# Finite distributive semilattices

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**Abstract**

The present article aims to develop a categorical duality for the category of finite distributive join-semilattices and $\wedge$-homomorphisms (maps that preserve the joins and the meets, when they exist). This dual equivalence is a generalization of the famous categorical duality given by Birkhoff for finite distributive lattices. Moreover, we show that every finite distributive semilattice is a Hilbert algebra with supremum. We obtain some applications from the dual equivalence. We provide a dual description of the 1-1 and onto $\wedge$-homomorphisms, and we obtain a dual characterization of some subalgebras. Finally, we present a representation for the class of finite semi-boolean algebras.

*Keywords:* Semilattice, Distributivity, Representation, Lattice

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## 1. Introduction and preliminaries

A join-semilattice $\langle A, \vee, 1\rangle$ is said to be *distributive* when for all $a_1, \dots, a_n, b \in A$, if the infimum $a_1 \wedge \cdots \wedge a_n$ exists in $A$, then the infimum $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ exists in $A$ and $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b) = (a_1 \wedge \cdots \wedge a_n) \vee b$. Distributive semilattices were first studied by Balbes [3] under the name of prime semilattices. Then, such semilattices were studied by Varlet [23] and Cornish and Hickman [12] under the name of weakly distributive semilattices. See also [19, 20]. An interesting class of distributive join-semilattices are those which satisfy the *lower bound property*, that is each finite non-empty subset which is bounded below has an infimum. Distributive join-semilattices satisfying the lower bound property are also known in the literature as *distributive nearlattices*, and they were studied from different

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points of view: algebraic, topological and logical, see [18, 11, 7, 5, 8, 14, 16, 9, 15, 6, 17]. The class of distributive join-semilattices satisfying the lower bound property (distributive nearlattices) contains all distributive lattices, and also all those join-semilattices in which every principal upset is a Boolean algebra. These semilattices are known as *semi-boolean algebras* and were studied by Abbott in [2, 1]. Abbott proved ([2]) that semi-boolean algebras are in a one-to-one correspondence with implication algebras. The class of implication algebras corresponds to the equivalent algebraic semantics of the implicational fragment of the classical propositional calculus.

In this paper, we focus on the class of finite distributive join-semilattices, which is a natural generalization of the class of finite distributive lattices and contains all finite semi-boolean algebras. In Section 2, it is shown that every element of a finite distributive join-semilattice is the infimum of all meet-irreducible elements (see Definition 2.3) above it. Then, we show that in every finite distributive join-semilattice a binary operation $\to$ can be defined in such a way that the resulting algebra is a Hilbert algebra with supremum [13, 10]. Section 3 is dedicated to reviewing the set-theoretic representation for finite distributive join-semilattices (finite distributive nearlattices) given in [17]. In Section 4, we extend the representation given in the previous section to a full dual equivalence for the category of finite distributive join-semilattices and $\wedge$-homomorphisms. Section 5 is concerned with some applications of the dual equivalence. We characterize the 1-1 and onto $\wedge$-homomorphisms. This leads to obtaining an effective method to produce subalgebras that are closed under finite existent infimum. We also obtain a representation for the class of all finite semi-boolean algebras.

We close this section by presenting some notations and basic definitions.

Let $P$ be a poset. A subset $X \subseteq P$ is called an *upset* of $P$ when for every $x \in X$ and $y \in P$, if $x \leq y$, then $y \in X$. Dually we have the notion of *downset* of $P$. For every subset $X \subseteq P$, let

$$[X)_P := \{a \in P : \exists x \in X(x \leq a)\} \quad \text{and} \quad (X]_P := \{a \in P : \exists x \in X(a \leq x)\}.$$

Notice that for every $x \in P$ we write $[x)_P := \{a \in P : x \leq a\}$ and $(x]_P := \{a \in P : a \leq x\}$. We also need the following notations. If $Q \subseteq P$ and $x \in P$, then

$$[x)_Q := \{y \in Q : x \leq y\} \quad \text{and} \quad (x]_Q := \{y \in Q : y \leq x\}.$$

For us, *semilattice* will mean a *join-semilattice with a top element* $\langle A, \vee, 1\rangle$. Thus, the partial order $\leq$ associated with a semilattice $\langle A, \vee, 1\rangle$ is given by: $a \leq b \iff a \vee b = b$, for all $a, b \in A$. Then, $a \vee b$ is the supremum of $a$ and $b$ in $A$. Clearly, the infimum of two elements of $A$ does not necessarily exist in $A$. Throughout the paper, we write $a_1 \wedge \cdots \wedge a_n$ meaning that the infimum of $a_1, \ldots, a_n$ exists and it is $a_1 \wedge \cdots \wedge a_n$.

Let $A$ be a semilattice. A subset $F \subseteq A$ is said to be a *filter* of $A$ if (i) $1 \in F$, (ii) $F$ is an upset of $A$, and (iii) if $a, b \in F$ and $a \wedge b$ exists in $A$, then $a \wedge b \in F$. Let us denote by $\mathrm{Fi}(A)$ the collection of all filters of $A$. It is straightforward to check directly that $\mathrm{Fi}(A)$ is an algebraic closure system, and thus $\mathrm{Fi}(A)$ is a complete lattice.

**Definition 1.1.** A semilattice $A$ is said to be *distributive* when for all $a_1, \ldots, a_n, b \in A$, if $a_1 \wedge \cdots \wedge a_n$ exists in $A$, then $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ exists and $(a_1 \wedge \cdots \wedge a_n) \vee b = (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$.

**Lemma 1.2** ([12]). *A semilattice $A$ is distributive if and only if the lattice* $\mathrm{Fi}(A)$ *is distributive.*

Let $A$ be a distributive semilattice and $X \subseteq A$. We denote by $\mathrm{Fig}_A(X)$ the *filter generated* by $X$. Hence, by [12], we have that

$$\mathrm{Fig}_A(X) = \{a \in A : \exists a_1, \ldots, a_n \in [X) \text{ s.t. } a = a_1 \wedge \cdots \wedge a_n\}. \tag{1.1}$$

Let $A$ be a semilattice. A subset $I \subseteq A$ is called an *ideal* of $A$ if it is a downset of $A$ and for all $a, b \in I$, $a \vee b \in I$[1]. For every subset $X \subseteq A$, the *ideal generated by* $X$ is denoted by $\mathrm{Idg}_A(X)$ and $\mathrm{Idg}_A(X) = \{a \in A : a \leq x_1 \vee \cdots \vee x_n, \text{ for some } x_1, \ldots, x_n \in X\}$.

Let $\langle A, \vee, 1\rangle$ and $\langle B, \vee, 1\rangle$ be semilattices. A map $h\colon A \to B$ is said to be a *homomorphism* if $h(1) = 1$ and for all $a, b \in A$, $h(a \vee b) = h(a) \vee h(b)$. We will say that $h$ is a $\wedge$-*homomorphism* if it is a homomorphism and for all $a, b \in A$, if $a \wedge b$ exists in $A$, then $h(a) \wedge h(b)$ exists in $B$ and $h(a \wedge b) = h(a) \wedge h(b)$. We will say that $h\colon A \to B$ is a $\wedge$-*embedding* if it is a 1-1 $\wedge$-homomorphism.

## 2. Finite distributive semilattices

A semilattice $A$ is said to have the *lower bound property* if any two elements that are bounded below have an infimum. Notice that a semilattice $A$ has the lower bound property if and only if for every $a \in A$, the principal upset $[a)$ is a lattice. Hence, the following result is straightforward.

[1]Notice that we are allowing that the empty set is an ideal.

**Lemma 2.1.** *Let $A$ be a semilattice. The following are equivalent.*

*(i) $A$ has the lower bound property and it is distributive.*

*(ii) For every $a \in A$, $[a)$ is a distributive lattice.*

The semilattices $A$ satisfying the above condition (ii) are called *distributive nearlattices*. By Lemma 2.1, we can apply all the results and facts known about distributive nearlattices to distributive semilattices satisfying the lower bound property. The following proposition is straightforward, but it will be important for us.

**Proposition 2.2.** *Let $A$ be a distributive semilattice. If $A$ is finite, then $A$ has the lower bound property. Hence, $A$ is a distributive nearlattice.*

From now on, all semilattices will be finite.

**Definition 2.3.** Let $A$ be a semilattice. An element $m \in A$ is said to be *meet-irreducible* (or simply *irreducible*) if $m \neq 1$ and for all $a_1, a_2 \in A$, if $a_1 \wedge a_2$ exists and $a_1 \wedge a_2 = m$, then $a_1 = m$ or $a_2 = m$.

Let $\mathrm{Irr}(A)$ be the set of all irreducible elements of a semilattice $A$.

**Lemma 2.4.** *Let $A$ be a distributive semilattice. An element $m \in A$ is irreducible if and only if for all $a_1, a_2 \in A$, $a_1 \wedge a_2 \leq m$ implies that $a_1 \leq m$ or $a_2 \leq m$.*

The following proposition is fundamental for the representation given in the next section.

**Proposition 2.5** ([17, Theo. 5.3])**.** *Let $A$ be a finite distributive semilattice. Then, for every $a \in A$, we have*

$$a = \bigwedge \{m \in \mathrm{Irr}(A) : a \leq m\}.$$

**Remark 2.6.** The above proposition tells us that every element is the infimum of the irreducible elements above it. But, notice that not every subset of irreducible elements has an infimum.

Now we will see that finite distributive semilattices are very closely related to finite distributive lattices. Recall that every finite distributive lattice $L$ is, in fact, a Heyting algebra, where the Heyting implication is given by:

$$a \to b = \bigvee \{x \in L : a \wedge x \leq b\}.$$

Let $\langle A, \vee, 1\rangle$ be a finite distributive semilattice and $a \in A$. We denote by $\wedge_a$ the meet on $[a)$. Thus $\langle [a), \wedge_a, \vee, a, 1\rangle$ is a finite distributive lattice. Hence, $[a)$ is a Heyting algebra, where the Heyting implication $\to_a$ on $[a)$ is given by:

$$x \to_a y = \bigvee\{z \in [a) : x \wedge_a z \leq y\} \tag{2.1}$$

for all $x, y \in [a)$. Then, we can define a binary operation $\to$ on $A$ as follows: for every $a, b \in A$,

$$a \to b := (a \vee b) \to_b b \tag{2.2}$$

Let us use the operation $\to$ to characterize the irreducible elements.

**Proposition 2.7.** *Let $A$ be a finite distributive semilattice. Let $1 \neq m \in A$. Then, $m$ is irreducible if and only if for every $a \in A$, $a \leq m$ or $a \to m = m$.*

*Proof.* Assume that $m$ is irreducible, and let $a \in A$. Since $[m)$ is a Heyting algebra and $m \vee a \in [m)$, it follows that $(m \vee a) \wedge ((m \vee a) \to_m m) = m$. Then, since $m$ is irreducible, we have that $m \vee a = m$ or $(m \vee a) \to_m m = m$. Hence $a \leq m$ or $a \to m = m$.

Conversely, assume that for every $a \in A$, $a \leq m$ or $a \to m = m$. Let $a, b \in A$ be such that $a \wedge b$ exists and suppose that $m = a \wedge b$. Suppose by contradiction that $m < a$ and $m < b$. Then $a \to m = m$ and $b \to m = m$. Given that $a, b \in [m)$, we obtain that $a \to m = a \to_m m$ and $b \to m = b \to_m m$. Thus $a \to_m m = m$ and $b \to_m m = m$. Then, $(a \wedge b) \to_m m = a \to_m (b \to_m m) = a \to_m m = m$. Thus we have $1 = m \to_m m = (a \wedge b) \to_m m = m$, which is a contradiction. Hence, $m = a$ or $m = b$. Therefore, $m$ is irreducible. □

**Definition 2.8.** Let $A$ be a distributive semilattice. A pair $\langle L_A, e_A\rangle$, where $L_A$ is a bounded distributive lattice and $e_A \colon A \to L_A$ is a $\wedge$-embedding, is said to be a *free distributive lattice extension of* $A$ if $e[A]$ is finitely meet-dense in $L_A$ and the following universal property holds: for every bounded distributive lattice $M$ and every $\wedge$-homomorphism $h \colon A \to M$, there exists a unique lattice homomorphism $\widehat{h} \colon L_A \to M$ such that $h = \widehat{h} \circ e_A$.

In [12] it is shown that every distributive semilattice $A$ has a free distributive lattice extension, and the finite meet-density implies that it is unique up to isomorphism.

The following proposition tells us how we can construct the free distributive lattice extension of a finite distributive semilattice.

**Proposition 2.9** ([17, Prop. 5.4])**.** *Let $A$ be a finite distributive semilattice and $\langle L_A, e_A\rangle$ its free distributive lattice extension. Then, $e[\mathrm{Irr}(A)] = \mathrm{Irr}(L_A)$.*

**Remark 2.10.** Let $A$ be a finite distributive semilattice. Then, its free distributive lattice extension $L_A$ is finite. It is well-known that $L_A \overset{D}{\cong} \mathrm{Up}(\mathrm{Irr}(L_A))$ (where $\mathrm{Up}(X)$ denotes the lattice of all upsets of a poset $X$). By the previous proposition we obtain that $\mathrm{Irr}(A)$ and $\mathrm{Irr}(L_A)$ are order-isomorphic. Hence, $L_A \overset{D}{\cong} \mathrm{Up}(\mathrm{Irr}(A))$.

Notice that if $A$ is a finite distributive semilattice, then its free distributive lattice extension $L_A$ is also finite. Thus $L_A$ is a Heyting algebra. Now we show that the $\wedge$-embedding $e_A$ preserves the operation on $A$ defined by (2.2). This is made clear by the next proposition.

**Proposition 2.11.** *Let $A$ be a finite distributive semilattice and $\langle L_A, e_A\rangle$ its free distributive lattice extension. Then, for all $a, b \in A$, $e_A(a \to b) = e_A(a) \to e_A(b)$.*

*Proof.* Let $a, b \in A$. Since $[b)$ is a Heyting algebra, it follows that $(a \vee b) \wedge_b ((a \vee b) \to_b b) = (a \vee b) \wedge_b b$. Thus we have $(a \vee b) \wedge (a \to b) = b$. Now, given that $e_A$ is a $\wedge$-homomorphism, we obtain that $(e_A(a) \vee e_A(b)) \wedge e_A(a \to b) = e_A(b)$. Then $(e_A(a) \wedge e_A(a \to b)) \vee (e_A(b) \wedge e_A(a \to b)) = e_A(b)$. Since $b \leq a \to b$, we have that $(e_A(a) \wedge e_A(a \to b)) \vee e_A(b) = e_A(b)$. Hence,

$$e_A(a) \wedge e_A(a \to b) \leq e_A(b).$$

Now we show that $e_A(a \to b)$ is the greatest element in $L_A$ satisfying the above condition. Let $u \in L_A$ be such that $e_A(a) \wedge u \leq e_A(b)$. We need to prove that $u \leq e_A(a \to b)$. By Proposition 2.5, it is enough to prove that for every $y \in \mathrm{Irr}(L_A)$, $e_A(a \to b) \leq y$ implies that $u \leq y$. Let $x \in \mathrm{Irr}(A)$ be such that $e_A(a \to b) \leq e_A(x)$ (recall that $e_A[\mathrm{Irr}(A)] = \mathrm{Irr}(L_A)$). On the one hand, notice that $a \to b = (a \vee b) \to_b b = \bigvee\{c \in [b) : (a \vee b) \wedge_b c \leq b\}$. Thus $e_A(a \to b) = \bigvee\{e_A(c) : c \in [b),\ (a \vee b) \wedge c \leq b\}$. Then, we have that $e_A(c) \leq e_A(x)$, for all $c \in [b)$ such that $(a \vee b) \wedge c \leq b$. On the other hand, since there are $a_1, \dots, a_n \in A$ such that $u = e_A(a_1) \wedge \cdots \wedge e_A(a_n)$ and since $e_A(a) \wedge u \leq e_A(b)$, it follows that

$$\begin{aligned} e_A(b) &= (e_A(a) \vee e_A(b)) \wedge (u \vee e_A(b)) \\ &= e_A(a \vee b) \wedge [(e_A(a_1) \vee e_A(b)) \wedge \cdots \wedge (e_A(a_n) \vee e_A(b))] \\ &= e_A((a \vee b) \wedge (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)). \end{aligned}$$

Then $b = (a \vee b) \wedge (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$. Let $c := (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$. Thus, $c \in [b)$ and $(a \vee b) \wedge c \leq b$. Hence $e_A(c) \leq e_A(x)$. That is, $e_A(a_1 \vee b) \wedge \cdots \wedge e_A(a_n \vee b) \leq e_A(x)$. Since $e_A(x)$ is irreducible, it follows that there is $i \in \{1, \dots, n\}$ such that $e_A(a_i) \leq e_A(a_i \vee b) \leq e_A(x)$. Then, $u \leq e_A(x)$. Hence $u \leq e_A(a \to b)$. Therefore, $e_A(a \to b) = e_A(a) \to e_A(b)$. □

Hilbert algebras with supremum correspond to the implication-disjunction subreducts of Heyting algebras. For further reading on Hilbert algebras, see [13, 22], and on Hilbert algebras with supremum, see [21, 10].

**Corollary 2.12.** *Every finite distributive semilattice $A$ is a Hilbert algebra with supremum, with the implication defined by* (2.2)*.*

*Proof.* Let $A$ be a finite distributive semilattice. Then, by Proposition 2.11, we have that the algebra $\langle A, \vee, \to, 1\rangle$ is isomorphic to the implication-disjunction subreduct $\langle e_A[A], \vee, \to, 1\rangle$ of the Heyting algebra $L_A$. Hence, the algebra $\langle A, \vee, \to, 1\rangle$, with $\to$ defined by (2.2), is a Hilbert algebra with supremum. $\square$

## 3. Representation

In this section, we present the representation for finite distributive semilattices given in [17]. Recall that finite distributive semilattice is equivalent to finite distributive nearlattice, as it was named in [17]. Also, we point out that in [17, Sec. 5] the authors worked with the lattice of downsets, ordered by inclusion, of a poset. In the present article, we choose (as will become clear in the following sections) to work dually with the lattice of upsets, ordered by reverse inclusion. Thus, the results and the definitions given in [17] are dually presented here.

From now on, given a poset $X$, let us consider the collection of all upsets of $X$, denoted by $\mathrm{Up}(X)$, ordered by $\supseteq$. Thus $\langle \mathrm{Up}(X), \sqcap, \sqcup\rangle$ is a distributive lattice, where the meet $\sqcap$ is $\cup$ and the join $\sqcup$ is $\cap$. Then, for all $U, V \in \mathrm{Up}(X)$, we have $U \sqsubseteq V \iff V \subseteq U$. Notice that $\mathrm{Irr}(\mathrm{Up}(X)) = \{[x)_X : x \in X\}$. The next definition corresponds dually to the definition of DN-structure given in [17, Def. 5.6].

**Definition 3.1** ([17, Def. 5.6])**.** A *DS-structure* is a pair $\langle X, \gamma\rangle$ such that $X$ is a poset and $\gamma\colon \mathrm{Up}(X) \to \{0, 1\}$ is a map satisfying the following:

(S1) $\gamma(\emptyset) = 1$;

(S2) $\gamma([x)) = 1$, for all $x \in X$;

(S3) for all $U, V \in \mathrm{Up}(X)$, $U \subseteq V$ implies $\gamma(V) \leq \gamma(U)$.

We say that a DS-structure $\langle X, \gamma\rangle$ is *finite* if the poset $X$ is finite.

Let $\langle X, \gamma \rangle$ be a DS-structure. Let

$$\mathcal{A}(X) := \{U \in \mathrm{Up}(X) : \gamma(U) = 1\}.$$

Thus $\mathcal{A}(X) \subseteq \mathrm{Up}(X)$, and by condition (S3), it follows that $\mathcal{A}(X)$ is closed under $\sqcup = \cap$.

**Proposition 3.2** ([17, Prop. 5.7]). *Let $\langle X, \gamma \rangle$ be a finite DS-structure. Then $\langle \mathcal{A}(X), \sqcup, \emptyset \rangle$ is a finite distributive semilattice and $\langle \mathrm{Irr}(\mathcal{A}(X)), \sqsubseteq \rangle \cong \langle X, \leq \rangle$.*

**Remark 3.3.** Let $\langle X, \gamma \rangle$ be a finite DS-structure. Let $U, V \in \mathcal{A}(X)$. If the infimum of $U$ and $V$ exists in $\mathcal{A}(X)$, then it is $U \cup V$. That is, $U \sqcap V = U \cup V \in \mathcal{A}(X)$.

Now, let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice and let

$$\mathcal{X}(A) := \langle \mathrm{Irr}(A), \gamma_A \rangle$$

be the pair where $\mathrm{Irr}(A)$ is the sub-poset of irreducible elements of $A$ and $\gamma_A \colon \mathrm{Up}(\mathrm{Irr}(A)) \to \{0, 1\}$ is the map defined by:

$$\gamma_A(U) = 1 \iff \bigwedge U \text{ exists in } A$$

for every $U \in \mathrm{Up}(\mathrm{Irr}(A))$.

**Proposition 3.4** ([17, Prop. 5.8]). *Let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice. Then, $\mathcal{X}(A) = \langle \mathrm{Irr}(A), \gamma_A \rangle$ is a DS-structure.*

Given a finite distributive semilattice $A$, we have that $\langle \mathcal{A}(\mathcal{X}(A)), \sqcup, \emptyset \rangle$ is a finite distributive semilattice, where $\mathcal{A}(\mathcal{X}(A)) = \{U \in \mathrm{Up}(\mathrm{Irr}(A)) : \gamma_A(U) = 1\}$ and $\sqcup = \cap$.

**Theorem 3.5** (Representation, [17, Theo. 5.9]). *Let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice. Then, the map $\alpha_A \colon A \to \mathcal{A}(\mathcal{X}(A))$ defined by $\alpha_A(a) = \{x \in \mathrm{Irr}(A) : a \leq x\}$ is an isomorphism.*

Roughly speaking, given a DS-structure $\langle X, \gamma \rangle$, $X$ represents the poset of irreducible elements of the distributive semilattice $\mathcal{A}(X)$, and the map $\gamma$ tells us which infima of irreducible elements exist in $\mathcal{A}(X)$.

**Example 3.6.** Let $A$ be the distributive semilattice given in Figure 1. Thus $X := \mathrm{Irr}(A) = \{x_1, x_2, x_3, x_4, x_5\}$. Figure 1 shows the poset $X$ partially ordered by the partial order induced by $A$. Then,

$$\mathcal{A}(X) = \gamma_A^{-1}[\{1\}] = \{\emptyset\} \cup \{[x_i)_X : i = 1, 2, 3, 4, 5\} \cup \{[x_3)_X \cup [x_4)_X, [x_4)_X \cup [x_5)_X\}.$$

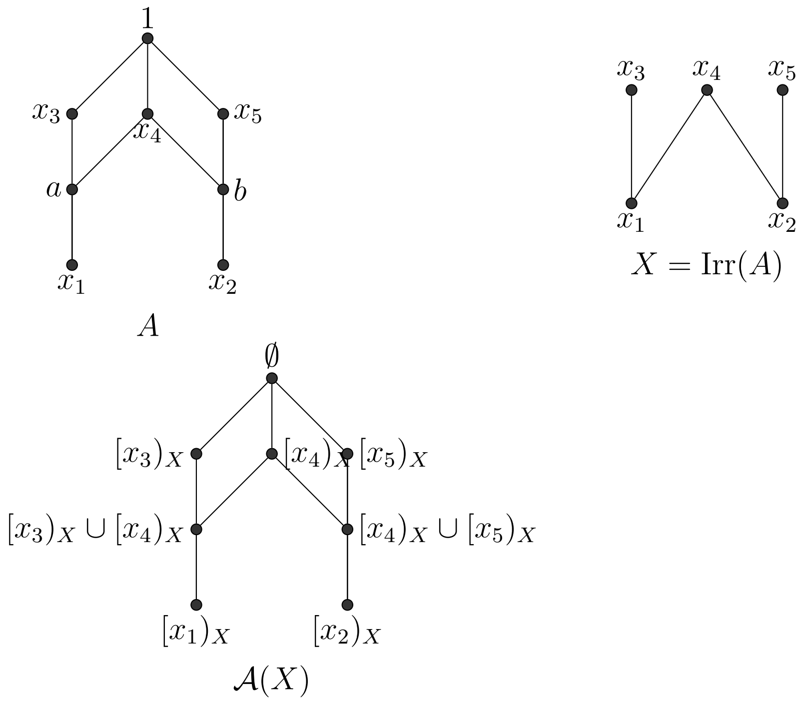


Figure 1:

**Remark 3.7.** Let $A$ be a finite distributive lattice. Then, $\gamma_A \colon \mathrm{Up}(\mathrm{Irr}(A)) \to \{0,1\}$ is such that $\gamma_A(U) = 1$, for all $U \in \mathrm{Up}(\mathrm{Irr}(A))$. Then, $A \cong \mathcal{A}(\mathcal{X}(A)) = \mathrm{Up}(\mathrm{Irr}(A))$. Thus, we obtain the representation given by Birkhoff for finite distributive lattices [4].

## 4. Categorical duality

Now we proceed to extend the representation developed in the previous section to a full categorical dual equivalence. Let us denote by $\mathcal{DS}^{\mathrm{f}}$ the category of finite distributive semilattices and $\wedge$-homomorphisms. We need to find the right morphisms between DS-structures that correspond to the $\wedge$-homomorphisms.

**Definition 4.1.** Let $\langle X, \gamma\rangle$ and $\langle Y, \tau\rangle$ be DS-structures. We will say that $f \colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ is a *DS-morphism* if $f \colon X \to Y$ is a partial function satisfying the following conditions:

(M1) For all $x_1, x_2 \in \mathrm{dom}\, f$, if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$;

(M2) $\mathrm{dom}\, f \in \mathrm{Up}(X)$;

(M3) $\gamma\left(f^{-1}[V]\right) = 1$, for every $V \in \mathrm{Up}(Y)$ such that $\tau(V) = 1$.

Notice that if $f\colon X \to Y$ is a partial function satisfying conditions (M1) and (M2), then $f^{-1}[V] \in \mathrm{Up}(X)$, for every $V \in \mathrm{Up}(Y)$. Moreover, recall that given two partial functions $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f\colon X \to Z$ is defined as follows: $\mathrm{dom}(g \circ f) = f^{-1}[\mathrm{dom}\, g] = \{x \in X : x \in \mathrm{dom}\, f \text{ and } f(x) \in \mathrm{dom}\, g\}$.

The proof of the following proposition is straightforward.

**Proposition 4.2.** *If $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ and $g\colon \langle Y, \tau\rangle \to \langle Z, \eta\rangle$ are two DS-morphisms, then the partial map $g \circ f\colon \langle X, \gamma\rangle \to \langle Z, \eta\rangle$ is also a DS-morphism.*

Now we are in a position to define the category $\mathcal{S}^{\mathrm{f}}$ of finite DS-structures and DS-morphisms.

For what follows, we need the following notion and some results. A proper ideal $I$ of $A$ is said to be *prime* if whenever $a, b \in A$ are such that $a \wedge b$ exists and $a \wedge b \in I$, then $a \in I$ or $b \in I$. Notice that if $A$ is a finite semilattice, then all ideals of $A$ are of the form $(a]$, for some $a \in A$.

**Lemma 4.3.** *Let $A$ be a finite distributive semilattice and $P$ an ideal of $A$. Then, $P$ is a prime ideal if and only if there is $x \in \mathrm{Irr}(A)$ such that $P = (x]$.*[2]

**Lemma 4.4** ([7])**.** *Let $A$ and $B$ be finite distributive semilattices and let $h\colon A \to B$ be a $\wedge$-homomorphism. Then, for every prime ideal $P$ of $B$, $h^{-1}[P]$ is a prime ideal of $A$.*

Let $h\colon A \to B$ be a $\wedge$-homomorphism between finite distributive semilattices. Let us define the partial function $f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$ as follows:

$$\mathrm{dom}\, f_h = \{y \in \mathrm{Irr}(B) : h^{-1}[(y]_B] \neq \emptyset\} \quad \text{and} \quad f_h(y) = \bigvee h^{-1}[(y]_B]$$

for every $y \in \mathrm{dom}\, f_h$. By Lemmas 4.3 and 4.4, it follows that $f_h$ is a well-defined partial function.

**Lemma 4.5.** *Let $h\colon A \to B$ be a $\wedge$-homomorphism. Then, for every $a \in A$ and $y \in \mathrm{Irr}(B)$, we have*

$$h(a) \leq y \iff y \in \mathrm{dom}\, f_h \;\text{ and }\; a \leq f_h(y).$$

*Proof.* If $h(a) \leq y$, then $a \in h^{-1}[(y]_B]$. Thus $y \in \mathrm{dom}\, f_h$ and $a \leq \bigvee h^{-1}[(y]] = f_h(y)$. Conversely, assume that $y \in \mathrm{dom}\, f_h$ and $a \leq f_h(y)$. Since $h^{-1}[(y]_B]$ is an ideal of $B$ and since $f_h(y) = \bigvee h^{-1}[(y]] \in h^{-1}[(y]]$, it follows that $a \in h^{-1}[(y]]$. Hence, $h(a) \leq y$. □

[2] This result is a direct generalization from the lattice case, and it was proved by Dr. Ismael Calomino.

**Proposition 4.6.** *Let $h\colon A \to B$ be a $\wedge$-homomorphism. Then the partial function $f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$ is a DS-morphism.*

*Proof.* Condition (M1) is straightforward. In order to prove condition (M2), let $y \in \operatorname{dom} f_h$ and $y' \in \operatorname{Irr}(B)$ be such that $y \leq y'$. Thus, we have $h^{-1}[(y]_B] \subseteq h^{-1}[(y']_B]$. Since $y \in \operatorname{dom} f_h$, it follows that $h^{-1}[(y]_B] \neq \emptyset$. Then, $h^{-1}[(y']_B] \neq \emptyset$. Thus $y' \in \operatorname{dom} f_h$. Hence $\operatorname{dom} f_h \in \operatorname{Up}(\operatorname{Irr}(B))$. Now, to prove condition (M3), recall from Theorem 3.5 that

$$\mathcal{A}(\mathcal{X}(A)) = \{U \in \operatorname{Up}(\operatorname{Irr}(A)) : \gamma_A(U) = 1\} = \{\alpha_A(a) : a \in A\}$$

where $\alpha_A(a) = \{x \in \operatorname{Irr}(A) : a \leq x\}$. Thus, by Lemma 4.5, we have $\alpha_B(h(a)) = f_h^{-1}[\alpha_A(a)]$, for every $a \in A$. Then, we obtain that

$$\gamma_B\left(f_h^{-1}[\alpha_A(a)]\right) = \gamma_B\left(\alpha_B(h(a))\right) = 1$$

for every $a \in A$. □

Now let $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ be a DS-morphism. We define the map $h_f\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ as follows: $h_f(V) := f^{-1}[V]$, for every $V \in \mathcal{A}(Y)$.

**Proposition 4.7.** *Let $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ be a DS-morphism. Then, the map $h_f\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ is a $\wedge$-homomorphism.*

*Proof.* By condition (M3) we obtain that $f_h$ is well-defined. Let $V_1, V_2 \in \mathcal{A}(Y)$. Then,

$$h_f(V_1 \sqcup V_2) = f^{-1}[V_1 \cap V_2] = f^{-1}[V_1] \cap f^{-1}[V_2] = h_f(V_1) \sqcup h_f(V_2).$$

Now suppose that there exists the infimum $V_1 \sqcap V_2$ in $\mathcal{A}(Y)$. Recall that $V_1 \sqcap V_2 = V_1 \cup V_2$. Then, it is clear that $h_f(V_1 \sqcap V_2) = h_f(V_1) \sqcap h_f(V_2)$. Moreover, we have $h_f(\emptyset) = \emptyset$. Therefore, $h_f$ is a $\wedge$-homomorphism. □

**Proposition 4.8.** *Let $h\colon A \to B$ and $k\colon B \to C$ be $\wedge$-homomorphisms. Then, $f_{k\circ h} = f_h \circ f_k$.*

*Proof.* First, we need to show that $\operatorname{dom} f_{k\circ h} = \operatorname{dom}(f_h \circ f_k)$. Let $z \in \operatorname{Irr}(C)$. Then,

$$\begin{aligned} z \in \operatorname{dom}(f_h \circ f_k) &\iff z \in \operatorname{dom} f_k \text{ and } f_k(z) \in \operatorname{dom} f_h \\ &\iff z \in \operatorname{dom} f_k \text{ and } h^{-1}[(f_k(z)]_B] \neq \emptyset \\ &\iff z \in \operatorname{dom} f_k \text{ and } \exists a \in A \text{ s.t. } h(a) \leq f_k(z) \end{aligned}$$

$$\overset{Lem. 4.5}{\iff} \exists a \in A \text{ s.t. } k(h(a)) \leq z$$
$$\iff (k \circ h)^{-1}[(z]_C] \neq \emptyset$$
$$\iff z \in \mathrm{dom}\, f_{k \circ h}.$$

Now let $z \in \mathrm{dom}\, f_{k\circ h} = \mathrm{dom}\,(f_h \circ f_k)$. For every $a \in A$, we have

$$a \in h^{-1}[(f_k(z)]_B] \iff h(a) \leq f_k(z) \iff k(h(a)) \leq z \iff a \in (k \circ h)^{-1}[(z]_C].$$

Then $h^{-1}[(f_k(z)]_B] = (k \circ h)^{-1}[(z]_C]$. Hence,

$$(f_h \circ f_k)(z) = f_h(f_k(z)) = \bigvee h^{-1}[(f_k(z)]_B] = \bigvee (k \circ h)^{-1}[(z]_C] = f_{k\circ h}(z). \qquad \square$$

**Proposition 4.9.** *Let $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ and $g\colon \langle Y, \tau\rangle \to \langle Z, \eta\rangle$ be DS-morphisms. Then, $h_{g\circ f} = h_f \circ h_g$.*

*Proof.* Let $h_g\colon \mathcal{A}(Z) \to \mathcal{A}(Y)$ and $h_f\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ be the dual $\wedge$-homomorphisms of $g$ and $f$, respectively. Let $W \in \mathcal{A}(Z)$. Then,

$$h_{g\circ f}(W) = (g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]] = h_f(h_g(W)) = (h_f \circ h_g)(W).$$

Hence, $h_{g\circ f} = h_f \circ h_g$. $\square$

Recall that a partial function $f\colon X \to Y$ is an isomorphism in the category of sets and partial functions if and only if $f$ is a bijective total function. Then, a DS-morphism $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$ is an isomorphism in the category $\mathcal{S}^{\mathrm{f}}$ if and only if $f\colon X \to Y$ is an order-isomorphism, and for every $U \in \mathrm{Up}(X)$, $\gamma(U) = 1 \iff \tau(f[U]) = 1$.

Let $\langle X, \gamma\rangle$ be a DS-structure. Recall that $\mathcal{A}(X) = \{U \in \mathrm{Up}(X) : \gamma(U) = 1\}$ and $\mathcal{X}(\mathcal{A}(X)) = \langle \mathrm{Irr}(\mathcal{A}(X)), \gamma_{\mathcal{A}(X)}\rangle$, where $\mathrm{Irr}(\mathcal{A}(X)) = \{[x)_X : x \in X\}$ and $\gamma_{\mathcal{A}(X)}\colon \mathrm{Up}(\mathrm{Irr}(\mathcal{A}(X)) \to \{0, 1\}$. We define the map $\theta_X\colon \langle X, \gamma\rangle \to \mathcal{X}(\mathcal{A}(X))$, as follows: $\theta_X(x) = [x)_X$, for every $x \in X$.

**Proposition 4.10.** *Let $\langle X, \gamma\rangle$ be a DS-structure. Then, the function $\theta_X\colon \langle X, \gamma\rangle \to \mathcal{X}(\mathcal{A}(X))$ is an isomorphism in $\mathcal{S}^{\mathrm{f}}$.*

*Proof.* It is straightforward by Proposition 3.2. $\square$

**Proposition 4.11.** *Let $f\colon \langle X,\gamma\rangle \to \langle Y,\tau\rangle$ be a DS-morphism, and let $h\colon A \to B$ be a $\wedge$-homomorphism. Then, the following diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\quad h \quad} & B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ \mathcal{A}(\mathcal{X}(A)) & \xrightarrow[h_{f_h}]{} & \mathcal{A}(\mathcal{X}(B)) \end{array} \qquad\qquad \begin{array}{ccc} \langle X,\gamma\rangle & \xrightarrow{\quad f \quad} & \langle Y,\tau\rangle \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ \mathcal{X}(\mathcal{A}(X)) & \xrightarrow[f_{h_f}]{} & \mathcal{X}(\mathcal{A}(Y)) \end{array}$$

*commute.*

*Proof.* Let $a \in A$. By Lemma 4.5, we have $\alpha_B(h(a)) = f_h^{-1}[\alpha_A(a)]$. Then, $(\alpha_B \circ h)(a) = \alpha_B(h(a)) = f_h^{-1}[\alpha_A(a)] = h_{f_h}(\alpha_A(a)) = (h_{f_h} \circ \alpha_A)(a)$.

In order to prove that the second diagram commutes, first we need to show that $\operatorname{dom}(f_{h_f} \circ \theta_X) = \operatorname{dom}(\theta_Y \circ f)$. Recall that $h_f\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ is given by $h_f(V) = f^{-1}[V]$, and $f_{h_f}\colon \mathcal{X}(\mathcal{A}(X)) \to \mathcal{X}(\mathcal{A}(Y))$ is defined by $\operatorname{dom} f_{h_f} = \{[x) \in \operatorname{Irr}(\mathcal{A}(X)) : h_f^{-1}[([x)]_{\mathcal{A}(X)}] \neq \emptyset\}$ and $f_{h_f}([x)) = \bigvee h_f^{-1}[([x)]_{\mathcal{A}(X)}]$. Then, on the one hand we obtain that

$$\begin{aligned} x \in \operatorname{dom}(f_{h_f} \circ \theta_X) &\iff x \in \theta_X^{-1}[\operatorname{dom} f_{h_f}] \iff [x)_X \in \operatorname{dom} f_{h_f} \\ &\iff h_f^{-1}[([x)]_{\mathcal{A}(X)}] \neq \emptyset \\ &\iff \exists V \in \mathcal{A}(Y) \text{ s.t. } h_f(V) \in ([x)]_{\mathcal{A}(X)} \\ &\iff \exists V \in \mathcal{A}(Y) \text{ s.t. } h_f(V) \sqsubseteq [x) \\ &\iff \exists V \in \mathcal{A}(Y) \text{ s.t. } [x) \subseteq f^{-1}[V] \\ &\iff \exists V \in \mathcal{A}(Y) \text{ s.t. } x \in \operatorname{dom} f \text{ and } f(x) \in V \\ &\iff x \in \operatorname{dom} f. \end{aligned}$$

Thus, $\operatorname{dom}(f_{h_f} \circ \theta_X) = \operatorname{dom} f$. On the other hand,

$$x \in \operatorname{dom}(\theta_Y \circ f) \iff x \in f^{-1}[\operatorname{dom}\theta_Y] \iff x \in f^{-1}[Y] \iff x \in \operatorname{dom} f.$$

Hence, $\operatorname{dom}(\theta_Y \circ f) = \operatorname{dom} f = \operatorname{dom}(f_{h_f} \circ \theta_X)$. Now let $x \in X$ and $V \in \mathcal{A}(Y)$. Then,

$$\begin{aligned} V \in h_f^{-1}[([x)]_{\mathcal{A}(X)}] &\iff h_f(V) \in ([x)]_{\mathcal{A}(X)} \\ &\iff f^{-1}[V] \sqsubseteq [x)_X \\ &\iff [x) \subseteq f^{-1}[V] \\ &\iff x \in \operatorname{dom} f \text{ and } f(x) \in V \end{aligned}$$

$$\iff [f(x))_Y \subseteq V$$
$$\iff V \sqsubseteq [f(x))_Y.$$

Hence, we obtain that $\bigvee h_f^{-1}[([x)]_{\mathcal{A}(X)}] = [f(x))_Y$. Therefore,

$$\begin{aligned}(f_{h_f} \circ \theta_X)(x) = f_{h_f}([x)) &= \bigvee h_f^{-1}[([x)]_{\mathcal{A}(X)}] \\ &= [f(x))_Y = \theta_Y(f(x)) = (\theta_Y \circ f)(x).\end{aligned}$$ □

Now we are ready to establish the main result of this section.

**Theorem 4.12.** *The categories $\mathcal{DS}^{\mathrm{f}}$ and $\mathcal{S}^{\mathrm{f}}$ are dually equivalent.*

*Proof.* Let $\Gamma\colon \mathcal{DS}^{\mathrm{f}} \to \mathcal{S}^{\mathrm{f}}$ be defined by: for every object $A$, $\Gamma(A) = \mathcal{X}(A)$, and for every morphism $h\colon A \to B$, $\Gamma(h) = f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$. By Propositions 3.4, 4.6 and 4.8, $\Gamma$ is a well-defined contravariant functor. Let $\Delta\colon \mathcal{S}^{\mathrm{f}} \to \mathcal{DS}^{\mathrm{f}}$ be defined by: for every object $\langle X, \gamma\rangle$, $\Delta(\langle X, \gamma\rangle) = \mathcal{A}(X)$, and for every morphism $f\colon \langle X, \gamma\rangle \to \langle Y, \tau\rangle$, $\Delta(f) = h_f\colon \mathcal{A}(Y) \to \mathcal{A}(X)$. By Propositions 3.2, 4.7 and 4.9, we have that $\Delta$ is a well-defined contravariant functor. Finally, let $\alpha\colon \mathbf{1}_{\mathcal{DS}^{\mathrm{f}}} \to \Delta \circ \Gamma$ given by: for every $A \in \mathcal{DS}^{\mathrm{f}}$, $\alpha(A) = \alpha_A\colon A \to (\Delta \circ \Gamma)(A)$; and let $\theta\colon \mathbf{1}_{\mathcal{S}^{\mathrm{f}}} \to \Gamma \circ \Delta$ be given by: for every $\langle X, \gamma\rangle \in \mathcal{S}^{\mathrm{f}}$, $\theta(X) = \theta_X\colon \langle X, \gamma\rangle \to (\Gamma \circ \Delta)(\langle X, \gamma\rangle)$. Then, from Theorem 3.5 and, by Propositions 4.10 and 4.11, we obtain that $\alpha$ and $\theta$ are natural isomorphisms. Therefore, the categories $\mathcal{DS}^{\mathrm{f}}$ and $\mathcal{S}^{\mathrm{f}}$ are dually equivalent. □

**Remark 4.13.** Let $\mathcal{DL}^{\mathrm{f}}$ be the category of finite distributive lattices and lattice-homomorphisms. It is clear that $\mathcal{DL}^{\mathrm{f}}$ is a full subcategory of $\mathcal{DS}^{\mathrm{f}}$. If we restrict the functor $\Gamma$ to $\mathcal{DL}^{\mathrm{f}}$, we obtain that $\Gamma\colon \mathcal{DL}^{\mathrm{f}} \to \mathcal{P}$ is a dual equivalence, where $\mathcal{P}$ is the category of posets and order-preserving maps.

## 5. Some applications of the dual equivalence between $\mathcal{DS}^{\mathrm{f}}$ and $\mathcal{S}^{\mathrm{f}}$

### *5.1. 1-1 and onto $\wedge$-homomorphisms*

In order to prove one of the main results of this subsection we need the following. Recall that a proper ideal $I$ of $A$ is said to be prime if whenever $a, b \in A$ are such that $a \wedge b$ exists and $a \wedge b \in I$, then $a \in I$ or $b \in I$.

**Lemma 5.1** ([18])**.** *Let $A$ be a finite distributive semilattice. Let $F$ be a filter and $I$ a non-empty ideal of $A$. If $F \cap I = \emptyset$, then there exists a prime ideal $P$ of $A$ such that $F \cap P = \emptyset$ and $I \subseteq P$.*

**Proposition 5.2.** *Let $A$ and $B$ be finite distributive semilattices and $h\colon A \to B$ a $\wedge$-homomorphism. Then $h$ is 1-1 if and only if its dual DS-morphism $f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$ is an onto partial function.*

*Proof.* Assume first that $h$ is 1-1. Let $x \in \mathrm{Irr}(A)$. Let

$$F := \mathrm{Fig}_B\left(h[(x]^c]\right) \quad \text{ and } \quad I := \mathrm{Idg}_B\left(h[(x]]\right).$$

Suppose that $F \cap I \neq \emptyset$. So, there is $b \in F \cap I$. On the one hand, by (1.1), there are $a_1, \ldots, a_n \in (x]^c$ and $b_1, \ldots, b_n \in B$ such that for every $i = 1, \ldots, n$, $h(a_i) \leq b_i$ and $b = b_1 \wedge \cdots \wedge b_n$. On the other hand, since $b \in I$, there is $a \in (x]$ such that $b \leq h(a)$. For every $i = 1, \ldots, n$, we obtain that $h(a_i \vee a) = h(a_i) \vee h(a) \leq b_i \vee h(a)$. Let $a' := (a_1 \vee a) \wedge \cdots \wedge (a_n \vee a)$. Since for every $i = 1, \ldots, n$ we have $a_i \vee a \in (x]^c$, it follows that $a' \in (x]^c$. Now notice that

$$\begin{aligned} h(a') &= h(a_1 \vee a) \wedge \cdots \wedge h(a_n \vee a) \\ &\leq (b_1 \vee h(a)) \wedge \cdots \wedge (b_n \vee h(a)) \\ &= (b_1 \wedge \cdots \wedge b_n) \vee h(a) = h(a). \end{aligned}$$

Then, since $h$ is an embedding, we have that $a' \leq a$. Hence $a' \in (x]$, which is a contradiction. Therefore, we conclude that $F \cap I = \emptyset$. Then, by Lemma 5.1, there exists a prime ideal $P$ of $B$ such that $F \cap P = \emptyset$ and $I \subseteq P$. Now from Lemma 4.3, there is $y \in \mathrm{Irr}(B)$ such that $P = (y]$. Thus $h[(x]^c] \cap (y] = \emptyset$ and $h[(x]] \subseteq (y]$. It follows that $h^{-1}[(y]] = (x]$. Hence, $y \in \mathrm{dom}\, f_h$ and $f_h(y) = \bigvee h^{-1}[(y]] = \bigvee (x] = x$. Therefore, $f_h$ is onto.

Conversely, assume that the DS-morphism $f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$ is onto. Let $a, a' \in A$ be such that $h(a) = h(a')$. Let $x \in \mathrm{Irr}(A)$ be such that $a' \leq x$. Since $f_h$ is onto, there is $y \in \mathrm{dom}\, f_h$ such that $f_h(y) = x$. Since $a' \leq x = f_h(y)$, it follows by Lemma 4.5 that $h(a') \leq y$. Thus $h(a) \leq y$. By Lemma 4.5, we obtain that $a \leq f_h(y) = x$. Then, we have proved that $\forall x \in \mathrm{Irr}(A)(a' \leq x \implies a \leq x)$. Hence $a \leq a'$. Similarly we have that $a' \leq a$. That is, $a = a'$. □

**Proposition 5.3.** *Let $A$ and $B$ be finite distributive semilattices and $h\colon A \to B$ a $\wedge$-homomorphism. Then, $h$ is onto if and only if the dual DS-morphism $f_h\colon \mathcal{X}(B) \to \mathcal{X}(A)$ satisfies the following conditions:*

*(i)* $\mathrm{dom}\, f_h = \mathrm{Irr}(B)$*;*

*(ii)* $f_h$ *is an order-embedding;*

*(iii) For every $V \in \mathcal{A}(\mathcal{X}(B))$,*

$$\gamma_A \left( \bigcup \left\{ [f_h(y))_{\mathrm{Irr}(A)} : y \in V \right\} \right) = 1.$$

*Proof.* Assume that $h$ is onto. Let $y \in \mathrm{Irr}(B)$. Since $h$ is onto, there is $a \in A$ such that $h(a) = y$. Then $a \in h^{-1}[(y]_B]$. Thus $h^{-1}[(y]_B] \neq \emptyset$. Hence $y \in \mathrm{dom}\, f_h$. We have proved that $\mathrm{dom}\, f_h = \mathrm{Irr}(B)$. In order to prove condition (ii), let $y_1, y_2 \in \mathrm{dom}\, f_h$ be such that $f_h(y_1) \leq f_h(y_2)$. By definition of $f_h$, we obtain that $\bigvee h^{-1}[(y_1]_B] \leq \bigvee h^{-1}[(y_2]_B]$. Now, since $h$ is onto, there are $a_1, a_2 \in A$ such that $h(a_1) = y_1$ and $h(a_2) = y_2$. Let

$$h^{-1}[(y_1]_B] = \{a_1^1, \ldots, a_n^1\} \quad \text{and} \quad h^{-1}[(y_2]_B] = \{a_1^2, \ldots, a_m^2\}.$$

Given that $h(a_1) = y_1$, we have $a_1 \in h^{-1}[(y_1]_B]$. Moreover, for every $i = 1, \ldots, n$, $h(a_i^1) \leq y_1 = h(a_1)$. Hence $h(a_1) = h(a_1^1) \vee \cdots \vee h(a_n^1)$. Similarly, we have that $h(a_2) = h(a_1^2) \vee \cdots \vee h(a_m^2)$. Since $a_1^1 \vee \cdots \vee a_n^1 \leq a_1^2 \vee \cdots \vee a_m^2$, it follows that

$$y_1 = h(a_1) = h(a_1^1) \vee \cdots \vee h(a_n^1) \leq h(a_1^2) \vee \cdots \vee h(a_m^2) = h(a_2) = y_2.$$

Hence $f_h$ is an order-embedding, and thus condition (ii) holds. Let $V \in \mathcal{A}(\mathcal{X}(B))$. Recall that $\mathcal{A}(\mathcal{X}(B)) = \{\alpha_B(b) : b \in B\}$. Thus, $V = \alpha_B(b)$ for some $b \in B$. Since $h$ is onto, there is $a \in A$ such that $h(a) = b$. Let us show that $\bigcup \left\{ [f_h(y))_{\mathrm{Irr}(A)} : y \in V \right\} \subseteq \alpha_A(a)$. Let $y \in V$. By Lemma 4.5, we have $V = \alpha_B(b) = \alpha_B(h(a)) = f_h^{-1}[\alpha_A(a)]$. Then, $f_h(y) \in \alpha_A(a)$. Since $\alpha_A(a)$ is an upset of $\mathrm{Irr}(A)$, it follows that $[f_h(y))_{\mathrm{Irr}(A)} \subseteq \alpha_A(a)$. Hence $\bigcup\{[f_h(y))_{\mathrm{Irr}(A)} : y \in V\} \subseteq \alpha_A(a)$. Now, since $\gamma_A(\alpha_A(a)) = 1$, and by condition (S3) of Definition 3.1, it follows that $\gamma_A \left(\bigcup\{[f_h(y))_{\mathrm{Irr}(A)} : y \in V\}\right) = 1$. Hence, condition (iii) holds.

Conversely, suppose that $f_h$ satisfies conditions (i)–(iii). Let $b \in B$. By condition (iii), we have that

$$\gamma_A \left( \bigcup \{[f_h(y))_{\mathrm{Irr}(A)} : y \in \alpha_B(b)\} \right) = 1.$$

Let $U := \bigcup\{[f_h(y))_{\mathrm{Irr}(A)} : y \in \alpha_B(b)\}$. Then, we obtain that $U \in \mathcal{A}(\mathcal{X}(A))$. Thus, there is $a \in A$ such that $U = \alpha_A(a)$. We prove that $h(a) = b$. Let $y \in \mathrm{Irr}(B)$. If $b \leq y$, then $f_h(y) \in [f_h(y))_{\mathrm{Irr}(A)} \subseteq U$. It follows that $a \leq f_h(y)$, and thus $h(a) \leq y$. Hence $h(a) \leq b$. Now suppose that $h(a) \leq y$. Thus $a \leq f_h(y)$. Then $f_h(y) \in \alpha_A(a) = U$. It follows that there is $y' \in \alpha_B(b)$ such that $f_h(y') \leq f_h(y)$. Since $f_h$ is order-embedding, we have that $y' \leq y$. Then $y \in \alpha_B(b)$, and thus $b \leq y$. Hence $b \leq h(a)$. Therefore, $h(a) = b$. □


Figure 2:

The following example shows that condition (iii) of the previous proposition is not a consequence of conditions (i) and (ii). In other words, conditions (i) and (ii) on $f_h$ does not imply that $h$ is onto.

**Example 5.4.** Consider the diagrams in Figure 2. It is clear that $f_h$ satisfies conditions (i) and (ii) of Proposition 5.3. It is also clear that $h$ is a $\wedge$-homomorphism and it is not an onto map. Let us see that $f_h$ does not satisfy condition (iii) of Proposition 5.3. Recall that for every $U \in \mathrm{Up}(\mathrm{Irr}(A))$, $\gamma_A(U) = 1$ iff $\bigwedge U$ exists in $A$. Moreover $U \in \mathcal{A}(\mathcal{X}(A))$ iff $U \in \mathrm{Up}(\mathrm{Irr}(A))$ and $\gamma_A(U) = 1$. Let $V := \{y_1, y_2\}$. Thus $V \in \mathcal{A}(\mathcal{X}(B))$. We have

$$\bigcup \left\{ [f_h(y))_{\mathrm{Irr}(A)} : y \in V \right\} = [f_h(y_1))_{\mathrm{Irr}(A)} \cup [f_h(y_2))_{\mathrm{Irr}(A)} = \{x_1, x_2\}.$$

Then, $\gamma_A \left( \bigcup \left\{ [f_h(y))_{\mathrm{Irr}(A)} : y \in V \right\} \right) \neq 1$. Hence, condition (iii) does not hold.

*5.2. $\wedge$-subalgebras*

Let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice. We will said that a subalgebra $\langle A_0, \vee, 1 \rangle$ of $\langle A, \vee, 1 \rangle$ is a $\wedge$-*subalgebra* when for all $a_1, \ldots, a_n \in A_0$, if $a_1 \wedge \cdots \wedge a_n$ exists in $A$, then $a_1 \wedge \cdots \wedge a_n \in A_0$.

Our main goal in this subsection is to obtain, from the dual equivalence, an effective method to characterize all the $\wedge$-subalgebras of a finite distributive semilattice.

**Remark 5.5.** Let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice and $\langle A_0, \vee, 1 \rangle$ a $\wedge$-subalgebra. Let $a_1, \ldots, a_n \in A_0$. If $a_1 \wedge_{A_0} \cdots \wedge_{A_0} a_n$ exists in $A_0$, then $a_1 \wedge_{A_0} \cdots \wedge_{A_0} a_n = a_1 \wedge \cdots \wedge a_n$. That is, the infimum of $\{a_1, \ldots, a_n\}$ exists in $A$ and it is $a_1 \wedge_{A_0} \cdots \wedge_{A_0} a_n$. Hence, every $\wedge$-subalgebra is also distributive.

Let $\langle A, \vee, 1 \rangle$ be a finite distributive semilattice and $\langle X, \gamma \rangle$ its dual DS-structure. From Proposition 5.2, we have that the $\wedge$-subalgebras of $A$ are dually characterized as those DS-structures $\langle Y, \tau \rangle$ for which there is an onto DS-morphism $f \colon \langle X, \gamma \rangle \to \langle Y, \tau \rangle$.

Let $\langle X, \gamma \rangle$ and $\langle Y, \tau \rangle$ be finite DS-structures. Let $f \colon \langle X, \gamma \rangle \to \langle Y, \tau \rangle$ be an onto DS-morphism. Let $X_0 := \operatorname{dom} f \in \operatorname{Up}(X)$. Let $\theta_f := \{(x, x') \in X_0^2 : f(x) = f(x')\}$. It is clear that $\theta_f$ is an equivalence relation on $X_0 = \operatorname{dom} f$. Let

$$X_0/\theta_f := \{x/\theta_f : x \in \operatorname{dom} f\}.$$

We define a binary relation $\preceq$ on $X_0/\theta_f$ as follows:

$$x/\theta_f \preceq x'/\theta_f \iff f(x) \leq f(x'),$$

for all $x, x' \in \operatorname{dom} f$. Notice that the definition of the relation $\preceq$ does not depend on the representatives in the equivalence classes.

**Proposition 5.6.** *The relation $\preceq$ is a partial order on $X_0/\theta_f$.*

We define the canonical partial function $\pi \colon X \to X_0/\theta_f$ as follows:

$$\operatorname{dom} \pi = \operatorname{dom} f \quad \text{ and } \quad \pi(x) = x/\theta_f,$$

for every $x \in \operatorname{dom} \pi$. It is straightforward to show directly that $\pi$ satisfies conditions (M1) and (M2) of Definition 4.1.

Now let us define a (total) function $\widehat{f} \colon X_0/\theta_f \to Y$ as follows:

$$\widehat{f}(x/\theta_f) = f(x),$$

for every $x \in \operatorname{dom} f$. By the definition of $\theta_f$, we have that $\widehat{f}$ is well-defined.

**Proposition 5.7.** *The map $\widehat{f} \colon \langle X_0/\theta_f, \preceq \rangle \to \langle Y, \leq \rangle$ is an order-isomorphism.*

*Proof.* Let $x/\theta_f, x'/\theta_f \in X_0/\theta_f$. Then, by the definition of $\preceq$ and $\widehat{f}$, we have

$$x/\theta \preceq x'/\theta_f \iff f(x) \leq f(x') \iff \widehat{f}(x/\theta_f) \leq \widehat{f}(x'/\theta_f).$$

Thus, $\widehat{f}$ is an order-embedding. Moreover, since $f$ is onto, it follows that $\widehat{f}$ is also an onto map. Hence, $\widehat{f}$ is an order-isomorphism. □

Now we define a map $\widehat{\gamma} \colon \operatorname{Up}(X_0/\theta_f) \to \{0, 1\}$ as follows: $\widehat{\gamma}(W) = \tau(\widehat{f}[W])$, for every $W \in \operatorname{Up}(X_0/\theta_f)$.

**Proposition 5.8.** *$\langle X_0/\theta_f, \widehat{\gamma} \rangle$ is a DS-structure.*

*Proof.* (S1) $\widehat{\gamma}(\emptyset) = \tau(\widehat{f}[\emptyset]) = \tau(\emptyset) = 1$.

(S2) Since $\widehat{f}$ is an order-isomorphism, it follows that $\widehat{f}[[x/\theta_f)_{X_0/\theta_f}] = [\widehat{f}(x/\theta_f))_Y$. Then,

$$\widehat{\gamma}\left([x/\theta_f)_{X_0/\theta_f}\right) = \tau\left(\widehat{f}[[x/\theta_f)_{X_0/\theta_f}]\right) = \tau\left([\widehat{f}(x/\theta_f))_Y\right) = 1.$$

(S3) Let $W_1, W_2 \in \mathrm{Up}(X_0/\theta_f)$ be such that $W_1 \subseteq W_2$. Thus $\widehat{f}[W_1] \subseteq \widehat{f}[W_2]$. Then, $\widehat{\gamma}(W_2) = \tau\left(\widehat{f}[W_2]\right) \leq \tau\left(\widehat{f}[W_1]\right) = \widehat{\gamma}(W_1)$. □

**Theorem 5.9.** *The function $\widehat{f}\colon \langle X_0/\theta_f, \widehat{\gamma}\rangle \to \langle Y, \tau\rangle$ is an isomorphism in the category $\mathcal{S}^{\mathrm{f}}$.*

*Proof.* From Proposition 5.7, we have that $\widehat{f}\colon X_0/\theta_f \to Y$ is an order-isomorphism, and by the definition of $\widehat{\gamma}$, it follows that

$$\widehat{\gamma}(W) = 1 \iff \tau\left(\widehat{f}[W]\right) = 1,$$

for every $W \in \mathrm{Up}(X_0/\theta_f)$. Therefore, $\widehat{f}$ is an isomorphism of $\mathcal{S}^{\mathrm{f}}$. □

Finally, we are ready to obtain a dual characterization of the $\wedge$-subalgebras. Let $A$ be a finite distributive semilattice and $\langle X, \gamma\rangle$ its dual DS-structure. Then, by Proposition 5.2 and Theorem 5.9, we obtain that the $\wedge$-subalgebras of $A$ are dually characterized by the DS-structures of the form $\langle X_0/\theta, \widehat{\gamma}\rangle$ satisfying the following conditions:

(1) $X_0 \in \mathrm{Up}(X)$;

(2) $\theta$ is an equivalence relation defined on $X_0$;

(3) $X_0/\theta$ is partially ordered by a partial order $\preceq$ such that the canonical partial function $\pi\colon X \to X_0/\theta$, defined by $\pi(x) = x/\theta$ for every $x \in X_0$, is order-preserving;

(4) $\widehat{\gamma}\colon \mathrm{Up}\,(X_0/\theta) \to \{0,1\}$ is a function such that $\langle X_0/\theta, \widehat{\gamma}\rangle$ is a DS-structure and $\pi\colon \langle X, \gamma\rangle \to \langle X_0/\theta, \widehat{\gamma}\rangle$ is a DS-morphism.

**Example 5.10.** Let $A$ be the distributive semilattice given in Figure 3. Then $X := \mathrm{Irr}(A) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $X$ is ordered by the order induced by the order of $A$, see Figure 3. Recall that $\gamma_A\colon \mathrm{Up}(X) \to \{0,1\}$ is defined as: $\gamma_A(U) = 1$ iff $\bigwedge U$ exists in $A$. Hence $\langle X, \gamma_A\rangle$ is the dual DS-structure of $A$. We recall that $\mathcal{A}(X) = \{U \in \mathrm{Up}(X) : \gamma_A(U) = 1\}$ and $A \cong \mathcal{A}(X)$. Thus $\mathcal{A}(X) = \{\emptyset\} \cup \{[x_i)_X : i = 1, \ldots, 6\} \cup \{[x_4)_X \cup [x_5)_X, [x_4)_X \cup [x_6)_X, [x_5)_X \cup [x_6)_X\}$. Now let us build up a $\wedge$-subalgebra of $A$ following the previous steps:


Figure 3:

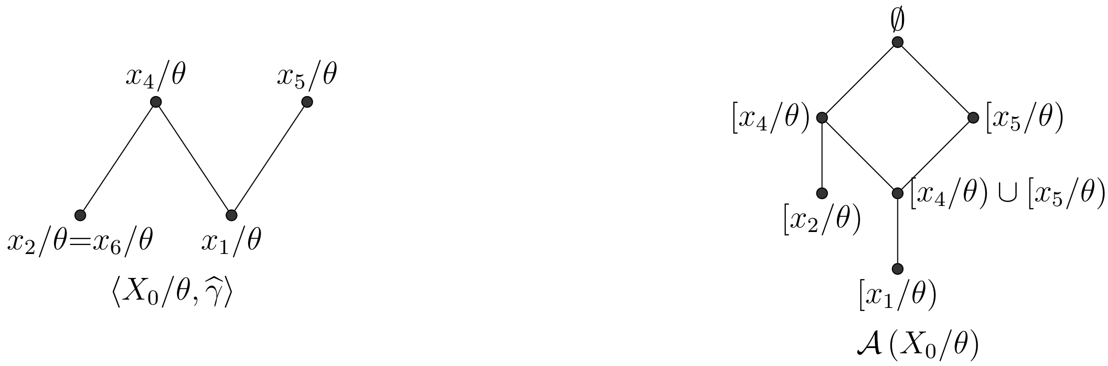


Figure 4:

(1) Let $X_0 = \{x_1, x_2, x_4, x_5, x_6\} \in \mathrm{Up}(X)$;

(2) Let $\theta$ be the least equivalence relation such that $(x_2, x_6) \in \theta$;

(3) Let $\preceq$ be the partial order on $X_0/\theta$ as given in Figure 4;

(4) Let $\widehat{\gamma}\colon \mathrm{Up}\,(X_0/\theta) \to \{0, 1\}$ be defined as follows:

$$\mathcal{A}\,(X_0/\theta) = \widehat{\gamma}^{-1}[\{1\}] = \{\emptyset, [x_1/\theta), [x_2/\theta), [x_4/\theta), [x_5/\theta), [x_4/\theta) \cup [x_5/\theta)\}.$$

Hence $\langle \mathcal{A}(X_0/\theta), \sqcup, \emptyset\rangle$, with $\sqcup = \cap$, is (isomorphic to) a $\wedge$-subalgebra of $A$, see Figure 4.

*5.3. Finite semi-boolean algebras*

A semilattice $\langle A, \vee, 1\rangle$ is called a *semi-boolean algebra* (see [2]) if every principal upset $[a)$ is a Boolean algebra. Thus, it is clear that semi-boolean algebras are, in particular, distributive semilattices satisfying the lower bound property. In this subsection, we obtain a representation for the class of finite semi-boolean algebras.

Let $A$ be a semilattice. An element $x \in A$ is said to be a *dual atom* if $x \neq 1$ and there is no element $a \in A$ such that $x < a < 1$. We denote by $\mathrm{At_d}(A)$ the collection of all dual atoms of $A$.

Let $\langle A, \vee, 1\rangle$ be a finite distributive semilattice. Recall that for every $a \in A$, $[a)$ is a Heyting algebra and $\to_a$ is the Heyting implication on $[a)$ (see (2.1)). In particular, if $A$ is a semi-boolean algebra, then $\to_a$ is the Boolean implication on the Boolean algebra $[a)$. Moreover, recall the Hilbert implication $\to$ on $A$ defined by (2.2), see also Corollary 2.12.

**Proposition 5.11.** *Let $\langle A, \vee, 1\rangle$ be a finite distributive semilattice. Then, $A$ is a semi-boolean algebra if and only if $\mathrm{Irr}(A) = \mathrm{At_d}(A)$.*

*Proof.* Assume that $A$ is a semi-boolean algebra. First, it is clear that $\mathrm{At_d}(A) \subseteq \mathrm{Irr}(A)$. Let $x \in \mathrm{Irr}(A)$, and let $a \in A$ be such that $x < a \leq 1$. Since $x$ is irreducible, it follows by Proposition 2.7 that $a \to x = x$. Then, $(a \vee x) \to_x x = x$. That is, $a \to_x x = x$. Since $[x)$ is a Boolean algebra, it follows that $a = 1$. Hence $x \in \mathrm{At_d}(A)$.

Conversely, assume that $\mathrm{At_d}(A) = \mathrm{Irr}(A)$. Let $a \in A$. It is straightforward to show that $\mathrm{At_d}([a)) = \mathrm{At_d}(A) \cap [a)$. Thus, by hypothesis, we have $\mathrm{At_d}([a)) = \mathrm{Irr}(A) \cap [a)$. Moreover, it is easy to show that $\mathrm{Irr}([a)) = \mathrm{Irr}(A) \cap [a)$. Then, we have that $\mathrm{At_d}([a)) = \mathrm{Irr}([a))$. Hence, since $[a)$ is a finite distributive lattice and $\mathrm{At_d}([a)) = \mathrm{Irr}([a))$, it follows that $[a)$ is a Boolean algebra. Therefore, $A$ is a semi-boolean algebra. $\square$

**Theorem 5.12.** *Let $A$ be a finite distributive semilattice and $\langle X, \gamma\rangle$ its dual DS-structure. Then, $A$ is a semi-boolean algebra if and only if $X$ is an antichain.*

*Proof.* Assume that $A$ is a semi-boolean algebra. By the previous proposition, we have that $\mathrm{Irr}(A) = \mathrm{At_d}(A)$. Then, since $X$ is order-isomorphic to $\mathrm{Irr}(A)$, it follows that $X$ is an antichain. Conversely, assume that $X$ is an antichain. Recall that $\langle A, \vee, 1\rangle \cong \langle \mathcal{A}(X), \sqcup, \emptyset\rangle$ and $\mathcal{A}(X) = \{U \in \mathrm{Up}(X) : \gamma(U) = 1\}$. Since $X$ is an antichain, we have that $\mathrm{Up}(X) = \mathcal{P}(X)$. Let $U \in \mathcal{A}(X)$ and $W \in [U)_{\mathcal{A}(X)}$. Consider $W^- := U \setminus W$. Thus, $W^- \subseteq U$ and $W^- \in \mathrm{Up}(X)$. Then, $W^- \in [U)_{\mathcal{A}(X)}$. We have that $W \sqcap W^- = W \cup W^- = U$ and $W \sqcup W^- = W \cap W^- = \emptyset$. Hence $W^-$ is the complement of $W$ in $[U)_{\mathcal{A}(X)}$. Thus, $[U)_{\mathcal{A}(X)}$ is a Boolean algebra. Then $\mathcal{A}(X)$ is a semi-boolean algebra. Therefore, $A$ is a semi-boolean algebra. $\square$

From the previous theorem and by Theorem 4.12, we obtain that the category of finite semi-boolean algebras and $\wedge$-homomorphisms is dually equivalent to the category of finite DS-structures $\langle X, \gamma\rangle$ such that $X$ is antichain and DS-morphisms. Moreover, notice that finite semi-boolean algebras correspond dually to structures $\langle X, \gamma \rangle$ where $X$ is a finite set and $\gamma\colon \mathcal{P}(X) \to \{0,1\}$ is a map satisfying conditions (S1)–(S3) of Definition 3.1 (considering the trivial order on $X$: $x \leq y \iff x = y$). Also, notice that if $\langle X, \gamma \rangle$ and $\langle Y, \tau \rangle$ are finite DS-structures such that $X$ and $Y$ are antichains, then a partial function $f\colon X \to Y$ is a DS-morphism if and only if it satisfies (M3) of Definition 4.1. That is, in this context, the conditions (M1) and (M2) are always valid.

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