π are order preserving extensions of f;*
(8) *f^σ and f^π coincide on $\mathsf{K}(L_1) \cup \mathsf{O}(L_1)$;*
(9) *$f^\sigma \le f^\pi$.*

PROPOSITION 5.4. *Let* $g\colon P_1 \to P_2$ *be an order reversing map. Then:*
(1) *for all* $y \in \mathsf{O}(L_1)$, $g^\sigma(y) = \bigwedge\{g(a) : y \geq a \in P_1\}$;
(2) *for all* $x \in \mathsf{K}(L_1)$, $g^\pi(x) = \bigvee\{g(a) : x \leq a \in P_1\}$;
(3) *for all* $y \in \mathsf{O}(L_1)$, $g^\sigma(y) \in \mathsf{K}(L_2)$;
(4) *for all* $u \in L_1$, $g^\sigma(u) = \bigvee\{g^\sigma(y) : u \leq y \in \mathsf{O}(L_1)\}$;
(5) *for all* $u \in L_1$, $g^\pi(u) = \bigwedge\{g^\pi(x) : u \geq x \in \mathsf{K}(L_1)\}$;
(6) g^σ *and* g^π *are order reversing extensions of* g;
(7) g^σ *and* g^π *coincide on* $\mathsf{K}(L_1) \cup \mathsf{O}(L_1)$;
(8) $g^\sigma \leq g^\pi$.

THEOREM 5.5. *Let* P_1 *and* P_2 *be* $\mathcal{F}_i$*-distributive posets and let* $h\colon P_1 \to P_2$ *be a map such that for all* $A \subseteq_\omega P_1$ *and* $b \in P_1$,

$$b \in C_{\mathcal{F}_1}(A) \implies h(b) \in C_{\mathcal{F}_2}(h[A]); \tag{H}$$

Then, $h^\sigma\colon L_1 \to L_2$ *preserves arbitrary nonempty meets.*

PROOF. Let $U \subseteq L_1$ be nonempty. We need to prove that $h^\sigma(\bigwedge U) = \bigwedge h^\sigma[U]$. Let $u_0 := \bigwedge U$. Since h^σ is order preserving, it follows that $h^\sigma(u_0) \leq \bigwedge h^\sigma[U]$. Now we prove the other inequality. By condition (5) of Proposition 5.3 and using the fact that L_1 is a completely distributive lattice, we have

$$\bigwedge h^\sigma[U] = \bigwedge_{u\in U} \bigvee \{\, h^\sigma(x) : u \geq x \in \mathsf{K}(L_1)\}$$

$$= \bigvee \Big\{ \bigwedge_{u\in U} h^\sigma(\gamma(u)) : \gamma\colon U \to \mathsf{K}(L_1) \text{ with } \gamma(u) \leq u \ \forall u \in U \Big\}.$$

Let us show that $\bigwedge_{u\in U} h^\sigma(\gamma(u)) \leq h^\sigma(u_0)$, for every $\gamma\colon U \to \mathsf{K}(L_1)$ such that $\gamma(u) \leq u$ for all $u \in U$. Thus, let $\gamma\colon U \to \mathsf{K}(L_1)$ such that $\gamma(u) \leq u$ for all $u \in U$. By condition (1) of Proposition 5.3, we have $h^\sigma(\gamma(u)) = \bigwedge\{h(a) : \gamma(u) \leq a \in P_1\}$. So,

$$\bigwedge_{u\in U} h^\sigma(\gamma(u)) = \bigwedge \{\, h(a) : \gamma(u) \leq a \in P_1, \text{ for some } u \in U\Big\} \tag{5.1}$$

Now we want to show that

$$\bigwedge \{\, h(a) : \gamma(u) \leq a \in P_1, \text{ for some } u \in U\} = \bigwedge \Big\{ h(a) : \bigwedge_{u\in U} \gamma(u) \leq a \in P_1 \Big\}. \tag{5.2}$$

The inequality $\geq$ is clear. Since $\gamma(u) \in \mathsf{K}(L_1)$ for all $u \in U$, it follows that $\gamma(u) = \bigwedge\{a \in P_1 : \gamma(u) \leq a\}$ for all $u \in U$. Let $a_0 \in P_1$ be such that

$\bigwedge_{u\in U}\gamma(u)\le a_0$. So, $\bigwedge_{u\in U}\big(\bigwedge\{a\in P_1:\gamma(u)\le a\}\big)\le a_0$. Then $\bigwedge\{a\in P_1 : \gamma(u)\le a$ for some $u\in U\}\le a_0$. Thus

$$a_0\in{\uparrow_P}\Big(\bigwedge\{a\in P_1:\gamma(u)\le a \text{ for some } u\in U\}\Big).$$

By Proposition 2.6, we have

$$a_0\in C_{\mathcal{F}_1}\big(\{a\in P_1:\gamma(u)\le a \text{ for some } u\in U\}\big).$$

Then, since $C_{\mathcal{F}_1}$ is finitary, it follows that there exists $A_0\subseteq_\omega\{a\in P_1:\gamma(u)\le a$ for some $u\in U\}$ such that $a_0\in C_{\mathcal{F}_1}(A_0)$. By condition (H), we have $h(a_0)\in C_{\mathcal{F}_2}(h[A_0])$. Thus, by Proposition 2.6, we obtain that $\bigwedge h[A_0]\le h(a_0)$. Hence $\bigwedge\{h(a):\gamma(u)\le a\in P_1$ for some $u\in U\}\le h(a_0)$. Thus

$$\bigwedge\big\{h(a):\gamma(u)\le a\in P_1 \text{ for some } u\in U\big\}\le\bigwedge\Big\{h(a):\bigwedge_{u\in U}\gamma(u)\le a\in P_1\Big\}.$$

Then we have proved (5.2). By (5.1), it follows that

$$\bigwedge_{u\in U}h^\sigma(\gamma(u))=\bigwedge\Big\{h(a):\bigwedge_{u\in U}\gamma(u)\le a\in P_1\Big\}.$$

By Corollary 2.10, we have that $\bigwedge_{u\in U}\gamma(u)\in\mathsf{K}(L_1)$. Thus, by condition (1) of Proposition 5.3,

$$h^\sigma\Big(\bigwedge_{u\in U}\gamma(u)\Big)=\bigwedge\Big\{h(a):\bigwedge_{u\in U}\gamma(u)\le a\in P_1\Big\}.$$

Hence $h^\sigma(\bigwedge_{u\in U}\gamma(u))=\bigwedge_{u\in U}h^\sigma(\gamma(u))$. Since $\gamma(u)\le u$ for each $u\in U$, it follows that $\bigwedge_{u\in U}\gamma(u)\le\bigwedge U=u_0$. Then, since h^σ is order preserving, we obtain $h^\sigma(\bigwedge_{u\in U}\gamma(u))\le h^\sigma(u_0)$. Hence $\bigwedge_{u\in U}h^\sigma(\gamma(u))\le h^\sigma(u_0)$. Thus we have proved that $\bigwedge_{u\in U}h^\sigma(\gamma(u))\le h^\sigma(u_0)$ for all $\gamma\colon U\to\mathsf{K}(L_1)$ such that $\gamma(u)\le u$. Then,

$$\bigvee\Big\{\bigwedge_{u\in U}h^\sigma(\gamma(u)):\gamma\colon U\to\mathsf{K}(L_1)\text{ s.t. }\gamma(u)\le u\Big\}\le h^\sigma(u_0).$$

Therefore, $h^\sigma(u_0)=\bigwedge h^\sigma[U]$. □

The above theorem may suggest what kind of maps from an $\mathcal{F}_1$-distributive poset P_1 into an $\mathcal{F}_2$-distributive poset P_2 should be called "homomorphism". The next proposition establishes a relation between condition (H) and the property of preserving existing finite meets.

Proposition 5.6. *Let* $h\colon P_1 \to P_2$ *be a map. Consider the following conditions*:

(1) *for all* $A \subseteq_\omega P_1$ *and* $b \in P_1$,

$$b \in C_{\mathcal{F}_1}(A) \implies h(b) \in C_{\mathcal{F}_2}(h[A]);$$

(2) h *preserves existing finite meets.*

Then:

(i) *If every* $F \in \mathcal{F}_1$ *is closed under existing finite meets, then* (1) $\Rightarrow$ (2).

(ii) *If all* $F \in \mathcal{F}_1$ *and all* $G \in \mathcal{F}_2$ *are closed under existing finite meets, respectively, in* P_1 *and* P_2, *and* P_1 *is* $\mathcal{F}_1$*-distributive, then* (2) $\Rightarrow$ (1).

Proof. (i) (1) $\Rightarrow$ (2). It should be noted that condition (1) implies that h is order preserving. Let $A \subseteq_\omega P_1$ be such that $\bigwedge A$ exists in P_1. Let $b := \bigwedge A$. By Proposition 4.8, we have that $C_{\mathcal{F}_1}(b) = C_{\mathcal{F}_1}(A)$. So $b \in C_{\mathcal{F}_1}(A)$, and then by (1) we get $h(b) \in C_{\mathcal{F}_2}(h[A])$. Thus $C_{\mathcal{F}_2}(h(b)) \subseteq C_{\mathcal{F}_2}(h[A])$. On the other hand, since $b = \bigwedge A$ and h is order preserving, it follows that $C_{\mathcal{F}_2}(h(a)) \subseteq C_{\mathcal{F}_2}(h(b))$ for all $a \in A$. Thus $\bigcup_{a\in A} C_{\mathcal{F}_2}(h(a)) \subseteq C_{\mathcal{F}_2}(h(b))$. Then $C_{\mathcal{F}_2}(h[A]) \subseteq C_{\mathcal{F}_2}(h(b))$. Hence $C_{\mathcal{F}_2}(h(b)) = C_{\mathcal{F}_2}(h[A])$. Therefore, by Proposition 4.7, we obtain $h(b) = \bigwedge h[A]$.

(ii) (2) $\Rightarrow$ (1). Let $A \subseteq_\omega P_1$ and $b \in P_1$. Assume that $b \in C_{\mathcal{F}_1}(A)$. Since P_1 is $\mathcal{F}_1$-distributive, it follows by Corollary 4.10 that there exists $B \subseteq_\omega \bigcup_{a\in A} C_{\mathcal{F}_1}(a)$ such that $b = \bigwedge B$. Then, by (2), we have $h(b) = \bigwedge h[B]$. Hence, by Proposition 4.8, $C_{\mathcal{F}_2}(h(b)) = C_{\mathcal{F}_2}(h[B])$. Notice that for every $b' \in B$, there is $a \in A$ such that $a \leq b'$. Since h is order preserving, it follows that $C_{\mathcal{F}_2}(h(b')) \subseteq C_{\mathcal{F}_2}(h[A])$ for all $b' \in B$. Then $C_{\mathcal{F}_2}(h[B]) \subseteq C_{\mathcal{F}_2}(h[A])$. Hence $h(b) \in C_{\mathcal{F}_2}(h[A])$. □

We end this section studying the extensions of residuated maps $f\colon P_1 \to P_2$. We show under what conditions the extensions are residuated. This issue was already studied by Morton [32] for some particular Δ_1-completions. Here we generalise several results of [32].

We recall the definition of residuated map.

Definition 5.7. A map $f\colon P_1 \to P_2$ is called *residuated* if there exists a map $g\colon P_2 \to P_1$, called the *residual* of f, such that for all $a \in P_1$ and $b \in P_2$ we have

$$f(a) \leq b \iff a \leq g(b).$$

Let $f\colon P_1 \to P_2$ be a residuated map with $g\colon P_2 \to P_1$ its residual. So, it is known (see [9, Ch. 7]) that f preserves arbitrary existing joins, and g preserves arbitrary existing meets. Hence, in particular, f and g are order preserving maps.

Lemma 5.8. *Let* P_1 *be an* $\mathcal{F}_1$*-distributive poset and* P_2 *an* $\mathcal{F}_2$*-distributive poset, where the algebraic closure systems* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are such that all their* *members are closed under existing finite meets (in P_1 and P_2, respectively). Let $f\colon P_1 \to P_2$ be a residuated map with $g\colon P_2 \to P_1$ its residual. Then, for all $F \in \mathcal{F}_1$ and $J \in \mathcal{I}_u^2$ we have*

$$C_{\mathcal{F}_2}(f[F]) \cap J \neq \emptyset \iff F \cap C_{\mathcal{I}_u^1}(g[J]) \neq \emptyset.$$

PROOF. ($\Rightarrow$) Assume $C_{\mathcal{F}_2}(f[F]) \cap J \neq \emptyset$. So let $b \in C_{\mathcal{F}_2}(f[F]) \cap J$. It is clear that $g(b) \in C_{\mathcal{I}_u^1}(g[J])$. As $b \in C_{\mathcal{F}_2}(f[F])$ and $\mathcal{F}_2$ is algebraic, then there is $A_0 \subseteq_\omega F$ such that $b \in C_{\mathcal{F}_2}(f[A_0])$. Thus, since P_2 is $\mathcal{F}_2$-distributive, there exists $B \subseteq_\omega \uparrow f[A_0]$ such that $b = \bigwedge B$. Given that g is the residual of f, we thus know that g preservers existing finite meets and so $g(b) = \bigwedge g[B]$. Notice that for every $b' \in B$, there is $a \in A_0$ such that $f(a) \leq b'$ and so $a \leq g(b')$. Thus, since $A_0 \subseteq F$ and F is an upset, we have $g[B] \subseteq F$ and hence $g(b) = \bigwedge g[B] \in F$. Then $g(b) \in F \cap C_{\mathcal{I}_u^1}(g[J])$.

($\Leftarrow$) Now assume that $F \cap C_{\mathcal{I}_u^1}(g[J]) \neq \emptyset$. Let $a \in F \cap C_{\mathcal{I}_u^1}(g[J])$. It is straightforward that $f(a) \in C_{\mathcal{F}_2}(f[F])$. Now, since J is up-directed and g is order preserving, it follows that $g[J]$ is up-directed. Thus, $\downarrow g[J] \in \mathcal{I}_u^1$. Then, we have $C_{\mathcal{I}_u^1}(g[J]) = \downarrow g[J]$. As $a \in \downarrow g[J]$, we have that there is $b \in J$ such that $a \leq g(b)$. So $f(a) \leq b$. Hence $f(a) \in J$. Therefore $f(a) \in C_{\mathcal{F}_2}(f[F]) \cap J$. $\square$

PROPOSITION 5.9. *Let P_1 be an $\mathcal{F}_1$-distributive poset and P_2 an $\mathcal{F}_2$-distributive poset, where the algebraic closure systems $\mathcal{F}_1$ and $\mathcal{F}_2$ are such that all their members are closed under existing finite meets (in P_1 and P_2, respectively). Let $f\colon P_1 \to P_2$ be a residuated map with $g\colon P_2 \to P_1$ its residual. Let L_i be the $\langle \mathcal{F}_i, \mathcal{I}_u^i\rangle$-completion of P_i, $i = 1, 2$. Then $f^\sigma\colon L_1 \to L_2$ is residuated with residual $g^\pi\colon L_2 \to L_1$.*

PROOF. Let $u \in L_1$ and $v \in L_2$. We need to prove that $f^\sigma(u) \leq v \iff u \leq g^\pi(v)$. First, let us show that for every $x \in \mathsf{K}(L_1)$ and every $y \in \mathsf{O}(L_2)$, $f^\sigma(x) \leq y \iff x \leq g^\pi(y)$. By the $\langle \mathcal{F}_2, \mathcal{I}_u^2\rangle$-compactness, Proposition 5.3 and Proposition 2.6, it follows that

$$\begin{aligned} f^\sigma(x) \leq y &\iff (\exists b \in P_2)(f^\sigma(x) \leq b \leq y) \\ &\iff (\exists b \in P_2)\Big(\bigwedge f[\uparrow_{P_1} x] \leq b \leq y\Big) \\ &\iff (\exists b \in P_2)\big(b \in C_{\mathcal{F}_2}(f[\uparrow_{P_1} x]) \text{ and } b \in \downarrow_{P_2} y\big) \\ &\iff C_{\mathcal{F}_2}(f[\uparrow_{P_1} x]) \cap \downarrow_{P_1} y \neq \emptyset. \end{aligned}$$

By a similar argumentation to the previous one, we obtain $x \leq g^\pi(y) \Leftrightarrow \uparrow_{P_1} x \cap C_{\mathcal{I}_u^1}(g[\downarrow_{P_2} y]) \neq \emptyset$. Hence, by Lemma 5.8, we have $f^\sigma(x) \leq y$ if and only if $x \leq g^\pi(y)$. Now we can prove the general case as follows:

$$f^\sigma(u) \leq v \iff \bigvee\{f^\sigma(x) : u \geq x \in \mathsf{K}(L_1)\} \leq \bigwedge\{y \in \mathsf{O}(L_2) : v \leq y\}$$

$$\iff f^\sigma(x) \le y,\ \forall x \in \mathsf{K}(L_1) \text{ s.t. } x \le u \text{ and } \forall y \in \mathsf{O}(L_2) \text{ s.t. } v \le y$$

$$\iff x \le g^\pi(y),\ \forall x \in \mathsf{K}(L_1) \text{ s.t. } x \le u \text{ and } \forall y \in \mathsf{O}(L_2) \text{ s.t. } v \le y$$

$$\iff \bigvee\{x \in \mathsf{K}(L_1) : x \le u\} \le \bigwedge\{g^\pi(y) : v \le y \in \mathsf{O}(L_2)\}$$

$$\iff u \le g^\pi(v). \quad \square$$

5.2. Extensions of n-ary maps. Let $P_1, \dots, P_n$ be posets. For every $i \in \{1, \dots, n\}$, let $\langle \mathcal{F}_i, \mathcal{I}_i \rangle$ be an arbitrary standard Δ_1-polarity of P_i, and let $\langle L_i, \alpha_i \rangle$ be the $\langle \mathcal{F}_i, \mathcal{I}_i \rangle$-completion of P_i. Let $L := \prod_{i=1}^n L_i$ and $\alpha\colon \prod_{i=1}^n P_i \to L$ defined by $\alpha(a_1, \dots, a_n) = (\alpha_1(a_1), \dots, \alpha_n(a_n))$.

PROPOSITION 5.10. *The map α is an order embedding. Therefore, $\langle L, \alpha \rangle$ is a completion of $\prod_{i=1}^n P_i$.*

Let us define the set of *closed elements* of L by $\mathsf{K}(L) := \prod_{i=1}^n \mathsf{K}(L_i)$, and the set of *open elements* of L by $\mathsf{O}(L) := \prod_{i=1}^n \mathsf{O}(L_i)$. We want to use these two sets of elements of L to define the corresponding π and σ-extensions of n-ary maps. For this purpose, the sets $\mathsf{K}(L)$ and $\mathsf{O}(L)$ should be dense on L. Thus, we need to know under what conditions on the Δ_1-polarities $\langle \mathcal{F}_i, \mathcal{I}_i \rangle$ the sets $\mathsf{K}(L)$ and $\mathsf{O}(L)$ are join-dense and meet-dense, respectively, on the lattice L.

PROPOSITION 5.11. *If for every $i \in \{1, \dots, n\}$ the bottom (top) element of L_i is an $\mathcal{F}_i$-closed ($\mathcal{I}_i$-open) element of L_i, then $\mathsf{K}(L)$ ($\mathsf{O}(L)$) is join-dense (meet-dense) in L.*

PROOF. Let $\overline{u} \in L = \prod_{i=1}^n L_i$. We need to prove that $\overline{u} = \bigvee\{\overline{x} \in \mathsf{K}(L) : \overline{x} \le \overline{u}\}$. It is clear that $\overline{u}$ is an upper bound of $\{\overline{x} \in \mathsf{K}(L) : \overline{x} \le \overline{u}\}$. Let $\overline{v} \in L$ be such that $\overline{x} \le \overline{v}$ for all $\overline{x} \in \mathsf{K}(L)$ such that $\overline{x} \le \overline{u}$. So, $x_i \le v_i$ for all $i = 1, \dots, n$. Let $i \in \{1, \dots, n\}$ and $x_i' \in \mathsf{K}(L_i)$ be such that $x_i' \le u_i$. Let us take $\overline{x}' := (0_{L_1}, \dots, x_i', \dots, 0_{L_n}) \in \mathsf{K}(L)$. It is clear that $\overline{x}' \le \overline{u}$. Then $\overline{x}' \le \overline{v}$ and hence $x_i' \le v_i$. Thus, by $\langle \mathcal{F}_i, \mathcal{I}_i \rangle$-density on L_i, $u_i \le v_i$. Hence $\overline{u} \le \overline{v}$. Therefore $\overline{u} = \bigvee\{\overline{x} \in \mathsf{K}(L) : \overline{x} \le \overline{u}\}$. $\square$

PROPOSITION 5.12. *Let P be a poset and $\langle \mathcal{F}, \mathcal{I} \rangle$ a standard Δ_1-polarity of P. Let L be the $\langle \mathcal{F}, \mathcal{I} \rangle$-completion of P. The bottom (top) element of L is a closed (open) element of L if and only if $P \in \mathcal{F}$ ($P \in \mathcal{I}$) and $I \ne \emptyset$, for all $I \in \mathcal{I}$, ($F \ne \emptyset$, for all $F \in \mathcal{F}$).*

PROOF. Suppose first that the bottom element of L, 0_L, is closed. So, there is $F \in \mathcal{F}$ such that $\bigwedge F = 0_L$. For each $a \in P$, $\bigwedge F = 0_L \le a$. Thus, since F is an upset and by $\langle \mathcal{F}, \mathcal{I} \rangle$-density, we have $a \in F$. Then $F = P$, and hence $P \in \mathcal{F}$. Let $I \in \mathcal{I}$. Since $P \in \mathcal{F}$ and $\bigwedge P = 0_L \le \bigvee I$, it follows by $\langle \mathcal{F}, \mathcal{I} \rangle$-compactness that $P \cap I \ne \emptyset$. Hence $I \ne \emptyset$.

Now assume that $P \in \mathcal{F}$ and $I \ne \emptyset$ for all $I \in \mathcal{I}$. Let us show that $\bigwedge P = 0_L$. Let $y \in \mathsf{O}(L)$. So, there is $I \in \mathcal{I}$ such that $y = \bigvee I$. Since $I \ne \emptyset$, let $a \in I$. Then $\bigwedge P \leq a \leq \bigvee I = y$. Hence $\bigwedge P \leq y$ for all $y \in \mathsf{O}(L)$. Therefore, by $\langle \mathcal{F}, \mathcal{I} \rangle$-density, we obtain $\bigwedge P = 0_L$. $\square$

From now on, unless otherwise stated, all the standard Δ_1-polarities $\langle \mathcal{F}, \mathcal{I} \rangle$ on a poset P will be considered to have the following properties: (i) $\mathcal{F}$ will be an algebraic closure system where all its members are nonempty; and (ii) $\mathcal{I} = \mathcal{I}_u \cup \{P\}$, that is, $\mathcal{I}$ will be the collection consisting of all nonempty up-directed downsets and $P \in \mathcal{I}$. Notice, since $\mathcal{F}$ is a closure system, that $P \in \mathcal{F}$.

Now we present the definitions for the extensions of maps $f\colon \prod_{i=1}^{n} P_i \to P_{n+1}$ that are order preserving or order reversing in each coordinate. For simplicity, we will consider only the case $n = 2$. Also, in the following definition we consider only the maps $f\colon P_1 \times P_2 \to P_3$ that are order reversing in the first coordinate and order preserving in the second coordinate. The other possible cases (order preserving and order reversing in both coordinate, and order preserving in the first coordinate and order reversing in the second) can be deduced without difficulty by the reader.

For what follows, let P_1, P_2 and P_3 be posets and $\langle \mathcal{F}_1, \mathcal{I}_1 \rangle$, $\langle \mathcal{F}_2, \mathcal{I}_2 \rangle$ and $\langle \mathcal{F}_3, \mathcal{I}_3 \rangle$ be standard Δ_1-polarities of P_1, P_2, P_3, respectively, satisfying the above conditions (i) and (ii). For $i \in \{1, 2, 3\}$, let L_i be the $\langle \mathcal{F}_i, \mathcal{I}_i \rangle$-completion of P_i.

Definition 5.13. Let $f\colon P_1 \times P_2 \to P_3$ be a map that is order reversing in the first coordinate and order preserving in the second coordinate. We define the maps $f^\sigma\colon L_1 \times L_2 \to L_3$ and $f^\pi\colon L_1 \times L_2 \to L_3$ as follows: let $u \in L_1$ and $v \in L_2$,

$$f^\sigma(u, v) = \bigvee \Big\{ \bigwedge \{ f(a, b) : y \geq a \in P_1,\ x \leq b \in P_2 \} : u \leq y \in \mathsf{O}(L_1),\ v \geq x \in \mathsf{K}(L_2) \Big\}$$

and

$$f^\pi(u, v) = \bigwedge \Big\{ \bigvee \{ f(a, b) : x \leq a \in P_1,\ y \geq b \in P_2 \} : u \geq x \in \mathsf{K}(L_1),\ v \leq y \in \mathsf{O}(L_2) \Big\}.$$

Now we establish the main properties of the maps f^σ and f^π.

Proposition 5.14. *Let $f\colon P_1 \times P_2 \to P_3$ be a map that is order reversing in the first coordinate and order preserving in the second coordinate. Then, the maps $f^\sigma\colon L_1 \times L_2 \to L_3$ and $f^\pi\colon L_1 \times L_2 \to L_3$ have the following properties*:

(1) *For every $y \in \mathsf{O}(L_1)$ and $x \in \mathsf{K}(L_2)$,*

$$f^\sigma(y, x) = \bigwedge \{ f(a, b) : y \geq a \in P_1,\ x \leq b \in P_2 \};$$

(2) *for every* $x \in \mathsf{K}(L_1)$ *and* $y \in \mathsf{O}(L_2)$,

$$f^\pi(x,y) = \bigvee \big\{ f(a,b) : x \leq a \in P_1,\ y \geq b \in P_2 \big\};$$

(3) *for every* $y \in \mathsf{O}(L_1)$ *and* $x \in \mathsf{K}(L_2)$, $f^\sigma(y,x) \in \mathsf{K}(L_3)$;
(4) *for every* $a \in P_1$ *and* $y \in \mathsf{O}(L_2)$, $f^\pi(a,y) \in \mathsf{O}(L_3)$;
(5) *for every* $u \in L_1$ *and* $v \in L_2$,

$$f^\sigma(u,v) = \bigvee \big\{ f^\sigma(y,x) : u \leq y \in \mathsf{O}(L_1),\ v \geq x \in \mathsf{K}(L_2) \big\}$$

and

$$f^\pi(u,v) = \bigwedge \big\{ f^\pi(x,y) : u \geq x \in \mathsf{K}(L_1),\ v \leq y \in \mathsf{O}(L_2) \big\};$$

(6) f^σ *and* f^π *are order reversing in the first coordinate and order preserving in the second coordinate and are both extensions of* f;
(7) f^σ *and* f^π *coincide on* $\mathsf{O}(L_1) \times \mathsf{K}(L_2)$ *and* $\mathsf{K}(L_1) \times \mathsf{O}(L_2)$;
(8) $f^\sigma \leq f^\pi$.

Proof. The proofs follow by similar argumentations to those that prove the properties of the extensions of unary maps, see Propositions 5.3 and 5.4. □

The above definitions of the π and σ-extensions of maps $f\colon P_1 \times P_2 \to P_3$ and their properties will be used in the next section when we will study some applications.

6. Applications

The extensions of additional operations on (distributive) lattices and Boolean algebras are well known and studied [18,21–23,29,30]. However, when the ordered algebraic structures do not have a lattice reduct, the situation of how obtaining adequate extensions for the additional operations become more complex. Some papers in this direction are [12,24,32].

In the following subsections, we will apply the results previously obtained to some ordered algebraic structures associated with some propositional logics.

6.1. An $\langle \mathcal{F}, \mathcal{I} \rangle$-completion for Tarski algebras. The variety of Tarski algebras $\mathbb{TA}$, also called implication algebras, is the equivalent algebraic semantics of the implication fragment of the classical propositional calculus (see [33]), and so $\mathbb{TA}$ can be obtained as the implication subreducts of the Boolean algebras. We refer the reader to [1,2,33] for more information about Tarski algebras.

Definition 6.1 [1]. An algebra $\langle A, \to, 1\rangle$ of type (2,0) is a *Tarski algebra* if satisfies the following identities:
(T1) $1 \to x \approx x$;
(T2) $x \to x \approx 1$;
(T3) $x \to (y \to z) \approx (x \to y) \to (x \to z)$;
(T4) $(x \to y) \to y \approx (y \to x) \to x$.
Let us denote by $\mathbb{TA}$ the variety of Tarski algebras.

Every Tarski algebra A has associated a natural partial order. The partial order $\leq_A$ (we omit the subscript when there is no danger of confusion) is defined as follows: Let $a, b \in A$,

$$a \leq_A b \iff a \to b = 1.$$

Therefore, for every Tarski algebra A, $\langle A, \leq_A\rangle$ is a partially ordered set where 1 is the greatest element.

Proposition 6.2. *Let A be a Tarski algebra. Then, the operation $\to$ on A is order reversing in the first coordinate and order preserving in the second coordinate.*

Proposition 6.3. *Let $\langle A, \to, 1\rangle$ be a Tarski algebra. Then $\langle A, \vee\rangle$, where $x \vee y := (x \to y) \to y$, is a join-semilattice. Moreover, $x \vee y = y$ if and only if $x \leq_A y$.*

Given a Tarski algebra A, we will denote by $\mathsf{Id}(A)$ the collection of all *ideals* of A (as a join-semilattice). That is, $I \in \mathsf{Id}(A)$ if and only if I is a nonempty up-directed downset of $\langle A, \leq_A\rangle$.

Definition 6.4. Let A be a Tarski algebra. A subset F of A is called an *implicative filter* (or *deductive system*) if $1 \in F$, and $a, a \to b \in F$ implies $b \in F$. Let us denote by $\mathsf{Fi}_{\to}(A)$ the collection of all implicative filters of A.

It is straightforward to show that every implicative filter is an upset of A, and each principal upset $\uparrow a$ of A is an implicative filter.

Proposition 6.5. *For every Tarski algebra A, $\mathsf{Fi}_{\to}(A)$ is an algebraic closure system on A. Moreover, the corresponding lattice $\mathsf{Fi}_{\to}(A)$ is distributive.*

Definition 6.6. Let A be a Tarski algebra. A proper implicative filter F of A is said to be *maximal* if for every proper implicative filter G of A, $F \subseteq G$ implies $F = G$.

We denote by $\mathsf{M}(A)$ the collection of all maximal implicative filters of a Tarski algebra A.

Proposition 6.7. *Let A be a Tarski algebra and $F \in \mathsf{Fi}_{\to}(A)$. The following are equivalent*:

(1) *F is maximal;*
(2) *F is a meet-prime element of the lattice $\mathsf{Fi}_{\to}(A)$;*
(3) *F^c is an ideal of A.*

COROLLARY 6.8. *Let $M \in \mathsf{M}(A)$ and $a, b \in A$. Then, $a \to b \in M$ if and only if $a \notin M$ or $b \in M$.*

THEOREM 6.9. *Let A be a Tarski algebra. Let $F \in \mathsf{Fi}_{\to}(A)$ and $I \in \mathsf{Id}(A)$. If $F \cap I = \emptyset$, then there exists $M \in \mathsf{M}(A)$ such that $F \subseteq M$ and $M \cap I = \emptyset$.*

Let A be a Tarski algebra. Then, we have that $\langle \mathsf{Fi}_{\to}(A), \mathsf{Id}(A)\rangle$ is a standard Δ_1-polarity of A, and A is $\mathsf{Fi}_{\to}(A)$-distributive. Hence, by Theorem 4.11, we obtain that $\langle \mathsf{Up}(\mathsf{M}(A)), \alpha\rangle$ is the $\langle \mathsf{Fi}_{\to}(A), \mathsf{Id}(A)\rangle$-completion of A (as a poset). Now notice, since all the members of $\mathsf{M}(A)$ are maximal (between the proper implicative filters), that the poset $\langle \mathsf{M}(A), \subseteq\rangle$ is an antichain, and thus $\mathsf{Up}(\mathsf{M}(A)) = \mathcal{P}(\mathsf{M}(A))$. Hence, we have proved the following result:

PROPOSITION 6.10. *The $\langle \mathsf{Fi}_{\to}(A), \mathsf{Id}(A)\rangle$-completion of a Tarski algebra A is a complete atomistic Boolean algebra.*

Given a Tarski algebra A, let us denote by A^* the $\langle \mathsf{Fi}_{\to}(A), \mathsf{Id}(A)\rangle$-completion of A. That is, $A^* := \langle \mathcal{P}(\mathsf{M}(A)), \cap, \cup, {}^c, \emptyset, \mathsf{M}(A)\rangle$. Recall that in a Boolean algebra $\langle B, \wedge, \vee, \neg, 0, 1\rangle$ the implication is defined as: $x \to y \approx \neg x \vee y$. So, for all $u, v \in A^*$, we have that $u \Rightarrow v = u^c \cup v$.

PROPOSITION 6.11. *Let A be a Tarski algebra. Then, the order embedding $\alpha\colon A \to \mathcal{P}(\mathsf{M}(A))$ is a Tarski homomorphism, that is, $\alpha(a \to b) = \alpha(a) \Rightarrow \alpha(b)$ for all $a, b \in A$.*

PROOF. It is an immediate consequence of Corollary 6.8. □

Let $\langle A, \to, 1\rangle$ be a Tarski algebra. Then the standard Δ_1-polarity $\langle \mathsf{Fi}_{\to}(A), \mathsf{Id}(A)\rangle$ satisfies the conditions (i) $\mathsf{Fi}_{\to}(A)$ is an algebraic closure system where all its elements are nonempty, and (ii) $\mathsf{Id}(A)$ is the collection of all nonempty up-directed downsets of A and $P \in \mathsf{Id}(A)$. Thus we can consider the π-extension $\to^{\pi}$ of the operation $\to$ as given in Definition 5.13. Let $u, v \in \mathcal{P}(\mathsf{M}(A))$. Then,

$$u \to^{\pi} v = \bigwedge \Big\{ \bigvee \{\alpha(a \to b) : x \subseteq \alpha(a),\ \alpha(b) \subseteq y,\ a, b \in A\} : u \supseteq x \in \mathsf{K}(A^*),\ v \subseteq y \in \mathsf{O}(A^*)\Big\}.$$

Thus,

$$u \to^{\pi} v = \bigcap \Big\{ \bigcup \{\alpha(a \to b) : a \in F,\ b \in I\} : F \in \mathsf{Fi}_{\to}(A),\ I \in \mathsf{Id}(A) \text{ and } \bigcap \alpha[F] \subseteq u,\ v \subseteq \bigcup \alpha[I]\Big\}.$$

Proposition 6.12. *Let A be a Tarski algebra and A^* its $\langle \mathsf{Fi}_{\rightarrow}(A), \mathsf{Id}(A)\rangle$-completion. Then, for all $u, v \in A^*$, we have that $u \rightarrow^{\pi} v = u \Rightarrow v$.*

Proof. Let $u, v \in A^*$. Recall that $u \Rightarrow v = u^c \cup v$. Let $P \in \mathsf{M}(A)$. Assume that $P \notin u^c \cup v$. So $P \in u$ and $P \notin v$, whence $\bigcap \alpha[P] \subseteq u$ and $v \subseteq \bigcup \alpha[P^c]$. Since P is maximal, we have $P^c \in \mathsf{Id}(A)$. By Corollary 6.8, we obtain that $a \rightarrow b \notin P$ for all $a \in P$ and $b \in P^c$. Thus, $P \notin \bigcup\{\alpha(a \rightarrow b) : a \in P, b \in P^c\}$. Hence $P \notin u \rightarrow^{\pi} v$. Therefore, $u \rightarrow^{\pi} v \subseteq u^c \cup v$.

Now suppose that $P \notin u \rightarrow^{\pi} v$. Then, there are $F \in \mathsf{Fi}_{\rightarrow}(A)$ and $I \in \mathsf{Id}(A)$ with $\bigcap \alpha[F] \subseteq u$ and $v \subseteq \bigcup \alpha[I]$ such that $P \notin \bigcup\{\alpha(a \rightarrow b) : a \in F, b \in I\}$. Thus, $a \rightarrow b \notin P$ for all $a \in F$ and $b \in I$. Then, by Corollary 6.8, $a \in P$ and $b \notin P$ for all $a \in F$ and $b \in I$. Hence $F \subseteq P$ and $P \cap I = \emptyset$. This implies that $P \in \bigcap \alpha[F]$ and $P \notin \bigcup \alpha[I]$, and hence $P \in u \cap v^c$. Thus $P \notin u^c \cup v$. Therefore $u^c \cup v \subseteq u \rightarrow^{\pi} v$. This completes the proof. □

Let $\langle A, \rightarrow, 1\rangle$ be a Tarski algebra. By Proposition 6.10, we know that the $\langle \mathsf{Fi}_{\rightarrow}(A), \mathsf{Id}(A)\rangle$-completion of A is a complete atomistic Boolean algebra $\langle A^*, \Rightarrow, \neg, 1_{A^*}\rangle$. Hence, the previous proposition shows that the π-extension of the operation $\rightarrow$ is the Boolean implication of A^*, and thus $\langle A^*, \rightarrow^{\pi}, \neg, 1_{A^*}\rangle$ is a complete atomistic Boolean algebra.

6.2. An $\langle \mathcal{F}, \mathcal{I}\rangle$-completion for Hilbert algebras. The variety of Hilbert algebras, also called positive implication algebras, is the equivalent algebraic semantics of the implication fragment of the intuitionistic propositional calculus. The aim of this section is to obtain an adequate $\langle \mathcal{F}, \mathcal{I}\rangle$-completion for Hilbert algebras. The next definitions and results can be found in [8,11,33].

Definition 6.13. A *Hilbert algebra* is an algebra $\langle A, \rightarrow, 1\rangle$ of type (2,0) such that satisfies the following conditions for all $a, b, c \in A$:

(H1) $a \rightarrow (b \rightarrow a) = 1$,

(H2) $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$,

(H3) if $a \rightarrow b = 1$ and $b \rightarrow a = 1$, then $a = b$.

In [11] Diego proved that the class of Hilbert algebras form a variety. Every Hilbert algebra $\langle A, \rightarrow, 1\rangle$ has associated in a natural way a partial order as follows. For every $a, b \in A$, $a \leq_A b \iff a \rightarrow b = 1$.

Proposition 6.14. *Let $\langle A, \rightarrow, 1\rangle$ be a Hilbert algebra. Then, the operation $\rightarrow$ is order reversing in the first coordinate and order preserving in the second coordinate.*

Definition 6.15. Let $\langle A, \rightarrow, 1\rangle$ be a Hilbert algebra. A subset $F \subseteq A$ is called an *implicative filter* of A if (i) $1 \in F$, and (ii) if $a, a \rightarrow b \in F$, then $b \in F$. Let us denote by $\mathsf{Fi}_{\rightarrow}(A)$ the collection of all implicative filters of A.

Proposition 6.16. *Let A be a Hilbert algebra. Then, $\mathsf{Fi}_{\to}(A)$ is an algebraic closure system on A. Moreover, the corresponding lattice $\mathsf{Fi}_{\to}(A)$ is distributive.*

Definition 6.17. A proper implicative filter F of a Hilbert algebra A is called *prime* if F is a meet-prime element of the lattice $\mathsf{Fi}_{\to}(A)$. We denote by $\mathsf{Pr}(A)$ the collection of all prime implicative filters of A.

Let A be a Hilbert algebra. Let us denote by $\mathcal{I}_u$ the collection of all nonempty up-directed downsets of $\langle A, \leq_A \rangle$.

Proposition 6.18. *Let A be a Hilbert algebra and $F \in \mathsf{Fi}_{\to}(A)$. Then, F is prime if and only if F^c is a nonempty up-directed downset of A. Hence, $\mathsf{Pr}(A) = \{F \in \mathsf{Fi}_{\to}(A) : F^c \in \mathcal{I}_u\}$.*

Proposition 6.19 [3]. *Let A be a Hilbert algebra and let $F \in \mathsf{Fi}_{\to}(A)$ and $I \in \mathcal{I}_u$ be such that $F \cap I = \emptyset$. Then, there exists $P \in \mathsf{Pr}(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Corollary 6.20 [4]. *Let A be a Hilbert algebra and $P \in \mathsf{Pr}(A)$. Then, $a \to b \notin P$ if and only if there exists $Q \in \mathsf{Pr}(A)$ such that $P \subseteq Q$, $a \in Q$ and $b \notin Q$.*

Let A be a Hilbert algebra. Then, we have that $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$ is a standard Δ_1-polarity of A, and A is (as a poset) $\mathsf{Fi}_{\to}(A)$-distributive. Hence, by Theorem 4.11, we obtain that $\langle \mathsf{Up}(\mathsf{Pr}(A)), \alpha \rangle$ is the $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$-completion of A. Moreover, it is known that $\langle \mathsf{Up}(\mathsf{Pr}(A)), \cap, \cup, \Rightarrow, \emptyset, \mathsf{Pr}(A) \rangle$ is a complete Heyting algebra, where $u \Rightarrow v := \{P \in \mathsf{Pr}(A) : u \cap {\uparrow} P \subseteq v\}$ (here ${\uparrow} P = \{Q \in \mathsf{Pr}(A) : P \subseteq Q\}$). Let us denote by $A^* := \langle \mathsf{Up}(\mathsf{Pr}(A)), \cap, \cup, \Rightarrow, \emptyset, \mathsf{Pr}(A) \rangle$ the $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$-completion of A.

Let $\langle A, \to, 1 \rangle$ be a Hilbert algebra. Then, $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$ is a standard Δ_1-polarity satisfying the following conditions: (i) $\mathsf{Fi}_{\to}(A)$ is an algebraic closure system where all its elements are nonempty, and (ii) $\mathcal{I}_u$ is the collection of all nonempty up-directed downsets of A and $A \in \mathcal{I}_u$. Thus, we can consider the extension of the operation $\to$ as in Definition 5.13:

$$u \to^{\pi} v = \bigwedge \Big\{ \bigvee \{ \alpha(a \to b) : x \subseteq \alpha(a),\ \alpha(b) \subseteq y,\ a, b \in A \} : u \supseteq x \in \mathsf{K}(A^*),\ v \subseteq y \in \mathsf{O}(A^*) \Big\}. \tag{6.1}$$

Hence,

$$u \to^{\pi} v = \bigcap \Big\{ \bigcup \{ \alpha(a \to b) : a \in F,\ b \in I \} : F \in \mathsf{Fi}_{\to}(A),\ I \in \mathcal{I}_u \text{ and } \bigcap \alpha[F] \subseteq u,\ v \subseteq \bigcup \alpha[I] \Big\}.$$

Recall that the map $\alpha\colon A \to \mathsf{Up}(\mathsf{Pr}(A))$ is defined by $\alpha(a) = \{P \in \mathsf{Pr}(A) : a \in P\}$.

PROPOSITION 6.21. *Let $\langle A, \to, 1\rangle$ be a Hilbert algebra. Then, we have that the map $\alpha\colon \langle A \to, 1\rangle \to \langle A^*, \to^\pi, 1_{A^*}\rangle$ is a homomorphism. Moreover, for all $a, b \in A$, $\alpha(a \to b) = \alpha(a) \Rightarrow \alpha(b)$.*

PROOF. Notice that $\alpha(1) = \{P \in \mathsf{Pr}(A) : 1 \in P\} = \mathsf{Pr}(A) = 1_{A^*}$. Let $a, b \in A$. We need to show that $\alpha(a \to b) = \alpha(a) \to^\pi \alpha(b)$. By (6.1), we have

$$\alpha(a) \to^\pi \alpha(b) = \bigcap\Big\{\bigcup\{\,\alpha(c \to d) : x \subseteq \alpha(c),\ \alpha(d) \subseteq y,\ c, d \in A\}\ :\ \alpha(a) \supseteq x \in \mathsf{K}(A^*) \text{ and } \alpha(b) \subseteq y \in \mathsf{O}(A^*)\Big\}.$$

Since $\alpha[A] \subseteq \mathsf{K}(A^*) \cap \mathsf{O}(A^*)$, it follows that $\alpha(a) \to^\pi \alpha(b) = \bigcup\{\alpha(c \to d) : \alpha(a) \subseteq \alpha(c), \alpha(d) \subseteq \alpha(b), c, d \in A\}$. Notice that for every $c, d \in A$ such that $\alpha(a) \subseteq \alpha(c)$ and $\alpha(d) \subseteq \alpha(b)$, we have $a \leq c$ and $d \leq b$. So $c \to d \leq a \to b$. Thus $\alpha(c \to d) \subseteq \alpha(a \to b)$. Hence $\alpha(a) \to^\pi \alpha(b) = \bigcup\{\alpha(c \to d) : \alpha(a) \subseteq \alpha(c),\ \alpha(d) \subseteq \alpha(b),\ c, d \in A\} = \alpha(a \to b)$.

Finally, $\alpha(a \to b) = \alpha(a) \Rightarrow \alpha(b)$ is a consequence of the definition of α and from Corollary 6.20. $\square$

We have shown that for all $a, b \in A$,

$$\alpha(a \to b) = \alpha(a) \to^\pi \alpha(b) = \alpha(a) \Rightarrow \alpha(b).$$

Now we will prove that the operations $\to^\pi$ and $\Rightarrow$ coincide on $A^* \times A^*$. Firstly, notice by Proposition 6.21 that for all $u, v \in A^*$,

$$u \to^\pi v := \bigcap\Big\{\bigcup\{\,\alpha(a) \Rightarrow \alpha(b) : a \in F, b \in I\}\ :\ F \in \mathsf{Fi}_\to(A),\ I \in \mathsf{Id}(A) \text{ and } \bigcap\alpha[F] \subseteq u, v \subseteq \bigcup\alpha[I]\Big\}. \tag{6.2}$$

PROPOSITION 6.22. *Let A be a Hilbert algebra and $u, v \in A^* = \mathsf{Up}(\mathsf{Pr}(A))$. Then $u \to^\pi v \subseteq u \Rightarrow v$.*

PROOF. Let $P \in u \to^\pi v$. From the definition of the operation $\Rightarrow$, we need to prove that $u \cap {\uparrow}P \subseteq v$. So, let $P_0 \in u$ be such that $P \subseteq P_0$. Since u is an upset, $\bigcap\alpha[P_0] \subseteq u$. Let $I := P_0^c \in \mathcal{I}_u$. Suppose towards a contradiction that $P_0 \notin v$. Thus $v \subseteq \bigcup\alpha[I]$. Hence, since $\bigcap\alpha[P_0] \subseteq u$, $v \subseteq \bigcup\alpha[I]$ and since $P \in u \to^\pi v$, it follows that there exist $a \in P_0$ and $b \in I$ such that $P \in \alpha(a) \Rightarrow \alpha(b)$. Then $P_0 \in \alpha(a) \cap {\uparrow}P \subseteq \alpha(b)$ and thus $b \in P_0$, which is impossible because $b \in I = P_0^c$. Hence $P_0 \in v$ and therefore $P \in u \Rightarrow v$. $\square$

In order to prove the inverse inclusion, that is, to prove that $u \Rightarrow v \subseteq u \to^\pi v$, we need the following about Hilbert algebras. Let A be a Hilbert algebra. We define inductively for $b \in A$ and for every sequence $a_0, \ldots, a_n$ of elements of A the element $(a_0, \ldots, a_n; b) \in A$ as follows:

$$(a_0; b) = a_0 \to b \quad \text{and} \quad (a_{n+1}, a_n, \ldots, a_0; b) = a_{n+1} \to (a_n, \ldots, a_0; b).$$

LEMMA 6.23 [7]. *Let A be a Hilbert algebra and let $a_0, \ldots, a_n, b \in A$.*
(1) $(a_n, \ldots, a_0; b) = (a_{\pi(n)}, \ldots, a_{\pi(0)}; b)$, *for every permutation π of $\{0, \ldots, n\}$.*
(2) $(a_n, \ldots, a_0; b) = (a_n, \ldots, a_m; (a_{m-1}, \ldots, a_0; b))$.

LEMMA 6.24 [7]. *Let F be an implicative filter.*
(1) *If $(a_n, \ldots, a_m, a_{m-1}, \ldots, a_0; b) \in F$ and $a_n, \ldots, a_m \in F$, then we obtain that $(a_{m-1}, \ldots, a_0; b) \in F$.*
(2) *If $(a_n, \ldots, a_0; b) \in F$ and $a_0, \ldots, a_n \in F$, then $b \in F$.*

Let A be a Hilbert algebra. Recall that $\mathsf{Fi}_{\to}(A)$ is a lattice. The join operation of $\mathsf{Fi}_{\to}(A)$ can be characterised as follows: let $F, G \in \mathsf{Fi}_{\to}(A)$, then

$$F \vee G = \{b \in A : (a_n, \ldots, a_0; b) = 1 \text{ for some } a_0, \ldots, a_n \in F \cup G\}.$$

Now we are ready to prove the inverse inclusion of Proposition 6.22.

PROPOSITION 6.25. *Let A be a Hilbert algebra and let $u, v \in A^* = \mathsf{Up}(\mathsf{Pr}(A))$. Then $u \Rightarrow v \subseteq u \to^{\pi} v$.*

PROOF. Recall that $u \Rightarrow v = \{P \in \mathsf{Pr}(A) : u \cap {\uparrow}P \subseteq v\}$. Let $P \in u \Rightarrow v$. So $u \cap {\uparrow}P \subseteq v$. Let $F \in \mathsf{Fi}_{\to}(A)$ and $I \in \mathcal{I}_u$ be such that $\bigcap \alpha[F] \subseteq u$ and $v \subseteq \bigcup \alpha[I]$. Then, $\bigcap \alpha[F] \cap {\uparrow}P \subseteq u \cap {\uparrow}P \subseteq v \subseteq \bigcup \alpha[I]$. Notice that ${\uparrow}P = \bigcap \alpha[P]$. Thus $\bigcap \alpha[F \vee P] = \bigcap \alpha[F] \cap \bigcap \alpha[P] \subseteq \bigcup \alpha[I]$. As $F \vee P \in \mathsf{Fi}_{\to}(A)$ and $I \in \mathcal{I}_u$, by the $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$-compactness, we have that there exists $b \in (F \vee P) \cap I$. Since $b \in F \vee P$, it follows that there are $a_0, \ldots, a_n \in F \cup P$ such that $(a_n, \ldots, a_0; b) = 1$. Let $X := \{a_0, \ldots, a_n\} \cap F$ and $Y := \{a_0, \ldots, a_n\} \cap P$. Without loss of generality, we can assume that $X \neq \emptyset$ and $Y \neq \emptyset$. Suppose that $X = \{a'_1, \ldots, a'_m\}$ and $Y = \{a''_1, \ldots, a''_k\}$. So, we have $(a'_1, \ldots, a'_m, a''_1, \ldots, a''_k; b) = 1$. Since $a'_1 \ldots, a'_m \in F$, it follows by Lemma 6.24 that $a := (a''_1, \ldots, a''_k; b) \in F$. Then, we obtain that $a \in F$ and $b \in I$. Now we show that $P \in \alpha(a) \Rightarrow \alpha(b)$. That is, we prove that $\alpha(a) \cap {\uparrow}P \subseteq \alpha(b)$. Let $Q \in \alpha(a) \cap {\uparrow}P$. Thus $a \in Q$ and $P \subseteq Q$. So $(a''_1, \ldots, a''_k; b) \in Q$ and $a''_1, \ldots, a''_k \in Q$. Then $b \in Q$, that is, $Q \in \alpha(b)$. Hence $\alpha(a) \cap {\uparrow}P \subseteq \alpha(b)$. Therefore, by (6.2), we have proved that $P \in u \to^{\pi} v$. $\square$

COROLLARY 6.26. *Let $\langle A, \to, 1 \rangle$ be a Hilbert algebra. Then, it follows that the $\langle \mathsf{Fi}_{\to}(A), \mathcal{I}_u \rangle$-completion of A with the π-extension of $\to$, $\langle A^*, \cap, \cup, \to^{\pi}, 0_{A^*}, 1_{A^*} \rangle$, is a complete Heyting algebra.*

6.3. An $\langle \mathcal{F}, \mathcal{I} \rangle$-completion for filter distributive finitary congruential logics. The aim of this section is to use the results of Section 4 to obtain an adequate $\langle \mathcal{F}, \mathcal{I}\rangle$-completion for the algebras that are canonically associated (in the sense of abstract algebraic logic) with every filter distributive finitary congruential logic. This completion, called the $\mathcal{S}$-canonical extension, was already obtained by Gehrke et al. in [19] following a different approach. We will see that our approach may be considered more straightforward than that in [19].

Let $\mathcal{S}$ be a finitary congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$. The *$\mathcal{S}$-canonical extension* of A ([19, Definition 4.17]) is defined as the canonical extension (in the sense of [12]) of the meet-semilattice $L^{\wedge}_{\mathcal{S}}(A)$. That is, the $\mathcal{S}$-canonical extension A is the $\langle \mathcal{F}, \mathcal{I}\rangle$-completion of the meet-semilattice $L^{\wedge}_{\mathcal{S}}(A)$ where $\mathcal{F}$ is the collection of filters of $L^{\wedge}_{\mathcal{S}}(A)$ and $\mathcal{I}$ is the family of nonempty up-directed downsets of $L^{\wedge}_{\mathcal{S}}(A)$. The meet-semilattice $L^{\wedge}_{\mathcal{S}}(A)$ can be consider (see [19, p. 1510]) as the meet-semilattice $\langle \mathsf{Fi}^{\mathrm{f}}_{\mathcal{S}}(A), \supseteq\rangle$, where $\mathsf{Fi}^{\mathrm{f}}_{\mathcal{S}}(A)$ is the collection of all finitely generated $\mathcal{S}$-filters of A. Then, in [19] (see Theorem 4.20) the authors prove, under the condition that the $\mathcal{S}$-canonical extension satisfies a distributive law, that the $\mathcal{S}$-canonical extension of A is isomorphic to the $\langle \mathsf{Fi}_{\mathcal{S}}(A), \mathcal{I}\rangle$-completion of $\langle A, \leq^{A}_{\mathcal{S}}\rangle$ such that $\mathcal{I}$ is the set of the *nonempty up-directed $\mathcal{S}$-ideals*. In the Preliminaries of abstract algebraic logic we will see that the set of nonempty up-directed $\mathcal{S}$-ideals is exactly the set of nonempty up-directed downsets $\mathcal{I}_u$ of $\langle A, \leq^{A}_{\mathcal{S}}\rangle$. Hence, the $\mathcal{S}$-canonical extension of A is isomorphic to the $\langle \mathsf{Fi}_{\mathcal{S}}(A), \mathcal{I}_u\rangle$-completion of $\langle A, \leq^{A}_{\mathcal{S}}\rangle$.

6.3.1. Preliminaries of abstract algebraic logic. Our main references for the preliminaries on Abstract Algebraic Logic are [14–16].

Let $\mathcal{L}$ be a propositional language (or algebraic language). Let us denote by $\mathrm{Fm}(\mathcal{L})$ the algebra of formulas (or term algebra) of $\mathcal{L}$ over a denumerable set of variables Var, that is, $\mathrm{Fm}(\mathcal{L})$ is the absolutely free $\mathcal{L}$-algebra over Var. A *sentential logic* of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathrm{Fm}(\mathcal{L}, \vdash_{\mathcal{S}}\rangle$ where the *consequence relation* $\vdash_{\mathcal{S}}$ is a relation between subsets of $\mathrm{Fm}(\mathcal{L})$ and elements of $\mathrm{Fm}(\mathcal{L})$ such that the operator $C_{\mathcal{S}} \colon \mathcal{P}(\mathrm{Fm}(\mathcal{L})) \to \mathcal{P}(\mathrm{Fm}(\mathcal{L}))$ defined by

$$\varphi \in C_{\mathcal{S}}(\Gamma) \quad \text{if and only if} \quad \Gamma \vdash_{\mathcal{S}} \phi$$

is a closure operator with the property of invariance under substitutions. A logic $\mathcal{S}$ is *finitary* if the consequence operation $C_{\mathcal{S}}$ is finitary.

From now on, we fix an arbitrary propositional language $\mathcal{L}$. When there is no danger of confusion we omit the reference to the language $\mathcal{L}$. Thus, we write Fm instead of $\mathrm{Fm}(\mathcal{L})$, the sentential logics are consider over $\mathcal{L}$ and the algebras are also of the same type $\mathcal{L}$.

Let $\mathcal{S}$ be a sentential logic and A an algebra. A subset $F \subseteq A$ is called an *$\mathcal{S}$-filter* of A if for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ and every homomorphism $h \colon \mathrm{Fm} \to A$,

$$\text{if } \ \Gamma \vdash_{\mathcal{S}} \varphi \ \text{ and } \ h[\Gamma] \subseteq F, \ \text{ then } \ h(\varphi) \in F.$$

Let us denote by $\mathsf{Fi}_{\mathcal{S}}(A)$ the collection of all $\mathcal{S}$-filters of A. It is known that $\mathsf{Fi}_{\mathcal{S}}(A)$ is a closure system, and if the logic S is finitary, then $\mathsf{Fi}_{\mathcal{S}}(A)$ is an algebraic closure system. We denote the closure operator associated with $\mathsf{Fi}_{\mathcal{S}}(A)$ by $C^A_{\mathcal{S}}$. The $\mathcal{S}$-*specialization quasi-order* of A is the binary relation $\leq^A_{\mathcal{S}}$ on A defined by

$$a \leq^A_{\mathcal{S}} b \quad \Longleftrightarrow \quad C^A_{\mathcal{S}}(b) \subseteq C^A_{\mathcal{S}}(a).$$

DEFINITION 6.27. A logic $\mathcal{S}$ is said to be *filter distributive* when for all algebra A, the lattice $\mathsf{Fi}_{\mathcal{S}}(A)$ is distributive.

We consider the following equivalence relation $\equiv^A_{\mathcal{S}}$ on A defined as follows:

$$a \equiv^A_{\mathcal{S}} b \quad \Longleftrightarrow \quad C^A_{\mathcal{S}}(a) = C^A_{\mathcal{S}}(b).$$

The relation $\equiv^A_{\mathcal{S}}$ is not in general a congruence on the algebra A.

DEFINITION 6.28. A sentential logic $\mathcal{S}$ is called *congruential* if for every algebra A, the relation $\equiv^A_{\mathcal{S}}$ is a congruence on A.

The class of algebras $\mathrm{Alg}(\mathcal{S})$ associated with a sentential logic $\mathcal{S}$ is defined as follows:

$$\mathrm{Alg}(\mathcal{S}) := \left\{ A : (\forall \theta \in \mathrm{Con}(A))(\text{if } \theta \subseteq \equiv^A_{\mathcal{S}},\ \text{then } \theta = \Delta_A) \right\}.$$

In particular, if $\mathcal{S}$ is congruential, $\mathrm{Alg}(\mathcal{S})$ can be equivalently defined as

$$\mathrm{Alg}(\mathcal{S}) = \left\{ A : \equiv^A_{\mathcal{S}} = \Delta_A \right\}.$$

In the theory of abstract algebraic logic, the class $\mathrm{Alg}(\mathcal{S})$ is consider as the canonical algebraic counterpart of the logic $\mathcal{S}$.

PROPOSITION 6.29. *If $\mathcal{S}$ is a congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$, then $\leq^A_{\mathcal{S}}$ is a partial order on A.*

Let $\mathcal{S}$ be a finitary congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$. Thus we have $\langle A, \leq^A_{\mathcal{S}} \rangle$ is a poset. Then, every $\mathcal{S}$-filter of A is an upset of $\langle A, \leq^A_{\mathcal{S}} \rangle$, and for every $a \in A$, $C^A_{\mathcal{S}}(a) = {\uparrow}a = \{x \in A : a \leq^A_{\mathcal{S}} x\}$. A subset $I \subseteq A$ is said to be an $\mathcal{S}$-*ideal* of A provided for all $X \subseteq_\omega I$ and all $a \in A$

$$\text{if } \bigcap_{x \in X} C^A_{\mathcal{S}}(x) \subseteq C^A_{\mathcal{S}}(a), \ \text{ then } \ a \in I.$$

Equivalently, I is an $\mathcal{S}$-ideal of A if for all $X \subseteq_\omega I$ and all $a \in A$,

$$\text{if } \bigcap_{x \in X} {\uparrow}x \subseteq {\uparrow}a, \ \text{ then } \ a \in I^2. \tag{6.3}$$

[2] A subset I of a poset P satisfying condition (6.3) is known as a Frink ideal ([17]) of P.

Thus, it is clear that every $\mathcal{S}$-ideal of A is a downset of $\langle A, \leq_{\mathcal{S}}^{A}\rangle$, and for every $a \in A$, $\downarrow a = \{x \in A : x \leq_{\mathcal{S}}^{A} a\}$ is an $\mathcal{S}$-ideal of A.

LEMMA 6.30. *Let $\mathcal{S}$ be a finitary congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$. If $I \subseteq A$ is a nonempty up-directed downset of $\langle A, \leq_{\mathcal{S}}^{A}\rangle$, then I is an $\mathcal{S}$-ideal of A.*

COROLLARY 6.31. *Let $I \subseteq A$. Then, I is a nonempty up-directed $\mathcal{S}$-ideal of A if and only if it is a nonempty up-directed downset of A.*

6.3.2. The $\mathcal{S}$-canonical extension for filter distributive finitary congruential logics. Let $\mathcal{S}$ be a finitary congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$. We thus know that $\mathsf{Fi}_{\mathcal{S}}(A)$ is a standard algebraic closure system of upsets of the poset $\langle A, \leq_{\mathcal{S}}^{A}\rangle$. Let $\mathcal{I}_u$ be the collection of all nonempty up-directed downsets of $\langle A, \leq_{\mathcal{S}}^{A}\rangle$. We denote by $\mathsf{Pr}_{\mathcal{S}}(A)$ the collection of all meet-prime elements of the lattice $\mathsf{Fi}_{\mathcal{S}}(A)$.

PROPOSITION 6.32. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic $\mathcal{S}$ and $A \in \mathrm{Alg}(\mathcal{S})$. If $F \in \mathsf{Fi}_{\mathcal{S}}(A)$ and $I \in \mathcal{I}_u$ such that $F \cap I = \emptyset$, then there is $P \in \mathsf{Pr}_{\mathcal{S}}(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

PROOF. It is a direct consequence of Corollary 4.6. $\square$

THEOREM 6.33. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic $\mathcal{S}$ and $A \in \mathrm{Alg}(\mathcal{S})$. Then, the $\langle \mathsf{Fi}_{\mathcal{S}}(A), \mathcal{I}_u\rangle$-completion of the poset $\langle A, \leq_{\mathcal{S}}^{A}\rangle$ is the completely distributive algebraic lattice $\mathsf{Up}(\mathsf{Pr}_{\mathcal{S}}(A))$ with the order embedding $\alpha\colon A \to \mathsf{Up}(\mathsf{Pr}_{\mathcal{S}}(A))$ defined as $\alpha(a) = \{P \in \mathsf{Pr}_{\mathcal{S}}(A) : a \in P\}$.*

PROOF. It is a consequence of Theorem 4.11. $\square$

Let $\mathcal{S}$ be a filter distributive finitary congruential logic and $A \in \mathrm{Alg}(\mathcal{S})$. By [19], the $\mathcal{S}$-*canonical extension* of A, denoted by $A^{\mathcal{S}}$, is defined as the canonical extension (in the sense of [12]) of the meet-semilattice $\langle \mathsf{Fi}_{\mathcal{S}}^{\mathrm{f}}(A), \supseteq\rangle$, where $\mathsf{Fi}_{\mathcal{S}}^{\mathrm{f}}(A)$ is the collection of all finitely generated $\mathcal{S}$-filters of A. Since $\mathsf{Fi}_{\mathcal{S}}(A)$ is a distributive lattice, it follows that $\langle \mathsf{Fi}_{\mathcal{S}}^{\mathrm{f}}(A), \supseteq\rangle$ is a distributive meet-semilattice. Hence, by Corollary 4.21, the canonical extension $A^{\mathcal{S}}$ of $\langle \mathsf{Fi}_{\mathcal{S}}^{\mathrm{f}}(A), \supseteq\rangle$ is completely distributive. Thus, in particular, $A^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law

$$a \vee \bigwedge X = \bigwedge_{x \in X} a \vee x.$$

Then, by Theorem 4.20 in [19], $A^{\mathcal{S}}$ is the $\langle \mathsf{Fi}_{\mathcal{S}}(A), \mathcal{I}\rangle$-completion of $\langle A, \leq_{\mathcal{S}}^{A}\rangle$, where $\mathcal{I}$ is the collection of the nonempty up-directed $\mathcal{S}$-ideals of A. Hence, by Corollary 6.31, we have that $A^{\mathcal{S}}$ is the $\langle \mathsf{Fi}_{\mathcal{S}}(A), \mathcal{I}_u\rangle$-completion of $\langle A, \leq_{\mathcal{S}}^{A}\rangle$. Therefore, by Theorem 6.33, we obtain that $A^{\mathcal{S}} \cong \mathsf{Up}(\mathsf{Pr}_{\mathcal{S}}(A))$.

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