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Congruences on near-Heyting algebras

Luciano J. González and Marina B. Lattanzi

Abstract. A near-Heyting algebra is a join-semilattice with a top element such that every principal upset is a Heyting algebra. We establish a one-to-one correspondence between the lattices of filters and congruences of a near-Heyting algebra. To attain this aim, we first show an embedding from the lattice of filters to the lattice of congruences of a distributive nearlattice. Then, we describe the subdirectly irreducible and simple near-Heyting algebras. Finally, we fully characterize the principal congruences of distributive nearlattices and near-Heyting algebras. We conclude that the varieties of distributive nearlattices and near-Heyting algebras have equationally definable principal congruences.

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1. Introduction

A distributive nearlattice (DN-algebra for short) is a join-semilattice with a top element such that every principal upset is a distributive lattice with respect to the order induced by the join operation. Thus, DN-algebras is a generalisation of semi-boolean algebras introduced by Abbott [1]. DN-algebras are polynomially equivalent to algebras with only one ternary connective satisfying some identities [11]; the variety of DN-algebras was studied in several papers [9,10,7,5,8,15]. Near-Heyting algebras are DN-algebras such that every principal upset is a pseudocomplemented distributive lattice. Near-Heyting algebras were introduced by Chajda and Kolařík in [11] and they proved that

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these algebras are polynomially equivalent to algebras of type (3,2,0) satisfying some identities. We show that near-Heyting algebras can be equivalently defined as DN-algebras in which every principal upset is a Heyting algebra.

The purpose of this work is to study the lattice of congruences of near-Heyting algebras and obtain some consequences for this variety.

2. Distributive nearlattices

In this section, we present the basic facts on distributive nearlattices. Our main references for the theory of nearlattices are [9,11,2,19,4]. We assume that the reader is familiar with elementary order and lattice theoretical notions [13,17,3].

Definition 2.1. A *distributive nearlattice* (*DN-algebra* for short) is an algebra $\langle A, m, 1\rangle$ of type (3,0) satisfying the following identities:

(N1) $m(x, y, x) = x$;
(N2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$;
(N3) $m(x, x, 1) = 1$;
(N4) $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

Let us denote by $\mathbb{DN}$ the variety of DN-algebras.

Theorem 2.2 [11]. *Let $\langle A, m, 1\rangle$ be an algebra of type (3,0) and let $\vee$ be the binary operation on A defined by $x \vee y := m(x, x, y)$. Then, $\langle A, m, 1\rangle$ is a DN-algebra if and only if $\langle A, \vee, 1\rangle$ is a join-semilattice with top element such that for every $a \in A$, the principal upset $[a) = \{ x \in A : a \leq x \}$ is a bounded distributive lattice with respect to the order induced by $\vee$, and $m(x, y, a) = (x \vee a) \wedge_a (y \vee a)$.*

For every DN-algebra $\langle A, m, 1\rangle$, we will consider the join operation $\vee$ on A as defined in the previous theorem and the partial order $\leq$ on A induced by $\vee$, that is, $x \leq y$ if and only if $x \vee y = y$. For every element $a \in A$, we denote the meet in $[a)$ by $\wedge_a$. Notice that the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A. Hence, $x \wedge y = x \wedge_a y$ for all $x, y \in [a)$. For each $X \subseteq A$, we denote by $[X)$ the set of all $a \in A$ such that $a \geq x$ for some $x \in X$. We say that a subset $X \subseteq A$ is an *upset* if $X = [X)$.

Let A be a DN-algebra. A subset $F \subseteq A$ is called a *filter* of A if: (i) $1 \in F$, (ii) F is an upset and (iii) if $a, b \in F$ and $a \wedge b$ exists in A, then $a \wedge b \in F$. Let us denote by $\mathsf{Fi}(A)$ the set of all filters of A. Arbitrary intersection of filters of A is again a filter of A. Thus, $\mathsf{Fi}(A)$ is a closure system. Let $\langle . \rangle$ be the closure operator associated with $\mathsf{Fi}(A)$. Thus

$$\langle X\rangle = \{ a \in A : \text{ there exist } a_1, \dots, a_n \in [X) \text{ such that } a = a_1 \wedge \cdots \wedge a_n \},$$

for any set $X \subseteq A$. Hence, $\langle \mathsf{Fi}(A), \cap, \vee\rangle$ is a complete lattice where for every $F, G \in \mathsf{Fi}(A)$,

$$F \vee G = \langle F \cup G\rangle = \{ a \in A : a = x \wedge y \text{ for some } x \in F, y \in G \}$$

(see [9, pp. 38]). For $X \in A$ and $a \in A$, the filter $\langle X \cup \{a\}\rangle$ is denoted by $\langle X, a\rangle$.

A proper filter P of A is said to be *prime* if $a \in P$ or $b \in P$ whenever $a \vee b \in P$.

Lemma 2.3 [12]. *Let A be a DN-algebra. Then $\mathsf{Fi}(A)$ is a distributive lattice.*

Lemma 2.4 [15]. *Let A be a DN-algebra and let $a, x_1, \ldots, x_n \in A$. Then, $a \in \langle \{x_1, \ldots, x_n\} \rangle$ if and only if $a = (x_1 \vee a) \wedge \cdots \wedge (x_n \vee a)$.*

Proposition 2.5 [18]. *Let A be a DN-algebra. Let F be a filter of A and $a \in A$. If $a \notin F$, then there exists a prime filter P of A such that $F \subseteq P$ and $a \notin P$.*

3. Near-Heyting algebras

Definition 3.1. An algebra $\langle A, m, n, 1 \rangle$ of type (3,2,0) is said to be a *near-Heyting algebra* if $\langle A, m, 1 \rangle$ is a DN-algebra and the following identities hold:

(NH1) $y \leq n(x, y)$;
(NH2) $n(x, x) = 1$;
(NH3) $n(1, x) = x$;
(NH4) $m(x, n(m(x, y, z), z), z) = m(x, n(y, z), z)$.

We denote by $\mathbb{NHA}$ the variety of near-Heyting algebras.

Proposition 3.2 [9, Theorem 5.5.1]. *Assume that $\langle A, m, n, 1 \rangle$ is an algebra of type* (3,2,0) *such that $\langle A, m, 1 \rangle$ is a DN-algebra. Then $\langle A, m, n, 1 \rangle$ is a near-Heyting algebra if and only if for every $a \in A$, the principal upset $[a)$ is a bounded distributive pseudocomplemented lattice and $n(x, a)$ is the pseudocomplement of $x \vee a$ in $[a)$.*

Near-Heyting algebras are called *sectionally pseudocomplemented nearlattices* in [11,9]. The name "near-Heyting algebra" is justified by the following result.

Proposition 3.3. *Assume that $\langle A, m, n, 1 \rangle$ is an algebra of type* (3,2,0) *such that $\langle A, m, 1 \rangle$ is a DN-algebra. Then $\langle A, m, n, 1 \rangle$ is a near-Heyting algebra if and only if for every $a \in A$, the principal upset $[a)$ is a Heyting algebra under the operations induced by the partial ordering of A and the Heyting implication $\rightarrow_a$ in $[a)$ is given by: $x \rightarrow_a y := n(x, x \wedge_a y)$ for all $x, y \in [a)$.*

Proof. It follows from Proposition 3.2 and [3, Theorem IX.2.8]. □

Recall that a join-semilattice $\langle A, \vee, 1 \rangle$ is called a *semi-boolean algebra* [1] if for every $a \in A$, $[a)$ is a Boolean algebra. It was proved in [1] that semi-boolean algebras are polynomially equivalent to implication algebras (also called *Tarski algebras*).

Proposition 3.4. *Let $\langle A, m, n, 1 \rangle$ be a near-Heyting algebra. Then, $\langle A, \vee, 1 \rangle$ is a semi-boolean algebra if and only if the identity $(x \vee z) \vee n(x, z) = 1$ holds.*

Proof. It is a consequence of Proposition 3.2 and [3, pp. 155]. □

4. Congruences on near-Heyting algebras

For an algebra A, we denote by $\mathsf{Con}(A)$ the lattice of its congruences. The primary purpose of this section is to establish an isomorphism between the lattice of filters of a near-Heyting algebra and the lattice of its congruences. To attain this, we first show an embedding from the lattice of filters of a DN-algebra into the lattice of its congruences. Finally, we characterize the subdirectly irreducible elements of $\mathbb{NHA}$.

Lemma 4.1 [19]. *Let $\langle A, m, 1\rangle$ be a DN-algebra. An equivalence relation θ is a congruence on A if and only if θ is a congruence of $\langle A, \vee\rangle$ and $(a \wedge c)\ \theta\ (b \wedge d)$ whenever $a \wedge c$, $b \wedge d$ exist in A and $(a, b), (c, d) \in \theta$.*

Let $\langle A, m, 1\rangle$ be a DN-algebra. Let $F \in \mathsf{Fi}(A)$. We define the binary relation $\theta_F \subseteq A \times A$ as follows:

$$a\ \theta_F\ b \iff \text{there is a finite set } X \subseteq F\ (\langle X, a\rangle = \langle X, b\rangle). \tag{C}$$

Proposition 4.2. *For every DN-algebra A, the map $\Gamma\colon \mathsf{Fi}(A) \to \mathsf{Con}(A)$ defined by $\Gamma(F) = \theta_F$ is a lattice embedding and $1/\theta_F = F$.*

Proof. Let $F \in \mathsf{Fi}(A)$ and θ_F be defined by (C). It is clear that θ_F is a reflexive and symmetric relation. In order to prove transitivity, assume that $a\ \theta_F\ b$ and $b\ \theta_F\ c$. Then, there are finite sets $X, Y \subseteq F$ such that $\langle X, a\rangle = \langle X, b\rangle$ and $\langle Y, b\rangle = \langle Y, c\rangle$. Thus $\langle X \cup Y, a\rangle = \langle X \cup Y, b\rangle = \langle X \cup Y, c\rangle$ and since $X \cup Y$ is a finite subset of F, it follows that $a\ \theta_F\ c$.

Now let us show that θ_F is a congruence. Assume that $a\ \theta_F\ b$ and $c\ \theta_F\ d$. Without loss of generality we can assume that there is a finite $Z \subseteq F$ such that $\langle Z\rangle \vee [a) = \langle Z\rangle \vee [b)$ and $\langle Z\rangle \vee [c) = \langle Z\rangle \vee [d)$. Then

$$(\langle Z\rangle \vee [a)) \cap (\langle Z\rangle \vee [c)) = (\langle Z\rangle \vee [b)) \cap (\langle Z\rangle \vee [d)).$$

Since $\mathsf{Fi}(A)$ is a distributive lattice, it follows that $\langle Z\rangle \vee [a \vee c) = \langle Z\rangle \vee [b \vee d)$. Hence, $(a \vee c)\ \theta_F\ (b \vee d)$. Now, suppose that $a \wedge c$ and $b \wedge d$ exist in A. Notice that $\langle Z\rangle \vee [a) \vee [c) = \langle Z\rangle \vee [b) \vee [d)$. Thus, $\langle Z\rangle \vee [a \wedge c) = \langle Z\rangle \vee [b \wedge d)$. Hence $(a \wedge c)\ \theta_F\ (b \wedge d)$. Therefore, by Lemma 4.1, θ_F is a congruence on A. It is straightforward to show that $1/\theta_F = F$ and $F \subseteq G$ if and only if $\theta_F \subseteq \theta_G$ for every $F, G \in \mathsf{Fi}(A)$. Next, we prove that Γ is a lattice homomorphism. Let $F, G \in \mathsf{Fi}(A)$. First we prove $\theta_{F\cap G} = \theta_F \cap \theta_G$. Let $x, y \in A$. If $(x, y) \in \theta_{F\cap G}$, then there exists a finite set $Z \subseteq F \cap G$ such that $\langle Z, x\rangle = \langle Z, y\rangle$. Since $F \cap G$ is a subset of F and also of G, it is clear that $(x, y) \in \theta_F$ and $(x, y) \in \theta_G$, hence $\theta_{F\cap G} \subseteq \theta_F \cap \theta_G$. Suppose now that $(x, y) \in \theta_F \cap \theta_G$. Then, there exist finite sets $X \subseteq F$ and $Y \subseteq G$ such that $\langle X, x\rangle = \langle X, y\rangle$ and $\langle Y, x\rangle = \langle Y, y\rangle$. Let $Z = \{\, f \vee g : f \in X \text{ and } g \in Y \,\}$. We assert $\langle Z, x\rangle = \langle X, x\rangle \cap \langle Y, x\rangle$. Indeed, since $Z \subseteq \langle X, x\rangle \cap \langle Y, x\rangle$, it follows that $\langle Z, x\rangle \subseteq \langle X, x\rangle \cap \langle Y, x\rangle$. Conversely, let $a \in \langle X, x\rangle \cap \langle Y, x\rangle$ and suppose $X = \{f_1, \ldots, f_k\}$ and $Y = \{g_1, \ldots, g_r\}$; then by Lemma 2.4 we have

$$a = (f_1 \vee a) \wedge \cdots \wedge (f_k \vee a) \wedge (x \vee a) = (g_1 \vee a) \wedge \cdots \wedge (g_r \vee a) \wedge (x \vee a).$$

Thus, since $[a)$ is a distributive lattice, we obtain

$$a = a \vee a = \bigwedge_{\substack{1\leq i\leq k\\ 1\leq j\leq r}} (f_i \vee g_j \vee a) \wedge (x \vee a).$$

Then $a \in \langle Z, x\rangle$ and so $\langle X, x\rangle \cap \langle Y, x\rangle \subseteq \langle Z, x\rangle$. In a similar way we get $\langle Z, y\rangle = \langle X, y\rangle \cap \langle Y, y\rangle$. Hence, there exists a finite set $Z \subseteq F \cap G$ such that $\langle Z, x\rangle = \langle X, x\rangle \cap \langle Y, x\rangle = \langle X, y\rangle \cap \langle Y, y\rangle = \langle Z, y\rangle$, therefore $(x, y) \in \theta_{F\cap G}$.

Now, we prove that $\theta_{F\vee G} = \theta_F \vee \theta_G$. It is clear that $\theta_F \subseteq \theta_{F\vee G}$ and $\theta_G \subseteq \theta_{F\vee G}$; we will show that $\theta_{F\vee G}$ is the least upper bound of $\{\theta_F, \theta_G\}$. Let $\theta \in \mathsf{Con}(A)$ be such that $\theta_F \subseteq \theta$ and $\theta_G \subseteq \theta$ and let $(a, b) \in \theta_{F\vee G}$. Then there exists a finite set $H = \{h_1, \ldots, h_m\} \subseteq F \vee G$ such that $\langle H, a\rangle = \langle H, b\rangle$. For each $1 \leq i \leq m$, $h_i = f_i \wedge g_i$ for some $f_i \in F$ and $g_i \in G$. As $\langle H, a\rangle = [h_1) \vee \ldots [h_m) \vee [a)$, for each $1 \leq i \leq m$, we can write

$$[f_i) \cap ([h_1) \vee \ldots [h_m) \vee [a)) = [f_i) \cap ([h_1) \vee \ldots [h_m) \vee [b))$$

and since $\mathsf{Fi}(A)$ is a distributive lattice, we obtain

$$\begin{aligned}[f_i) \cap ([h_1) \vee \ldots [h_m) \vee [a)) &= ([f_i) \cap [h_1)) \vee \cdots \vee ([f_i) \cap [h_m)) \vee ([f_i) \cap [a))\\ &= [f_i \vee h_1) \vee \cdots \vee [f_i \vee h_m) \vee [f_i \vee a)\\ &= \langle \{f_i \vee h_1, \ldots, f_i \vee h_m\}, f_i \vee a\rangle.\end{aligned}$$

Hence,

$$\langle \{f_i \vee h_1, \ldots, f_i \vee h_m\}, f_i \vee a\rangle = \langle \{f_i \vee h_1, \ldots, f_i \vee h_m\}, f_i \vee b\rangle$$

for each $1 \leq i \leq m$. Since $f_i \vee h_j \in F$ for all $1 \leq i, j \leq m$, we have $(f_i \vee a, f_i \vee b) \in \theta_F \subseteq \theta$, and analogously we obtain $(g_i \vee a, g_i \vee b) \in \theta_G \subseteq \theta$ for every $1 \leq i \leq m$. So

$$h_i \vee a = (f_i \vee a) \wedge (g_i \vee a)\ \theta\ (f_i \vee b) \wedge (g_i \vee b) = h_i \vee b$$

for every $1 \leq i \leq m$ and thus

$$(h_1 \vee a) \wedge \cdots \wedge (h_m \vee a)\ \theta\ (h_1 \vee b) \wedge \cdots \wedge (h_m \vee b).$$

Now, since $\langle H, a\rangle = \langle H, b\rangle$, it follows by Lemma 2.4 that

$$\begin{aligned} a &= (h_1 \vee a) \wedge \cdots \wedge (h_m \vee a) \wedge (b \vee a) \text{ and}\\ b &= (h_1 \vee b) \wedge \cdots \wedge (h_m \vee b) \wedge (a \vee b).\end{aligned}$$

Hence, we obtain $a\ \theta\ b$. Therefore $\theta_{F\vee G} \subseteq \theta$. This completes the proof. □

Let us note that the lattice embedding Γ is not, in general, an isomorphism. Next, we show that in the setting of near-Heyting algebras the congruences are in a one-to-one correspondence with the filters. We start with the following results.

Proposition 4.3 [6]. *Let $\langle H, \wedge, \vee, \rightarrow, 0, 1\rangle$ be a Heyting algebra and F a filter of H. Then, $a \rightarrow b \notin F$ if and only if there exists a prime filter P of H such that $F \subseteq P$, $a \in P$ and $b \notin P$.*

Next, we prove in the setting of near-Heyting algebras a similar result to the previous one that will be very useful in what follows. Notice that the notions of filter and prime filter on near-Heyting algebras are those given for DN-algebras.

Proposition 4.4. *Let $\langle A, m, n, 1\rangle$ be a near-Heyting algebra. Let $F \in \mathsf{Fi}(A)$ and $a, c \in A$. If $n(c, a) \notin F$, then there exists a prime filter P of A such that $F \subseteq P$, $c \in P$ and $a \notin P$.*

Proof. Suppose that $n(c, a) \notin F$. So, it is clear that $n(c, a) \notin F \cap [a)$ and $F \cap [a) \in \mathsf{Fi}([a))$. By Propositions 3.2 and 3.3, we have $n(c, a) = (c \vee a) \to_a a$. Since $(c \vee a) \to_a a \notin F \cap [a) \in \mathsf{Fi}([a))$ and $\langle [a), \wedge_a, \vee, \to_a, a, 1\rangle$ is a Heyting algebra, it follows by Proposition 4.3 that there is a prime filter Q of $[a)$ such that $F \cap [a) \subseteq Q$, $c \vee a \in Q$ and $a \notin Q$. It is clear that Q is in fact a filter of A. Then, by Proposition 2.5, there is a prime filter P of A such that $Q \subseteq P$ and $a \notin P$. Thus, we have $F \cap [a) \subseteq P$. Let us show that $F \subseteq P$. Let $x \in F$. So $x \vee a \in F \cap [a)$ and then $x \vee a \in P$. Thus, since P is prime and $a \notin P$, it follows that $x \in P$. Similarly, since $c \vee a \in P$, it follows that $c \in P$. Hence, we have that P is a prime filter of A such that $F \subseteq P$, $c \in P$ and $a \notin P$. □

Proposition 4.5. *Let $\langle A, m, n, 1\rangle$ be a near-Heyting algebra. Let $X \subseteq A$ be finite and $a, b, c \in A$. Then $\langle X, n(c, a)\rangle = \langle X, n(c, b)\rangle$ whenever $\langle X, a\rangle = \langle X, b\rangle$.*

Proof. Let $a, b, c \in A$ and assume that $\langle X, a\rangle = \langle X, b\rangle$. Let us prove that $n(c, a) \in \langle X, n(c, b)\rangle$. Suppose that $n(c, a) \notin \langle X, n(c, b)\rangle$. By Proposition 4.4, there is a prime filter P of A such that $\langle X, n(c, b)\rangle \subseteq P$, $c \in P$ and $a \notin P$. Notice that $X \subseteq P$ and $n(c, b) \in P$. If $b \in P$, then $X \cup \{b\} \subseteq P$ and thus $\langle X, b\rangle \subseteq P$; by hypotheses, this implies that $\langle X, a\rangle \subseteq P$ and then $a \in P$, which is a contradiction. Hence $b \notin P$. Since $n(c, b) \wedge (c \vee b) = b \notin P$ and $n(c, b) \in P$, it follows that $c \vee b \notin P$. Thus $c \notin P$ and this is a contradiction. Hence $n(c, a) \in \langle X, n(c, b)\rangle$. By an analogous argument we have $n(c, b) \in \langle X, n(c, a)\rangle$. Therefore $\langle X, n(c, a)\rangle = \langle X, n(c, b)\rangle$. □

Theorem 4.6. *Let $\langle A, m, n, 1\rangle$ be a near-Heyting algebra and $F \in \mathsf{Fi}(A)$. Then, the binary relation θ_F defined by (C) is a congruence of A.*

Proof. By Proposition 4.2, we only need to prove that θ_F is a congruence with respect to the operation n. Let $a, b, c, d \in A$ and suppose that $a\ \theta_F\ b$ and $c\ \theta_F\ d$. Let us show that (1) $n(a, c)\ \theta_F\ n(b, c)$ and (2) $n(b, c)\ \theta_F\ n(b, d)$.

(1) Since θ_F is a congruence of the DN-algebra $\langle A, m, 1\rangle$, it follows that $(a \vee c)\ \theta_F\ (b \vee c)$. Thus, there is a finite set $X \subseteq F$ such that $\langle X, a \vee c\rangle = \langle X, b \vee c\rangle$. Then, $\langle X, a \vee c\rangle \cap [c) = \langle X, b \vee c\rangle \cap [c)$ and thus

$$[\langle X\rangle \cap [c)] \vee [a \vee c) = [\langle X\rangle \cap [c)] \vee [b \vee c)$$

because $\mathsf{Fi}(A)$ is a distributive lattice. Assume that $X = \{x_1, \ldots, x_n\}$ and let $z := (x_1 \vee c) \wedge \cdots \wedge (x_n \vee c)$. Then $\langle X\rangle \cap [c) = ([x_1) \vee \cdots \vee [x_n)) \cap [c) = [z)$. Notice that $z \in [c)$ and also $z \in F$. Hence,

$$[z \wedge (a \vee c)) = [z) \vee [a \vee c) = [z) \vee [b \vee c) = [z \wedge (b \vee c))$$

and so $z \wedge (a \vee c) = z \wedge (b \vee c)$. As $[c)$ is a Heyting algebra and $z \in [c)$, we have $z \wedge n(a, c) = z \wedge n(b, c)$ (see [3, Lemma VIII.4.2]). Then $\langle z, n(a, c)\rangle = \langle z, n(b, c)\rangle$, with $z \in F$. Hence $n(a, c)\ \theta_F\ n(b, c)$.

(2) It follows by Proposition 4.5.

Hence, from (1), (2) and transitive property we obtain $n(a, c)\ \theta_F\ n(b, d)$. Therefore θ_F is a congruence relation of the near-Heyting algebra A. □

Now, we are ready to show one of the main results of this section.

Theorem 4.7. *Let $\langle A, m, n, 1\rangle$ be a near-Heyting algebra. Then, the map $\Gamma\colon \mathsf{Fi}(A) \to \mathsf{Con}(A)$ defined in Proposition 4.2 is a lattice isomorphism.*

Proof. By Proposition 4.2 and Theorem 4.6, we have that Γ is a lattice embedding. It only remains to prove that Γ is onto. Let $\theta \in \mathsf{Con}(A)$. Let $F := 1/\theta = \{x \in A : x\ \theta\ 1\}$. It is easy to check that $F \in \mathsf{Fi}(A)$. Let us prove that $\theta_F = \theta$. Suppose that $a\ \theta_F\ b$. So, there is a finite set $X := \{x_1, \dots, x_n\} \subseteq F$ such that $\langle X, a\rangle = \langle X, b\rangle$. Since $a \in \langle X, b\rangle$, it follows by Lemma 2.4 that $a = (x_1 \vee a) \wedge \cdots \wedge (x_n \vee a) \wedge (b \vee a)$. As F is a filter and $X \subseteq F$, we have $x_i \vee a \in F$ for all $i = 1, \dots, n$. So $(x_i \vee a)\ \theta\ 1$ for all $i = 1, \dots, n$ and thus $(x_1 \vee a) \wedge \cdots \wedge (x_n \vee a)\ \theta\ 1$. Then $(x_1 \vee a) \wedge \cdots \wedge (x_n \vee a) \wedge (b \vee a)\ \theta\ (b \vee a)$. Hence $a\ \theta\ (b \vee a)$. Analogously, we can show that $b\ \theta\ (a \vee b)$. Therefore, $a\ \theta\ b$. Now, assume that $a\ \theta\ b$. So $n(a, b)\ \theta\ n(b, b)$. By condition (NH2), we have $n(a, b)\ \theta\ 1$. Similarly, we have $n(b, a)\ \theta\ 1$. Then $n(a, b), n(b, a) \in F$. Notice that

$$\begin{aligned}(n(a, b) \vee a) \wedge (n(b, a) \vee a) \wedge (b \vee a) &= (n(a, b) \vee a) \wedge ((n(b, a) \wedge (b \vee a)) \vee a) \\ &= (n(a, b) \vee a) \wedge a = a.\end{aligned}$$

Then, by Lemma 2.4, we obtain $a \in \langle\{n(a, b), n(b, a)\}, b\rangle$. Similarly, $b \in \langle\{n(a, b), n(b, a)\}, a\rangle$. Thus

$$\langle\{n(a, b), n(b, a)\}, a\rangle = \langle\{n(a, b), n(b, a)\}, b\rangle$$

with $\{n(a, b), n(b, a)\} \subseteq F$, hence $a\ \theta_F\ b$. Thus $\Gamma(F) = \theta_F = \theta$ and therefore Γ is onto. □

Now, as a consequence of the previous theorem, we can characterize subdirectly irreducible algebras in $\mathbb{NHA}$ with a similar argument to that used in the setting of Heyting algebras. We leave the details to the reader.

Corollary 4.8. *An algebra A is subdirectly irreducible in $\mathbb{NHA}$ if and only if $A = A_1 \oplus \mathbf{1}$ where $A_1 \in \mathbb{NHA}$.*

Corollary 4.9. *Two-element near-Heyting algebra $\mathbf{2}$ is the only simple algebra in $\mathbb{NHA}$.*

5. Principal congruences in $\mathbb{DN}$ and $\mathbb{NHA}$

The following proposition generalises the result in [9, Lemma 2.7.3] and fully characterises the principal congruences on DN-algebras.

Proposition 5.1. *Let $\langle A, m, 1\rangle$ be a DN-algebra and $a, b, x, y \in A$. Then,*

$$(x,y) \in \theta(a,b) \iff \begin{cases} a \vee b \vee x = a \vee b \vee y, \\ x = (a \vee x) \wedge (b \vee x) \wedge (y \vee x), \\ y = (a \vee y) \wedge (b \vee y) \wedge (x \vee y). \end{cases} \tag{P}$$

Proof. Let us denote by Ψ the binary relation on A defined by the three above identities. By Lemma 4.1, it is straightforward to show directly that Ψ is a congruence on A and $(a, b) \in \Psi$. Thus $\theta(a, b) \subseteq \Psi$. Now we prove that $\Psi \subseteq \theta(a, b)$. Let $(x, y) \in \Psi$. Thus, the identities in (P) hold. Since $(a, b) \in \theta(a, b)$, we have $(a \vee x, a \vee b \vee x), (a \vee y, a \vee b \vee y) \in \theta(a, b)$. Then $(a \vee x, a \vee y) \in \theta(a, b)$ by using the first identity in (P). Similarly, $(b \vee x, b \vee y) \in \theta(a, b)$. Hence, by (P), $(x, y) \in \theta(a, b)$. Therefore, $\Psi = \theta(a, b)$. □

Let $\langle A, m, 1\rangle$ be a DN-algebra and $a, b, x \in A$. Notice that $(a \vee x) \wedge (b \vee x) \wedge (y \vee x) = m(m(a, b, x), y, x)$. Thus, the identities in (P) are actually equations in the language of DN-algebras. Hence, we obtain the following two corollaries. The reader can find in [16, 14] the concepts of Universal Algebra mentioned here.

Corollary 5.2. *The variety $\mathbb{DN}$ has equationally definable principal congruences.*

Corollary 5.3. *The variety $\mathbb{DN}$ has the congruence extension property.*

Our next purpose is to obtain a characterization of the principal congruences on near-Heyting algebras. We need the following facts.

Lemma 5.4. *Let $\langle A, m, n, 1\rangle \in \mathbb{NHA}$ and let F be a filter of A. If $a, n(a, b) \in F$, then $b \in F$.*

Proof. Assume that $a, n(a, b) \in F$. Then $(a \vee b) \wedge n(a, b) \in F$. Since $n(a, b)$ is the pseudocomplement of $a \vee b$ in $[b)$, it follows that $(a \vee b) \wedge n(a, b) = b$. Thus, $b \in F$. □

Proposition 5.5. *Let $A \in \mathbb{NHA}$. Let $a, b \in A$ be such that $a \wedge b$ exists in A. Then, for every $x \in A$, $n(x, a \wedge b) = n(x, a) \wedge n(x, b)$.*

Proof. Let $x \in A$. Notice that $n(x, a) \wedge n(x, b)$ exists in A. Thus,

$$\begin{aligned}[x \vee (a \wedge b)] \wedge [n(x, a) \wedge n(x, b)] &= [(x \vee a) \wedge (x \vee b)] \wedge [n(x, a) \wedge n(x, b)] \\ &= [(x \vee a) \wedge n(x, a)] \wedge [(x \vee b) \wedge n(x, b)] \\ &= a \wedge b.\end{aligned}$$

Then, since $n(x, a \wedge b)$ is the pseudocomplement of $x \vee (a \wedge b)$ in $[a \wedge b)$, we have $n(x, a) \wedge n(x, b) \leq n(x, a \wedge b)$. In order to prove the inverse inequality, suppose that $n(x, a \wedge b) \nleq n(x, a) \wedge n(x, b)$. Then, by Proposition 2.5, there is a prime filter P of A such that $n(x, a \wedge b) \in P$ and $n(x, a) \wedge n(x, b) \notin P$. So $n(x, a) \notin P$ or $n(x, b) \notin P$. Suppose that $n(x, a) \notin P$ (similarly if $n(x, b) \notin P$). Thus, by Proposition 4.4, there is a prime filter P_1 such that $P \subseteq P_1$, $x \in P_1$ and $a \notin P_1$. So $a \wedge b \notin P_1$. Since $x, n(x, a \wedge b) \in P_1$, it follows that $a \wedge b \in P_1$, which is a contradiction. This completes the proof. □

Theorem 5.6. *Let $\langle A, m, n, 1\rangle$ be a near-Heyting algebra and $a, b \in A$. Then,*

$$(x, y) \in \theta(a, b) \iff n(n(a, b), n(n(b, a), x)) = n(n(a, b), n(n(b, a), y)).$$

Proof. Let Ψ be the equivalence relation on A defined by:

$$(x, y) \in \Psi \iff n(n(a, b), n(n(b, a), x)) = n(n(a, b), n(n(b, a), y)).$$

In order to prove that $\theta(a, b) = \Psi$, it is enough to show that Ψ is a congruence on A, $(a, b) \in \Psi$ and $\Psi \subseteq \theta(a, b)$. We set $a' := n(a, b)$ and $b' := n(b, a)$. Assume that $(x, y), (u, v) \in \Psi$. Then

$$n(a', n(b', x)) = n(a', n(b', y)) \quad \text{and} \quad n(a', n(b', u)) = n(a', n(b', v)). \tag{5.1}$$

The arguments to prove that $(x \vee u, y \vee v) \in \Psi$ and $(n(x, u), n(y, v)) \in \Psi$ are similar, so we only show the first statement and leave the details of the last one to the reader. We need to check that $n(a', n(b', x \vee u)) = n(a', n(b', y \vee v))$. Suppose, towards a contradiction, that $n(a', n(b', x \vee u)) \nleq n(a', n(b', y \vee v))$. Thus, by Proposition 2.5, there exists a prime filter P such that $n(a', n(b', x \vee u)) \in P$ and $n(a', n(b', y \vee v)) \notin P$. Now, by Proposition 4.4, there exists a prime filter P_1 such that $P \subseteq P_1$, $a' \in P_1$ and $n(b', y \vee v) \notin P_1$. By Proposition 4.4 again, there exists a prime filter P_2 such that $P_1 \subseteq P_2$, $b' \in P_2$ and $y \vee v \notin P_2$. So $y, v \notin P_2$. Then, we have $n(a', n(b', x \vee u)) \in P_2$, $a', b' \in P_2$ and $y, v \notin P_2$. Now, applying Lemma 5.4 twice, we obtain $x \vee u \in P_2$. Since P_2 is prime, it follows that $x \in P_2$ or $u \in P_2$. Suppose that $x \in P_2$. By (NH1), $n(b', x) \in P_2$ and then $n(a', n(b', x)) \in P_2$. Thus, by (5.1), $n(a', n(b', y)) \in P_2$. Then, by applying Lemma 5.4 again twice we obtain $y \in P_2$, which is a contradiction. Analogously, if $u \in P_2$. Hence $n(a', n(b', x \vee u)) \leq n(a', n(b', y \vee v))$. In a similar way, we obtain the inverse inequality and thus $n(a', n(b', x \vee u)) = n(a', n(b', y \vee v))$. Hence $(x \vee u, y \vee v) \in \Psi$.

Now, suppose that $x \wedge u$ and $y \wedge v$ exist in A. Then, by Proposition 5.5, it follows that $n(a', n(b', x \wedge u)) = n(a', n(b', y \wedge v))$. Hence $(x \wedge u, y \wedge v) \in \Psi$. Therefore, we have proved that Ψ is a congruence on $\langle A, m, n, 1\rangle$.

Now let us prove that $(a, b) \in \Psi$. Suppose, towards a contradiction, that

$$n(n(a, b), n(n(b, a), a)) \nleq n(n(a, b), n(n(b, a), b)).$$

Thus, there is a prime filter P such that $n(n(a, b), n(n(b, a), a)) \in P$ and $n(n(a, b), n(n(b, a), b)) \notin P$. Then, by applying Proposition 4.4 twice, there exists a prime filter Q such that $n(n(a, b), n(n(b, a), a)) \in Q$, $n(a, b), n(b, a) \in Q$ and $b \notin Q$. Thus, $n(n(b, a), a) \in Q$ and then $a \in Q$. So, since $a, n(a, b) \in Q$, it follows that $b \in Q$, which is a contradiction. Hence

$$n(n(a, b), n(n(b, a), a)) \leq n(n(a, b), n(n(b, a), b)).$$

In a similar way, we get the inverse inequality. Then

$$n(n(a, b), n(n(b, a), a)) = n(n(a, b), n(n(b, a), b)).$$

Therefore, $(a, b) \in \Psi$.

Finally, we show $\Psi \subseteq \theta(a, b)$. Let $(x, y) \in \Psi$. So $n(a', n(b', x)) = n(a', n(b', y))$. We set $\theta := \theta(a, b)$. Since $a \,\theta\, b$, it follows that $n(a, a) \,\theta\, n(a, b)$. By (NH2), $1 \,\theta\, n(a, b)$. Analogously, $1 \,\theta\, n(b, a)$. Thus $n(1, y) \,\theta\, n(n(b, a), y)$. By

(NH3), we obtain that $y\ \theta\ n(n(b,a),y)$. Similarly, we have $x\ \theta\ n(n(b,a),x)$. Since $1\ \theta\ n(a,b)$, it follows that

$$n(1,n(n(b,a),x))\ \theta\ n(n(a,b),n(n(b,a),x)).$$

By axiom (NH3), it follows that

$$n(n(b,a),x)\ \theta\ n(n(a,b),n(n(b,a),x))$$

and thus $x\ \theta\ n(a',n(b',x))$. Similarly, we get $y\ \theta\ n(a',n(b',y))$. Thus, $x\ \theta\ y$. Hence $\Psi \subseteq \theta(a,b)$. This completes the proof. □

Corollary 5.7. *The variety* $\mathbb{NHA}$ *has equationally definable principal congruences.*

Corollary 5.8. *The variety* $\mathbb{NHA}$ *has the congruence extension property.*

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Luciano J. González and Marina B. Lattanzi
Universidad Nacional de La Pampa.
Facultad de Ciencias Exactas y Naturales
Santa Rosa
Argentina
e-mail [L. J. González]: lucianogonzalez@exactas.unlpam.edu.ar
e-mail [M. B. Lattanzi]: mblatt@exactas.unlpam.edu.ar

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