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NOTES ON MILDLY DISTRIBUTIVE SEMILATTICES

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ABSTRACT. In this paper we shall investigate the mildly distributive meet-semilattices by means of the study of their filters and Frink-ideals as well as applying the theory of annihilator. We recall some characterizations of the condition of mildly-distributivity and we give several new characterizations. We prove that the definition of strong free distributive extension, introduced by Hickman in 1984, can be simplified and we show a correspondence between (prime) Frink-ideals of a mildly distributive semilattice and (prime) ideals of its strong free distributive extension.

1. Introduction

Mildly distributive semilattices were introduced and studied in [12] by Hickman. This class of semilattices lies between two important classes of semilattices: distributive semilattices [11] and weakly distributive semilattices [6] (also called prime semilattices in [1]). The class of weakly distributive semilattices was introduced in [1] by Balbes with the name of prime semilattices and this class was intensively studied in [6] due to Cornish and Hickman. In [11] Grätzer introduces distributive join-semilattices and gives a topological representation generalizing the spectral-style topological representation for distributive lattices proved by Stone. There are several works studying the class of distributive semilattices such as [5], [11], [14] and [15]. In [3] (also see [4]) Celani presented a full spectral-style topological duality for distributive semilattices extending the previous topological representation given by Grätzer. On the other hand, in [2] Bezhanishvili and Jansana introduced a Priestley-style topological duality for distributive semilattices that generalizes the well-known topological duality for distributive lattices due to Priestley.

The main purpose of this paper is to study from an algebraic point of view the class of mildly distributive semilattices and investigate different characterizations of the condition of mildly-distributivity. We will recall some known characterizations in the literature of the condition of mildly-distributivity on semilattices and we show some new characterizations.

The paper is organized as follows. In Section 2 we introduce some notations and we recall some definitions and basic properties of filters and ideals for posets which are needed in the paper. In Section 3 we recall the notions of meet-prime, irreducible, prime and optimal filters of a semilattice and we also consider the notions of irreducible and prime Frink-ideal. We prove a separation theorem between filters and Frink-ideals by means of irreducible Frink-ideals. In Section 4 we will study the class of mildly distributive semilattices introduced by Hickman [12]. For this purpose we put our attention on the several notions of prime filters and prime Frink-ideals that we introduced

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in the previous section. Also, we present several characterizations of the mildly-distributivity condition on semilattices.

In [12] (see also [6]), Hickman introduces the notion of *strong free distributive extension* of a semilattice. A similar notion is considered in [2] for distributive semilattices (under the name of 'distributive envelope'). In Section 5 we will provide a useful simple characterization of the strong free distributive extension of a mildly distributive semilattice. Finally, we will prove that the ordered set of (prime) Frink-ideals of a mildly distributive semilattice is order-isomorphic to the ordered set of (prime) ideals of its strong free distributive extension.

2. Preliminaries

Let $P = \langle P, \leq \rangle$ be a poset. A set $Y \subseteq P$ is a *down-set* provided that for every $b \in P$ if $b \leq a$ for some $a \in Y$, then $b \in Y$. Dually, a subset $X \subseteq P$ is called an *up-set* when for every $b \in P$ if $a \leq b$ for some $a \in Y$, then $b \in Y$. If $a \in P$, $\downarrow a$ or $(a]$ denotes the down-set $\{b \in P : b \leq a\}$ and $\uparrow a$ or $[a)$ denotes the up-set $\{b \in P : a \leq b\}$. If $Y \subseteq P$, let Y^u denote the set of all upper bounds of Y and Y^l the set of all lower bounds of Y. Note that if $x \in P$, then $\downarrow x = (\{x\})^{ul}$ and $\uparrow x = (\{x\})^{lu}$.

A subset $I \subseteq P$ is called a *Frink-ideal* of P when for every finite $X \subseteq I$, $(X)^{ul} \subseteq I$ (cf. [10]). Dually, a subset $F \subseteq P$ is called a *Frink-filter* of P when for every finite $X \subseteq F$, $(X)^{lu} \subseteq F$. It should be noted that the empty set may be a Frink-ideal or a Frink-filter. It immediately follows that Frink-filters are up-sets and Frink-ideals are down-sets. Note that P is both a Frink-filter and a Frink-ideal. A Frink-filter is proper if it is not P and similarly we say that a Frink-ideal is proper.

It is not hard to see that a subset $I \subseteq P$ is a Frink-ideal if and only if for every $a_1, \ldots, a_n \in I$ and $c \in P$, whenever $[a_1) \cap \cdots \cap [a_n) \subseteq [c)$, we have $c \in I$ and if c is a smallest element of P then also $c \in I$. Dually we can state the similar condition for Frink-filters. Let L be a semilattice. We denote by $\mathsf{FId}(P)$ the family of all Frink-ideals of P. It is known, and is not hard to check, that $\mathsf{FId}(P)$ is an algebraic closure system. So for every subset X of P we have the *Frink-ideal generated by* X, which it is denoted by $\mathrm{Id_F}(X)$. Moreover, it is straightforward to show

$$\mathrm{Id_F}(X) = \{a \in P : [x_1) \cap \cdots \cap [x_n) \subseteq [a) \text{ for some } x_1, \ldots, x_n \in X\}.$$

Then, we have that $\mathsf{FId}(P) = \langle \mathsf{FId}(P), \cap, \vee \rangle$ is a complete lattice. A Frink-ideal of P is called *finitely generated* if there exists a non-empty finite subset $X \subseteq P$ such that $I = \mathrm{Id_F}(X)$. Let us denote by $\mathsf{FId}^{\mathsf{f}}(P)$ the collection of all finitely generated Frink-ideals of P.

A non-empty subset $I \subseteq P$ is said to be an *order-ideal* of P if it is an up-directed down-set, that is, a down-set of P such that for every $a, b \in I$ there is $c \in I$ such that $a, b \leq c$. Similarly a non-empty subset $F \subseteq P$ is said to be an *order-filter* of P if it is a down-directed up-set, that is, an up-set of P such that for every $a, b \in F$ there is $c \in F$ with $c \leq a, b$. Every order-ideal is a Frink-ideal and every order-filter is a Frink-filter.

3. Semilattices

In this section we introduce the basic notions about semilattices as well as some properties that we will need throughout the paper.

A *meet-semilattice* is an algebra $L = \langle L, \wedge \rangle$ of type (2) such that the operation $\wedge$ is idempotent, commutative, associative. In this paper, unless stated otherwise, *semilattice* means meet-semilattice. As usual, the binary relation $\leq$ defined by $a \leq b$ if and only if $a \wedge b = a$ is a partial order and for every $a, b \in L$, $a \wedge b$ is the infimum of a and b. A semilattice with top element is an algebra $\langle L, \wedge, 1\rangle$ of type $(2, 0)$ such that $\langle L, \wedge\rangle$ is a semilattice and $a \wedge 1 = a$ for all $a \in L$ and a *bounded* semilattice is an algebra $\langle L, \wedge, 0, 1\rangle$ of type $(2, 0, 0)$ such that $\langle L, \wedge, 1\rangle$ is a semilattice with top element and $a \wedge 0 = 0$ for all $a \in L$.

Let L be a semilattice. A subset $F \subseteq L$ is said to be a *filter* of L if it is an order-filter of the semilattice order. This holds if and only if (1) for every $a, b \in F$, $a \wedge b \in F$ and (2) for every $a \in F$ and $b \in L$, if $a \leq b$, then $b \in F$. We denote the collection of all filters of L by $\mathsf{Fi}(L)$. Notice that if L has top element, then $\mathsf{Fi}(L)$ is an algebraic closure system and if L has no top element, then $\mathsf{Fi}(L) \cup \{\emptyset\}$ is a closure system. So, in any case, for every non-empty subset X of L we can take the least filter of L that contains to X and we denote it by $\mathrm{Fi}(X)$. A filter $F \in \mathsf{Fi}(L)$ is *proper* if $F \neq L$. A proper filter F of L is called *meet-prime* if it is a meet-prime element of the lattice of filters, i.e., for all filters F_1, F_2 of L, if $F_1 \cap F_2 \subseteq F$ then $F_1 \subseteq F$ or $F_2 \subseteq F$. We denote by $\mathsf{Fi}_{\mathsf{mpr}}(L)$ the family of all meet-prime filters of L. A proper filter F of L is called *irreducible* if for all filters F_1, F_2 of L, if $F_1 \cap F_2 = F$ then $F_1 = F$ or $F_2 = F$ and we denote by $\mathsf{Fi}_{\mathsf{irr}}(L)$ the collection of all irreducible filters of L. It is clear that a filter F of L is irreducible if and only if for any finite subfamily $F_1, \dots, F_n$ of $\mathsf{Fi}(A)$, if $F = F_1 \cap \dots \cap F_n$ then $F = F_i$ for some $1 \leq i \leq n$. Notice that every meet-prime filter is irreducible. The converse is valid in the class of distributive semilattice (see [3]).

The following two results will be needed in Theorem 4.5, where we prove several characterizations of mildly distributive semilattice. The next lemma is a generalization of [3: Lemma 6].

Lemma 3.1. *Let L be a semilattice. A proper filter F of L is irreducible if and only if for every $a_1, \dots, a_n \notin F$ there exists $c \notin F$ and $f \in F$ such that $a_i \wedge f \leq c$ for all $1 \leq i \leq n$.*

Theorem 3.2 ([3])**.** *Let L be a semilattice. Let $F \in \mathsf{Fi}(L)$ and I be an order-ideal of L. If $F \cap I = \emptyset$, then there exists $P \in \mathsf{Fi}_{\mathsf{irr}}(L)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

The following two definitions together with the notion of meet-prime filter are the most natural notions that generalize the notion of prime filter in Lattice Theory.

Definition 3.3. Let L be a semilattice. A proper filter F of L is said to be *optimal* if $L \setminus F$ is a Frink-ideal. We denote by $\mathsf{Opt}(L)$ the collection of all optimal filters of L.

Definition 3.4. Let L be a semilattice. A proper filter F of L is called *prime* if for each non-empty finite subset $\{a_1, \dots, a_n\}$ of L such that there exists $a_1 \vee \dots \vee a_n$ in L and $a_1 \vee \dots \vee a_n \in F$, then $a_i \in F$ for some $1 \leq i \leq n$. Let us denote by $\mathsf{Fi}_{\mathsf{pr}}(L)$ the family of all prime filters of a semilattice L.

The following lemma shows the relation between the notions of meet-prime, optimal and prime filters of a semilattice.

Lemma 3.5. *For each semilattice L, we have*

$$\mathsf{Fi}_{\mathsf{mpr}}(L) \subseteq \mathsf{Opt}(L) \subseteq \mathsf{Fi}_{\mathsf{pr}}(L),$$

and

$$\mathsf{Fi}_{\mathsf{mpr}}(L) \subseteq \mathsf{Fi}_{\mathsf{irr}}(L) \cap \mathsf{Fi}_{\mathsf{pr}}(L).$$

Proof. Let $F \in \mathsf{Fi}_{\mathsf{mpr}}(L)$. Let $a_1, \dots, a_n \in F^c$ and let $a \in L$ be such that $[a_1) \cap \dots \cap [a_n) \subseteq [a)$. We suppose that $a \in F$. So, we have that $[a_1) \cap \dots \cap [a_n) \subseteq [a) \subseteq F$ and, since F is a meet-prime filter, it follows that there is $1 \leq i \leq n$ such that $[a_i) \subseteq F$. Then $a_i \in F$, which is a contradiction. Then, $a \in F^c$. Therefore, F is optimal.

Now let $F \in \mathsf{Opt}(L)$. Let $\{a_1, \dots, a_n\}$ be a finite subset of L and assume that there exists $a_1 \vee \dots \vee a_n$ in L and $a_1 \vee \dots \vee a_n \in F$. Suppose towards a contradiction that $a_1, \dots, a_n \notin F$.

Since $[a_1) \cap \cdots \cap [a_n) = [a_1 \vee \cdots \vee a_n)$ and F is optimal, we obtain that $a_1 \vee \cdots \vee a_n \notin F$. So we arrive to an absurd. Hence $a_i \in F$, for some $1 \leq i \leq n$.

The proof of the inclusion $\mathsf{Fi}_{\mathsf{mpr}}(L) \subseteq \mathsf{Fi}_{\mathsf{irr}}(L) \cap \mathsf{Fi}_{\mathsf{pr}}(L)$ is easy and left to the reader. □

Let L be a semilattice. Given a proper Frink-ideal I of L, we say that I is *prime* when for all $I_1, I_2 \in \mathsf{FId}(L)$, if $I_1 \cap I_2 \subseteq I$ then $I_1 \subseteq I$ or $I_2 \subseteq I$. We also say that I is *irreducible* when for all $I_1, I_2 \in \mathsf{FId}(L)$, if $I_1 \cap I_2 = I$ then $I_1 = I$ or $I_2 = I$. We denote by $\mathsf{FId}_{\mathsf{pr}}(L)$ the family of all prime Frink-ideals of L and by $\mathsf{FId}_{\mathsf{irr}}(L)$ the family of all irreducible Frink-ideals of L. It is clear that $\mathsf{FId}_{\mathsf{pr}}(L) \subseteq \mathsf{FId}_{\mathsf{irr}}(L)$. The following three results are generalizations of well-known properties in Lattice Theory.

Lemma 3.6. *Let L be a semilattice and $I \subseteq L$. Then, $I \in \mathsf{FId}_{\mathsf{pr}}(L)$ if and only if $I^c \in \mathsf{Opt}(L)$.*

The proof of the following lemma is analogous to the case of prime ideal in Lattice Theory.

Lemma 3.7. *Let L be a semilattice. Let I be a proper Frink-ideal of L. Then, I is prime if and only if for all $a, b \in L$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$.*

Lemma 3.8 (Irreducible Frink-ideal theorem)**.** *Let L be a semilattice. If I is a Frink-ideal and F is a filter such that $F \cap I = \emptyset$, then there exists an irreducible Frink-ideal J such that $I \subseteq J$ and $F \cap J = \emptyset$.*

Proof. Consider the family

$$\mathcal{F} = \{J \in \mathsf{FId}(L) : I \subseteq J \text{ and } F \cap J = \emptyset\}.$$

It is clear that the family $\mathcal{F}$ is non-empty and closed under chains. Then, by Zorn's lemma there is M a maximal element of $\mathcal{F}$. We show that M is irreducible. Let J_1, $J_2 \in \mathsf{FId}(L)$ be such that $M = J_1 \cap J_2$. Suppose towards a contradiction that $M \subset J_1$ and $M \subset J_2$. Then, by the maximality of M, we have there are $x \in J_1 \cap F$ and $y \in J_2 \cap F$. We thus obtain that $x \wedge y \in F \cap M$, which is a contradiction. Hence M is irreducible. □

4. Characterizations of the condition of mildly distributivity

Definition 4.1. A *mildly distributive semilattice*, or *md-semilattice*, is a semilattice $L = \langle L, \wedge \rangle$ such that the lattice $\mathsf{FId}(L)$ is distributive.

Let L be a semilattice. Consider the set

$$\mathsf{Fi}^{\omega}(L) = \{[a_1) \cap \cdots \cap [a_n) : a_1, \ldots, a_n \in L\}.$$

If L has no top element, then $\mathsf{Fi}^{\omega}(L)$ is a sub meet-semilattice of $\mathsf{Fi}(L) \cup \{\emptyset\}$ and if L has a top element, then $\mathsf{Fi}^{\omega}(L)$ is a sub meet-semilattice of $\mathsf{Fi}(L)$. Moreover, Hickman in [12] proved that $\mathsf{Fi}^{\omega}(L)$ is dually isomorphic to $\mathsf{FId}^{\mathsf{f}}(L)$.

Theorem 4.2 ([12])**.** *Let L be a semilattice. Then, the following conditions are equivalent:*

(1) *L is an md-semilattice,*

(2) *$\mathsf{FId}^{\mathsf{f}}(L)$ is a distributive sublattice of $\mathsf{FId}(L)$,*

(3) *$\mathsf{FId}^{\mathsf{f}}(L)$ is a distributive lattice,*

(4) *$\mathsf{Fi}^{\omega}(L)$ is a distributive lattice,*

(5) *for all $a_1, \ldots, a_n \in L$ and $a \in L$, if $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$, then*

$$a = (a \wedge a_1) \vee (a \wedge a_2) \vee \cdots \vee (a \wedge a_n).$$

We denote by $\dot{\vee}$ the join in $\mathsf{Fi}^\omega(L)$, when it exists and this should be kept in mind, because it will be repeatedly used later on.

Lemma 4.3. *Let L be an md-semilattice and let F be a filter of L. Then F is prime if and only if it is optimal.*

Proof. By Lemma 3.5 we have that every optimal filter is prime. Now assume that F is prime. Let $a \in L$ and let $a_1, \dots, a_n \notin F$ be such that $[a_1) \cap \dots \cap [a_n) \subseteq [a)$. Then $a = (a \wedge a_1) \vee (a \wedge a_2) \vee \dots \vee (a \wedge a_n)$. Since $a_1, \dots, a_n \notin F$, we obtain that $a \wedge a_i \notin F$, for each $1 \leq i \leq n$. As F is prime, $(a \wedge a_1) \vee (a \wedge a_2) \vee \dots \vee (a \wedge a_n) \notin F$ and thus $a \notin F$. Then F is optimal. □

The following theorem provides several new characterizations of md-semilattices. One of them characterizes md-semilattices as those semilattices where the elements can be separated by means of optimal filters, other one is by mean of irreducible and optimal filters and other of the characterizations uses the notion of relative maximal filter. Thus, we introduce the corresponding definition.

Definition 4.4. Let L be a semilattice and let S be a subset of L closed under meets. A proper filter F of L is called a *relative maximal filter with respect to* S, when F is maximal among filters which are disjoint to S. If L is a semilattice with zero, then a relative maximal filter respect to $(0]$ is simply referred as a *maximal filter or ultrafilter.*

Theorem 4.5. *Let L be a semilattice. Then, the following conditions are equivalent:*

(1) *L is an md-semilattice;*
(2) *every relative maximal filter F respect to a Frink-ideal I of L is optimal;*
(3) *for every pair $(F, I) \in \mathsf{Fi}(L) \times \mathsf{FId}(L)$, if $F \cap I = \emptyset$, then there exists $P \in \mathsf{Opt}(L)$ such that $F \subseteq P$ and $P \cap I = \emptyset$;*
(4) *every irreducible filter of L is an optimal filter of L;*
(5) *for every $x, y \in L$, if $x \nleq y$, then there exists $P \in \mathsf{Opt}(L)$ such that $x \in P$ and $y \notin P$.*

Proof. (1) ⇒ (2) Let $I \in \mathsf{FId}(L)$ and let F be a relative maximal filter respect to I. Let us prove that F is optimal. Let $a_1, \dots, a_n, a \in L$ be such that $[a_1) \cap \ldots \cap [a_n) \subseteq [a)$ and $a_1, \dots, a_n \notin F$. Consider the filters $F_{a_i} = \mathrm{Fi}(F \cup \{a_i\})$, for $1 \leq i \leq n$. Since F is a relative maximal filter respect to I, it follows that $F_{a_i} \cap I \neq \emptyset$ for each $1 \leq i \leq n$. Then, for every $1 \leq i \leq n$ there are elements $f_i \in F$ and $x_i \in I$ such that $f_i \wedge a_i \leq x_i$. As F is a filter, $f := f_1 \wedge \dots \wedge f_n \in F$. So, $f \wedge a_i \leq x_i$ for every $1 \leq i \leq n$. Then,

$$\bigcap_{i=1}^{n} [x_i) \subseteq \bigcap_{i=1}^{n} [f \wedge a_i) = \bigcap_{i=1}^{n} ([f) \dot{\vee} [a_i)) = [f) \dot{\vee} \bigcap_{i=1}^{n} [a_i) \subseteq [f) \dot{\vee} [a) = [f \wedge a).$$

Since I is a Frink-ideal, we get that $f \wedge a \in I$. As $F \cap I = \emptyset$, we have that $f \wedge a \notin F$ and, as $f \in F$, we deduce that $a \notin F$. Thus, $F \in \mathsf{Opt}(L)$.

(2) ⇒ (3) It is not hard and thus we left the details to the reader.

(3) ⇒ (4) Let F be an irreducible filter of L. To show that F is optimal, let $a_1, \dots, a_n \in L \smallsetminus F$ and $a \in L$ be such that $[a_1) \cap \dots \cap [a_n) \subseteq [a)$. Since $a_1, \dots, a_n \notin F$, it follows by Lemma 3.1 that there exist $c \notin F$ and $f \in F$ such that $a_i \wedge f \leq c$ for all $i = 1, \dots, n$. Now, by (3), there is $P \in \mathsf{Opt}(L)$ such that $F \subseteq P$ and $c \notin P$. Since $a_i \wedge f \notin P$ for all $i = 1, \dots, n$ and $f \in P$, it follows that $a_1, \dots, a_n \notin P$. Then $a \in L \smallsetminus P$, because $L \smallsetminus P$ is a Frink-ideal. Hence $a \notin F$ and therefore $L \smallsetminus F$ is a Frink-ideal of L. We thus obtain that F is optimal.

(4) ⇒ (5). It is a direct consequence from Theorem 3.2.

(5) $\Rightarrow$ (1) Let $a_1, \dots, a_n \in L$ and $a \in L$ be such that $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$. We prove that $[a) = [a \wedge a_1) \cap \cdots \cap [a \wedge a_n)$. Otherwise suppose that $[a \wedge a_1) \cap \cdots \cap [a \wedge a_n) \nsubseteq [a)$. Then there exists $z \in [a \wedge a_1) \cap \cdots \cap [a \wedge a_n)$ such that $a \nleq z$. So, there exists $P \in \mathsf{Opt}(L)$ such that $a \in P$ and $z \notin P$. As P is optimal, $a_i \in P$ for some $1 \leq i \leq n$. But then we have that $a \wedge a_i \in P$. So $z \in P$, which is impossible. So, $[a) = [a \wedge a_1) \cap \cdots \cap [a \wedge a_n)$ and thus $a = (a \wedge a_1) \vee (a \wedge a_2) \vee \cdots \vee (a \wedge a_n)$. Then L is an md-semilattice. □

Remark 4.6. Let L be a semilattice. By Lemmas 3.5 and 4.3 and Theorem 4.5 we have that L is an md-semilattice if and only if

$$\mathsf{Fi}_{\mathsf{mpr}}(L) \subseteq \mathsf{Fi}_{\mathsf{irr}}(L) \subseteq \mathsf{Opt}(L) = \mathsf{Fi}_{\mathsf{pr}}(L).$$

We note that $\mathsf{Fi}_{\mathsf{mpr}}(L) = \mathsf{Fi}_{\mathsf{irr}}(L)$ if and only if L is distributive [3: Theorem 10]. As always $\mathsf{Fi}_{\mathsf{mpr}}(L) \subseteq \mathsf{Opt}(L)$, if L is distributive, then $\mathsf{Fi}_{\mathsf{mpr}}(L) = \mathsf{Fi}_{\mathsf{irr}}(L) \subseteq \mathsf{Opt}(L)$ and thus, by the last inclusion, we deduce that L is an md-semilattice. Hence we have shown that every distributive semilattice is an md-semilattice.

The following characterization of mildly-distributivity is a more or less a direct consequence of the definition itself. First we show that for an md -semilattice L, $\mathsf{FId}(L)$ is more than a distributive complete lattice. To see this, we define on $\mathsf{FId}(L)$ the following binary operation $\to$ as follows:

$$I \to J := \{a \in L : (a] \cap I \subseteq J\}$$

for each pair $I, J \in \mathsf{FId}(L)$. It is easy to see that the operation $\to$ can be also defined as

$$I \to J := \{a \in L : a \wedge b \in J \text{ for all } b \in I\}.$$

Lemma 4.7. *Let L be an md-semilattice. Then, $\langle \mathsf{FId}(L), \cap, \vee, \to, I_0, L\rangle$, where $I_0 = \mathrm{Id}_{\mathrm{F}}(\emptyset)$, is a complete Heyting algebra.*

Proof. We only need to prove that the operation $\to$ is well defined and satisfies the property $I \cap J \subseteq K \Leftrightarrow J \subseteq I \to K$, for every $I, J, K \in \mathsf{FId}(L)$.

Let $I, J \in \mathsf{FId}(L)$ and let $a_1, \dots, a_n \in I \to J$ and $a \in L$ be such that $\bigcap_{i=1}^{n} [a_i) \subseteq [a)$. Let $x \in I$. Then $a_i \wedge x \in J$ for all $1 \leq i \leq n$. So,

$$\bigcap_{i=1}^{n} [a_i \wedge x) = \bigcap_{i=1}^{n} ([a_i) \dot{\vee} [x)) = \Big(\bigcap_{i=1}^{n} [a_i)\Big) \dot{\vee} [x) \subseteq [a) \dot{\vee} [x) = [a \wedge x).$$

As J is a Frink-ideal and $a_i \wedge x \in J$, for all $1 \leq i \leq n$, we get that $a \wedge x \in J$. Thus, $a \in I \to J$, and consequently $I \to J$ is a Frink-ideal of L.

Now, let $I, J, K \in \mathsf{FId}(L)$. Assume that $I \cap J \subseteq K$. Let $a \in J$ and $x \in (a] \cap I$. So, $x \in I \cap J$ and then $x \in K$. Hence, $J \subseteq I \to K$. Conversely, suppose that $J \subseteq I \to K$ and let $a \in I \cap J$. Then, $a \in I \to K$. Which implies that $(a] \cap I \subseteq K$. Since $a \in (a] \cap I$, we obtain that $a \in K$. Then, $I \cap J \subseteq K$. □

Theorem 4.8. *Let L be a semilattice. Then, L is an md-semilattice if and only if $\langle \mathsf{FId}(L), \cap, \vee, \to, I_0, L\rangle$, is a complete Heyting algebra.*

Proof. The direction $\Rightarrow$ is the above lemma. If $\langle \mathsf{FId}(L), \cap, \vee, \to, I_0, L\rangle$ is a Heyting algebra then, $\langle \mathsf{FId}(L), \cap, \vee\rangle$ is a distributive lattice and thus L is an md-semilattice. □

In [13] Mandelker studied the properties of relative annihilator and characterized distributive lattices in terms of their relative annihilators. Later, Varlet in [16] gave a similar characterization for distributive semilattices. Here we present a similar result for md-semilattices. First we recall the definition of annihilator on semilattices.

Let L be a semilattice. For each $a, b \in L$, the *annihilator* $\langle a, b\rangle$ *of* a *relative to* b is defined by

$$\langle a, b\rangle := \{c \in L : a \wedge c \leq b\}.$$

For $X, Y \subseteq L$ we denote by $\langle X, Y\rangle$ the set

$$\langle X, Y\rangle := \bigcup\{\langle a, b\rangle : (a, b) \in X \times Y\},$$

and we write $\langle a, Y\rangle$ instead of $\langle\{a\}, Y\rangle$. Moreover, notice that $\langle [a), (b]\rangle = \langle a, b\rangle$ for all $a, b \in L$. If I is a Frink-ideal of L, then the set $\langle a, I\rangle$ is called the *annihilator of* a *relative to* I and it should be noted that $\langle a, I\rangle = \{c \in L : a \wedge c \in I\}$. In particular, if L has zero 0, the set $a^{\perp} = \langle a, 0\rangle = \{c \in A : a \wedge c = 0\}$ is called the *annulet* or *annihilator* of a (see [7], [8], and [9]).

Lemma 4.9. *Let L be a semilattice. Let I be a down-set of L. If $\langle a, I\rangle \in \mathsf{FId}(L)$ for all $a \in L$, then $I \in \mathsf{FId}(L)$.*

Proof. Let $x_1, \ldots, x_n \in I$ and $x \in L$ be such that $[x_1) \cap \cdots \cap [x_n) \subseteq [x)$. Since $x \wedge x_i \leq x_i$ for all $i = 1, \ldots, n$ and I is a down-set, it follows that $x \wedge x_i \in I$ for all $i = 1, \ldots, n$. So, $x_i \in \langle x, I\rangle$ for every $1 \leq i \leq n$ and, because $\langle x, I\rangle$ is a Frink-ideal we have $x \in \langle x, I\rangle$. Then $x \in I$. Thus, $I \in \mathsf{FId}(L)$. □

Theorem 4.10. *Let L be a semilattice. Then L is an md-semilattice if and only if $\langle a, I\rangle$ is a Frink-ideal for each $a \in L$ and for each $I \in \mathsf{FId}(L)$.*

Proof. Assume that L is an *md*-semilattice. Let $a \in L$ and $I \in \mathsf{FId}(L)$. It is clear that $\langle a, I\rangle$ is a down-set of L. We prove that it is a Frink-ideal. Let $x_1, \ldots, x_n \in \langle a, I\rangle$ and let $x \in L$ be such that $[x_1) \cap \cdots \cap [x_n) \subseteq [x)$. Then $x_i \wedge a \in I$ for every $1 \leq i \leq n$. From Theorem 4.2, we obtain

$$\begin{aligned}[x_1 \wedge a) \cap \cdots \cap [x_n \wedge a) &= ([x_1) \dot{\vee} [a)) \cap \cdots \cap ([x_n) \dot{\vee} [a)) \\ &= ([x_1) \cap \cdots \cap [x_n)) \dot{\vee} [a) \subseteq [x) \dot{\vee} [a) = [x \wedge a).\end{aligned}$$

As $x_i \wedge a \in I$ for every $1 \leq i \leq n$ and I is a Frink-ideal, we obtain that $x \wedge a \in I$, i.e., $x \in \langle a, I\rangle$. Thus $\langle a, I\rangle$ is a Frink-ideal.

Conversely, assume that $\langle a, I\rangle$ is a Frink-ideal for each $a \in L$ and for each $I \in \mathsf{FId}(L)$. Let $a, a_1, \ldots, a_n \in L$ and assume that $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$. We prove that $[a \wedge a_1) \cap \cdots \cap [a \wedge a_n) \subseteq [a)$. Let $b \in [a \wedge a_1) \cap \cdots \cap [a \wedge a_n)$. So $a \wedge a_i \leq b$ for all $1 \leq i \leq n$ and then $a_i \in \langle a, b\rangle$ for all $1 \leq i \leq n$. As $\langle a, b\rangle = \langle a, (b]\rangle$ is a Frink-ideal and $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$, we get that $a \in \langle a, b\rangle$, i.e., $a \wedge a = a \leq b$. So, $[a \wedge a_1) \cap \cdots \cap [a \wedge a_n) \subseteq [a)$. Since the other inclusion is always valid, we have that $[a \wedge a_1) \cap \cdots \cap [a \wedge a_n) = [a)$. So, $a = (a \wedge a_1) \vee (a \wedge a_2) \vee \cdots \vee (a \wedge a_n)$ and thus L is an *md*-semilattice. □

The next characterization of prime Frink-ideal is used in the Theorem 4.12.

Lemma 4.11. *Let L be a semilattice. Let I be a Frink-ideal. Then I is a prime if and only if $\langle x, I\rangle = I$, for every $x \notin I$.*

Proof. Assume that I is a prime Frink-ideal of L. Let $x \notin I$ and $y \in \langle x, I\rangle$. So, $x \wedge y \in I$. As I is prime and $x \notin I$, we get that $y \in I$. Thus, $\langle x, I\rangle = I$.

Assume that $\langle x, I\rangle = I$, for every $x \notin I$. Let $a, b \in L$ such that $a \wedge b \in I$. Suppose $a \notin I$ and $b \notin I$. By hypothesis, $I = \langle a, I\rangle$. Since $b \notin I = \langle a, I\rangle$, we get that $a \wedge b \notin I$, which is a contradiction. Hence, I is prime. □

Theorem 4.12. *Let L be a semilattice. Then the following conditions are equivalent:*

(1) *L is an md-semilattice,*

(2) *for every $x, y \in L$, if $\langle y, I\rangle \subseteq \langle x, I\rangle$ for all prime Frink-ideal I, then $x \leq y$.*

Proof. (1) ⇒ (2) Let $x, y \in L$ be such that $\langle y, I\rangle \subseteq \langle x, I\rangle$ for all prime Frink-ideal I of L. Assume that $x \nleq y$. By Theorem 4.5 there exist $P \in \mathsf{Opt}(L)$ such that $x \in P$ and $y \notin P$. Consider the prime Frink-ideal $I = P^c$. As $y \in I$, we get that $L = \langle y, I\rangle \subseteq \langle x, I\rangle$. But this implies that $\langle x, I\rangle = L$, i.e., $x \in I$, which is a contradiction. Thus, $x \leq y$.

(2) ⇒ (1) We apply Theorem 4.5. Let $x, y \in L$. Suppose that $x \nleq y$. Then there exists a prime Frink-ideal I such that $\langle y, I\rangle \nsubseteq \langle x, I\rangle$. So, there exists $z \in L$ such that $y \wedge z \in I$ and $x \wedge z \notin I$, i.e., $y \in \langle z, I\rangle$ and $x \notin \langle z, I\rangle$. We note that $z \notin I$. Since I is prime and using the previous lemma, it follows that $\langle z, I\rangle = I$. Then $y \in I$ and $x \notin I$. As $P = I^c$ is an optimal filter, we have that $x \in P$ and $y \notin P$. So, by Theorem 4.5 we conclude that L is an md-semilattice. □

5. Distributive lattice envelope of an md-semilattice

The main purpose of this section is to obtain an extension of an md-semilattice to a distributive lattice where the md-semilattice is embedded in a very nice way. To this end, we need to introduce the notions of strong homomorphism and strong embedding.

Let L and M be semilattices. We call a map $h\colon L \to M$ *homomorphism, or meet-homomorphism,* if for all $a, b \in L$ we have that $h(a \wedge b) = h(a) \wedge h(b)$. A homomorphism h between semilattices is called *join-homomorphism* if h preserves all existing finite joins (in [12] join-homomorphisms are called join partial homomorphism). That is, if $a_1, \ldots, a_n \in L$ and $a_1 \vee \cdots \vee a_n$ exists in L, then $h(a_1) \vee \cdots \vee h(a_n)$ exists in M and equals to $h(a_1 \vee \cdots \vee a_n)$. We say that a map $h\colon L \to M$ is a *strong homomorphism* (see [12] and [2]) if it is a homomorphism and satisfies that for all $a_1, \ldots, a_n, a \in L$,

$$[a_1) \cap \cdots \cap [a_n) \subseteq [a) \implies [h(a_1)) \cap \cdots \cap [h(a_n)) \subseteq [h(a)).$$

Moreover, if h is injective, we say that h is a *strong embedding.*

Let L and M be semilattices. If $h\colon L \to M$ is a strong homomorphism, then it is not hard to show that h is a join-homomorphism. But, if L is an md-semilattice and M is an arbitrary semilattice, then $h\colon L \to M$ is a strong homomorphism if and only if it is a join-homomorphism (this can be seen in [12]). Thus, strong homomorphisms and join-homomorphisms coincide in md-semilattices setting. Now, we give a characterization of the strong homomorphisms in terms of optimal filters.

Lemma 5.1. *Let L and M be two bounded md-semilattices and let $h\colon L \to M$ be an order-preserving map such that preserves top and bottom. Then the following conditions are equivalent:*

(1) *h is a strong homomorphism.*

(2) *$h^{-1}(Q) \in \mathsf{Opt}(L)$, for all $Q \in \mathsf{Opt}(M)$.*

Proof. (1) ⇒ (2) Let $Q \in \mathsf{Opt}(M)$. Since h is a homomorphism that preserves top and bottom, it is clear that $h^{-1}(Q) \in \mathsf{Fi}(L)$ and it is proper. Let $a_1, \ldots, a_n \notin h^{-1}(Q)$ and $a \in L$ be such that $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$. As h is a strong homomorphism, we have $[h(a_1)) \cap \cdots \cap [h(a_n)) \subseteq [h(a))$. So, $h(a_1), \ldots, h(a_n) \notin Q$ and then $h(a) \notin Q$. Thus, $a \notin h^{-1}(Q)$. Therefore, $h^{-1}(Q)$ is an optimal filter of L.

(2) ⇒ (1) Let $a, b \in L$. As h is order-preserving, we have $h(a \wedge b) \leq h(a) \wedge h(b)$. Suppose that $h(a) \wedge h(b) \nleq h(a \wedge b)$. Then there exists $Q \in \mathsf{Opt}(M)$ such that $h(a), h(b) \in Q$ and $h(a \wedge b) \notin Q$. So, $a, b \in h^{-1}(Q)$ and as $h^{-1}(Q) \in \mathsf{Fi}(L)$, we get that $a \wedge b \in h^{-1}(Q)$, i.e., $h(a \wedge b) \in Q$, which is a contradiction. Thus, $h(a) \wedge h(b) \leq h(a \wedge b)$, and consequently h is a meet-homomorphism. Now let $a_1, \ldots, a_n \in L$ and $a \in L$ be such that $[a_1) \cap \cdots \cap [a_n) \subseteq [a)$. If $[h(a_1)) \cap \cdots \cap [h(a_n)) \nsubseteq [h(a))$, there exists $b \in M$ such that $h(a_i) \leq b$ for all $1 \leq i \leq n$ and $h(a) \nleq b$. So, there exists

$Q \in \mathsf{Opt}(M)$ such that $h(a) \in Q$ and $b \notin Q$. As $a \in h^{-1}(Q)$ and $h^{-1}(Q) \in \mathsf{Opt}(L)$, it follows that $a_i \in h^{-1}(Q)$ for some $1 \leq i \leq n$. So, $h(a_i) \in Q$ and consequently $b \in Q$, which is impossible. Thus, $[h(a_1)) \cap \cdots \cap [h(a_n)) \subseteq [h(a))$. Therefore h is a strong homomorphism. □

Lemma 5.2. *Let L and M be semilattices. Let $h \colon L \to M$ be a strong homomorphism. If h is an embedding and $\bigcap_{i=1}^{n} [h(a_i)) \subseteq [h(a))$, then $\bigcap_{i=1}^{n} [a_i) \subseteq [a)$, for each $a_1, \ldots, a_n, b \in L$.*

P r o o f. It is easy and left to the reader. □

The following definition is due to R. Hickman [12].

Definition 5.3. A distributive lattice D is called a *strong free distributive extension* of an semilattice L if:

(1) there is a strong embedding $e \colon L \to D$;

(2) $e\,[L]$ generates D as a lattice, i.e., for each $a \in D$ there exists a non-empty finite subset X of L such that $a = \bigvee e[X]$;

(3) if D_1 is a distributive lattice and if $f \colon L \to D_1$ is a strong homomorphism, then there exists a unique lattice homomorphism $\bar{f} \colon D \to D_1$, such that $f = \bar{f} \circ e$.

As was pointed out by Hickman [12], a strong free distributive extension of a semilattice, if it exists, is unique up to isomorphism. He also proved that a semilattice is md-distributive if and only if has a strong free distributive extension and the strong free distributive extension of a md-distributive semilattice L is isomorphic to $\mathsf{Fld}^{\mathsf{f}}(L)$.

Now we introduce a slightly different definition of an extension of a semilattice, namely we omit the third condition in the definition of strong free distributive extension. And we show in the next theorem that the new definition is equivalent to the definition of strong free distributive extension, in other words, we prove that the third condition in Definition 5.3 follows from the first two.

Definition 5.4. Let L be a semilattice. A pair $\langle D, e\rangle$, where D is a distributive lattice and e a strong embedding from L to D, is a *distributive lattice envelope* of L if for every $a \in D$ there is a non-empty finite subset $X \subseteq L$ such that $a = \bigvee e[X]$.

Theorem 5.5. *Let L be a semilattice. Then $\langle D, e\rangle$ is a distributive lattice envelope of L if and only if D is a strong free distributive extension.*

P r o o f. Assume that $\langle D, e\rangle$ is a distributive lattice envelope of L. Let D_1 be a distributive lattice and $f \colon L \to D_1$ a strong homomorphism. We prove that there exists a unique lattice homomorphism $\bar{f} \colon D \to D_1$, such that $f = \bar{f} \circ e$.

Let $a, b \in D$. Then there exist finite subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_k\}$ of L such that $a = e(x_1) \vee \cdots \vee e(x_n)$ and $b = e(y_1) \vee \cdots \vee e(y_k)$. Suppose that $a = b$, i.e.,

$$e(x_1) \vee \cdots \vee e(x_n) = e(y_1) \vee \cdots \vee e(y_k).$$

So,

$$[e(x_1)) \cap \cdots \cap [e(x_n)) = [e(y_1)) \cap \cdots \cap [e(y_k))\,.$$

Then,

$$[e(x_1)) \cap \cdots \cap [e(x_n)) \subseteq [e(y_j))$$

for every $1 \leq j \leq k$. As e is a strong embedding, $[x_1) \cap \cdots \cap [x_n) \subseteq [y_j)$ for every $1 \leq j \leq k$. As f is a strong homomorphism,

$$[f(x_1)) \cap \cdots \cap [f(x_n)) \subseteq [f(y_j))$$

for every $1 \leq j \leq k$ and thus

$$[f(x_1)) \cap \cdots \cap [f(x_n)) \subseteq [f(y_1)) \cap \cdots \cap [f(y_k)) \,.$$

With a similar argument we can show the reverse inclusion. So,

$$[f(x_1)) \cap \cdots \cap [f(x_n)) = [f(y_1)) \cap \cdots \cap [f(y_k)) \,,$$

i.e.,

$$f(x_1) \vee \cdots \vee f(x_n) = f(y_1) \vee \cdots \vee f(y_k).$$

Hence, we can define a map $\bar{f}\colon D \to D_1$ by

$$\bar{f}(a) = f(x_1) \vee \cdots \vee f(x_n),$$

when $a = e(x_1) \vee \cdots \vee e(x_n)$ for some $x_1, \ldots, x_n \in L$. It is easy to see that $\bar{f}$ is a lattice homomorphism from D to D_1 and that $f = \bar{f} \circ e$. We prove that $\bar{f}$ is unique. Suppose that there exists other map $g\colon D \to D_1$ such that $f = g \circ e$. Let $a \in D$. Then there exists a finite $\{x_1, \ldots, x_n\} \subseteq L$ such that $a = e(x_1) \vee \cdots \vee e(x_n)$. So,

$$\begin{aligned} g(a) = g(e(x_1) \vee \cdots \vee e(x_n)) &= g(e(x_1)) \vee \cdots \vee g(e(x_n)) \\ &= f(x_1) \vee \cdots \vee f(x_n) = \bar{f}(e(x_1)) \vee \cdots \vee \bar{f}(e(x_n)) \\ &= \bar{f}(e(x_1) \vee \cdots \vee e(x_n)) = \bar{f}(a). \end{aligned}$$

Lastly, it is straightforward prove the implication from right to left. This completes the proof. □

From the previous theorem we can conclude that if L is a semilattice and $\langle D, e\rangle$ is a distributive lattice envelope of L, then for every distributive lattice D_1 and for every strong homomorphism $f\colon L \to D_1$ there exists a unique lattice homomorphism $\bar{f}\colon D \to D_1$ such that $f = \bar{f} \circ e$. Now we prove that certain properties of f are preserved by $\bar{f}$.

Lemma 5.6. *Let L be a semilattice and $\langle D, e\rangle$ a distributive lattice envelope of L. Let D_1 be a distributive lattice and let $f\colon L \to D_1$ be a strong homomorphism.*

(1) *If f is an embedding, then $\bar{f}$ is an embedding.*

(2) *If f is onto, then $\bar{f}$ is onto.*

Proof. (1) Let $a, b \in D$ be such that $\bar{f}(a) = \bar{f}(b)$. Then there exist two finite subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_k\}$ of L such that $a = e(x_1) \vee \cdots \vee e(x_n)$ and $b = e(y_1) \vee \cdots \vee e(y_k)$. So $\bar{f}(a) = f(e(x_1) \vee \cdots \vee e(x_n)) = f(e(y_1) \vee \cdots \vee e(y_k)) = \bar{f}(b)$. As f is a strong homomorphism we have that

$$f(e(x_1)) \vee \cdots \vee f(e(x_n)) = f(e(y_1)) \vee \cdots \vee f(e(y_k))$$

and consequently

$$[f(e(x_1))) \cap \cdots \cap [f(e(x_n)) = [f(e(y_1))) \cap \cdots \cap [f(e(y_k)) \,.$$

As

$$[f(e(x_1))) \cap \cdots \cap [f(e(x_n)) \subseteq [f(e(y_j)))$$

for each $1 \leq j \leq k$ and f is a strong embedding, we obtain that

$$[e(x_1)) \cap \cdots \cap [e(x_n)) = [e(x_1) \vee \cdots \vee e(x_n)) \subseteq [e(y_j))$$

for each $1 \leq j \leq k$. So, $e(y_j) \leq e(x_1) \vee \cdots \vee e(x_n)$ for each $1 \leq j \leq k$ and hence

$$e(y_1) \vee \cdots \vee e(y_k) \leq e(x_1) \vee \cdots \vee e(x_n).$$

By a similar argument we obtain that

$$e(x_1) \vee \cdots \vee e(x_n) \leq e(y_1) \vee \cdots \vee e(y_k).$$

Thus, $a = b$. Therefore $\bar{f}$ is an embedding.

(2) It is easy and left to the reader. □

We finish this section by showing a correspondence between (prime) Frink-ideals of a md-semilattice and (prime) ideals of its distributive lattice envelope. And from this, it follows a correspondence between optimal filters of the md-semilattice and prime filters of its distributive lattice envelope.

Let L be an md-semilattice and $\langle D, e\rangle$ its distributive lattice envelope. Without loss of generality we can assume that L is a sub-semilattice of D with e the identity map. So, by Definition 5.4, we have the following properties:

(P1) for all $a_1, \ldots, a_n, a \in L$,

$$[a_1)_L \cap \cdots \cap [a_n)_L \subseteq [a)_L \iff [a_1)_D \cap \cdots \cap [a_n)_D \subseteq [a)_D;$$

(P2) for every $a \in D$ there is a non-empty finite subset $X \subseteq L$ such that $a = \bigvee X$.

Let us denote by $\mathrm{Id}_D(X)$ the ideal of D generated by a subset $X \subseteq D$.

Theorem 5.7. *Let L be an md-semilattice and D its distributive lattice envelope.*

(1) *If J is a (prime) ideal of D, then $J \cap L$ is a (prime) Frink-ideal of L and $J = \mathrm{Id}_D(J \cap L)$. Moreover, for every Frink-ideal I of L, $I = \mathrm{Id}_D(I) \cap L$.*

(2) *If I is a prime Frink-ideal of L, then $\mathrm{Id}_D(I)$ is a prime ideal of D.*

Proof. (1) Let J be an ideal of D. Let $a_1, \ldots, a_n \in J \cap L$ and let $a \in L$ be such that $[a_1)_L \cap \cdots \cap [a_n)_L \subseteq [a)_L$. So, $[a_1)_D \cap \cdots \cap [a_n)_D \subseteq [a)_D$ and this implies that $a \in J \cap L$. Hence $J \cap L$ is a Frink-ideal of L. It is clear that $\mathrm{Id}_D(J \cap L) \subseteq J$. Let $x \in J$. So, there is a non-empty finite subset A of L such that $x = \bigvee A$. Then $A \subseteq J \cap L$ and hence $x \in \mathrm{Id}_D(J \cap L)$. Therefore $J = \mathrm{Id}_D(J \cap L)$. It is straightforward to check directly that if J is a prime ideal of D then $J \cap L$ is a prime Frink-ideal of L. Finally, by Property (P1) in the previous paragraph to the theorem, it is not hard to check that for every Frink-ideal I of L, $I = \mathrm{Id}_D(I) \cap L$.

(2) Let I be a prime Frink-ideal of L and let $x \wedge y \in \mathrm{Id}_D(I)$. Thus, by Property (P2), there are non-empty finite subsets $A, B \subseteq L$ such that $x = \bigvee A$ and $y = \bigvee B$. Since $x \wedge y = \bigvee A \wedge \bigvee B = \bigvee\{a \wedge b : a \in A \text{ and } b \in B\} \in \mathrm{Id}_D(I)$, it follows by (1) that $a \wedge b \in I$ for all $a \in A$ and $b \in B$. Suppose towards a contradiction that $A \nsubseteq I$ and $B \nsubseteq I$. So, there are $a \in A \smallsetminus I$ and $b \in B \smallsetminus I$. Since $a, b \notin I$, it follows by primality of I that $a \wedge b \notin I$, which is a contradiction. Then $A \subseteq I$ or $B \subseteq I$ and hence $x = \bigvee A \in \mathrm{Id}_D(I)$ or $y = \bigvee B \in \mathrm{Id}_D(I)$. Therefore $\mathrm{Id}_D(I)$ is a prime ideal of D. □

Therefore, given an md-semilattice L and its distributive lattice envelope D, we can conclude that the set of Frink-ideals of L and the set of ideals of D, both ordered with the set-theoretic inclusion, are order-isomorphic and the set of prime Frink-ideals is also order-isomorphic to the set of prime ideals. Consequently, by Lemma 3.6, we obtain that the set of optimal filters of L is order-isomorphic to the set of prime filters of D.

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