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PERFECT HILBERT ALGEBRAS

A b s t r a c t. In [*S. Celani and L. Cabrer. Duality for finite Hilbert algebras. Discrete Math., 305(1-3):74–99, 2005.*] the authors proved that every finite Hilbert algebra A is isomorphic to the Hilbert algebra $\mathrm{H}_{\mathcal{K}}(X) = \{w \Rightarrow_i v : w \in \mathcal{K} \text{ and } v \subseteq w\}$, where X is a finite poset, $\mathcal{K}$ is a distinguished collection of subsets of X, and the implication $\Rightarrow_i$ is defined by: $w \Rightarrow_i v = \{x \in X : w \cap [x) \subseteq v\}$, where $[x) = \{y \in X : x \leq y\}$. The Hilbert implication on $\mathrm{H}_{\mathcal{K}}(X)$ is the usual Heyting implication $\Rightarrow_i$ (as just defined) given on the increasing subsets. In the same article, Celani and Cabrer extended this representation to a full categorical duality. The aim of the present article is to obtain an algebraic characterization of the Hilbert algebras $\mathrm{H}_{\mathcal{K}}(X)$ for all structures $\langle X, \leq, \mathcal{K}\rangle$ defined by Celani and Cabrer but not necessarily finite. Then, we shall extend this representation to a full dual equivalence generalizing the finite setting given by Celani and Cabrer.

1 Introduction and preliminaries

A *Hilbert algebra* is an algebra $\langle A, \to, 1\rangle$ of type (2,0) satisfying the following conditions:

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(H1) $x \to (y \to x) = 1$;

(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

(H3) if $x \to y = 1$ and $y \to x = 1$, then $x = y$.

It is well-known that the class of Hilbert algebras forms a variety (see Diego [11]) in the sense of universal algebra. The variety of Hilbert algebras has deserved a lot of attention in the setting of algebraic logic since it is the algebraic counterpart (in the sense of Algebraic Logic, see [16]) of the implicative fragment of intuitionistic logic. In particular, in [8], the authors presented a representation and a full categorical duality for the class of finite Hilbert algebras by means of structures $\langle X, \leq, \mathcal{K}\rangle$, where $\langle X, \leq\rangle$ is a finite partially ordered set and $\mathcal{K}$ is a distinguished collection of subsets of X. We shall call these structures *H-sets*.

Our first aim (Section 2) will be to characterize algebraically the Hilbert algebras that are isomorphic to Hilbert algebras of the form $\mathrm{A}(X) = \{w \Rightarrow v : w \in \mathcal{K} \text{ and } v \subseteq w\}$, with $w \Rightarrow v = \{x \in X : w \cap (x] \subseteq v\}$ and where $(x] = \{y \in X : y \leq x\}$, for some H-set $\langle X, \leq, \mathcal{K}\rangle$ (but not necessarily finite). We shall call these Hilbert algebras *perfect*[1]. In [8] the authors used completely irreducible deductive systems to attain their aims. They proved that in the finite case there is a dual order-isomorphism between completely irreducible deductive systems and irreducible elements. In the present paper, we use the concept of irreducible elements instead of completely irreducible deductive systems. Thus, we work dually in the framework of decreasing subsets instead of increasing subsets as was done in [8]. As we shall notice below (see on page 31), if $\langle X, \leq, \mathcal{K}\rangle$ is an H-set, then $\langle X, \geq, \mathcal{K}\rangle$ is also an H-set, and the Hilbert algebra $H_{\mathcal{K}}(X)$ (as defined in the abstract, or see on page 31) defined by the H-set $\langle X, \leq, \mathcal{K}\rangle$ coincides with the Hilbert algebra $\mathrm{A}(X)$ (see also on page 31) defined by the H-set $\langle X, \geq, \mathcal{K}\rangle$.

In Section 3, we present some examples of perfect Hilbert algebras. Section 4 is devoted to develop a full dual categorical equivalence between the category of perfect Hilbert algebras and certain algebraic morphisms and the category of H-sets and H-functional morphism (see [8] or Definition 4.1).

Let us overview the representation given in [8] for finite Hilbert algebras. For this, we need to introduce some concepts and results which will be needed throughout the article. Thus, firstly, we will give all we need and then we overview the representation given in [8].

Let $\langle X, \leq\rangle$ be a poset. Let $A \subseteq X$. Let $(A]_X = \{x \in X : x \leq a \text{ for some } a \in A\}$. For every $x \in X$, we write $(x]_X$ instead of $(\{x\}]_X$. We drop the subscript in $(A]_X$ when confusion is unlikely, and we write $(A]$ simply. A subset $A \subseteq X$ is called *decreasing* if $A =$

[1]The adjective *perfect* is chosen by likeness with the framework of lattice. In [12], a distributive lattice is called *perfect* if it is isomorphic to a set-theoretic lattice based on the collection of decreasing subsets of some partial order.

$(A]$. We denote by $\mathcal{P}_d(X)$ the collection of all decreasing subsets of X. It is well-known that $\langle \mathcal{P}_d(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ is a complete Heyting algebra, where for all $U, V \in \mathcal{P}_d(X)$, $U \Rightarrow V = \{x \in X : U \cap (x] \subseteq V\}$. Thus, in particular, $\langle \mathcal{P}_d(X), \Rightarrow, X\rangle$ is a Hilbert algebra. Notice that for all $A, B \subseteq X$, not necessarily decreasing subsets, the implication

$$A \Rightarrow B = \{x \in X : A \cap (x] \subseteq B\} \tag{1}$$

is always a decreasing subset of X.

Dually, we have the concept of *increasing* subset and the Heyting algebra $\langle \mathcal{P}_i(X), \cap, \cup, \Rightarrow_i, \emptyset, X\rangle$ over the collection $\mathcal{P}_i(X)$ of all increasing subsets of the poset X with the implication

$$U \Rightarrow_i V = \{x \in X : U \cap [x) \subseteq V\}. \tag{2}$$

Notice that the set $U \Rightarrow_i V$ is always an increasing subset for U and V not necessarily increasing.

Let $\langle A, \to, 1\rangle$ be a Hilbert algebra. A partial order can be defined on A as follows: $a \leq b$ if and only if $a \to b = 1$. A subset $F \subseteq A$ is called *implicative filter* (also known as *deductive system*) if $1 \in F$, and if $a, a \to b \in F$, then $b \in F$. Let us denote by $\mathrm{ImFi}(A)$ the collection all implicative filters of A. It is well-known that $\mathrm{ImFi}(A)$ is an algebraic closure system, where the implicative filter generated by a set X is characterized as follows:

$$\langle X\rangle = \{a \in A : \exists x_1, \ldots, x_n \in X (x_1 \to (x_2 \to (\ldots (x_n \to a) \ldots))) = 1\}$$

It follows that $\langle \{a\}\rangle = [a)$. A useful property of the implicative filters generated by a set is the following. If F is an implicative filter and $x_1, \ldots, x_n \in A$, then

$$\langle F, x_1, \ldots, x_n\rangle = \langle F \cup \{x_1, \ldots, x_n\}\rangle = \{a \in A : x_1 \to (x_2 \to (\ldots (x_n \to a) \ldots)) \in F\}$$

(see [11]).

A proper implicative filter F is called *irreducible* when for any $D_1, D_2 \in \mathrm{ImFi}(A)$ such that $D_1 \cap D_2 = F$, it follows that $D_1 = F$ or $D_2 = F$. We say that F is *completely irreducible* if for any family $\{D_i : i \in I\} \subseteq \mathrm{ImFi}(A)$ such that $F = \cap\{D_i : i \in I\}$, it follows that $F = D_i$ for some $i \in I$. We denote by $\mathrm{CIrr}(A)$ the collection of all completely irreducible implicative filter of A.

The proofs of the following three results can be found in [11, 3, 14].

Theorem 1.1. *Let A be a Hilbert algebra and $F \in \mathrm{ImFi}(A)$. Then the following are equivalent:*

1. *F is irreducible.*
2. *If $a, b \notin F$, then there is $c \notin F$ such that $a, b \leq c$.*

3. *If $a, b \notin F$, then there is $c \notin F$ such that $a \to c, b \to c \in F$.*

Theorem 1.2. *Let A be a Hilbert algebra and $F \in \mathrm{ImFi}(A)$. Then the following are equivalent:*

1. *$F \in \mathrm{CIrr}(A)$.*
2. *There exists $a \notin F$ such that if $F \subsetneq D \in \mathrm{ImFi}(A)$, then $a \in D$.*
3. *There exists $a \notin F$ such that $b \to a \in F$, for all $b \notin F$.*

Let $F \in \mathrm{ImFi}(A)$ and $a \in A$. We say that F is *maximal relative to a* if F is a maximal element in the set $\{D \in \mathrm{ImFi}(A) : a \notin D\}$ (maximal with respect to the set inclusion).

Theorem 1.3. *Let A be a Hilbert algebra. Then:*

1. *If $F \in \mathrm{ImFi}(A)$ and $a \notin F$, then there exists $P \in \mathrm{CIrr}(A)$ maximal relative to a such that $F \subseteq P$.*
2. *If $a, b \in A$ are such that $a \nleq b$, then there exists $P \in \mathrm{CIrr}(A)$ maximal relative to b such that $a \in P$.*
3. *If $F \in \mathrm{ImFi}(A)$ and $a \to b \notin F$, then there exists $P \in \mathrm{CIrr}(A)$ maximal relative to b such that $a \in P$ and $F \subseteq P$.*

Let A be a Hilbert algebra. Following the notation of [8], we define the relation $K_A \subseteq \mathrm{CIrr}(A) \times A$ as follows:

$$(P, a) \in K_A \text{ if and only if } P \text{ is maximal relative to } a.$$

Considering the poset $\langle \mathrm{CIrr}(A), \subseteq \rangle$, it defines the map $\varphi \colon A \to \mathcal{P}_i(\mathrm{CIrr}(A))$ as follows:

$$\varphi(a) = \{P \in \mathrm{CIrr}(A) : a \in P\}.$$

Theorem 1.4 ([11])**.** *The map $\varphi \colon A \to \mathcal{P}_i(\mathrm{CIrr}(A))$ is a Hilbert embedding into $\langle \mathcal{P}_i(\mathrm{CIrr}(A)), \Rightarrow_i, \mathrm{CIrr}(A) \rangle$.*

Now we are ready to establish the representation for the *finite* Hilbert algebras given in [8]. For the missing details, we refer the reader to [8]. We begin with the definition of the dual structures of the finite Hilbert algebras.

Definition 1.5 ([8])**.** A triple $\langle X, \leq, \mathcal{K} \rangle$ is called an *H-set*[2] if:

[2] In [8], the H-sets are called H-spaces, but the term *H-space* was also used later in [9] to refer to certain ordered topological spaces which were used to obtain a duality for the algebraic category of Hilbert algebras. Thus, we prefer the term *H-set* instead of *H-space* because the definition in [8] and here there is not a topology involved.

(HS1) $\langle X, \leq \rangle$ is a poset.

(HS2) $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$ such that $\{x\} \in \mathcal{K}$, for all $x \in X$.

(HS3) For all $w \in \mathcal{K}$, w is an antichain of X.

Given an H-set $\langle X, \leq, \mathcal{K} \rangle$, it defines the set

$$\mathrm{H}(X) = \{w \Rightarrow_i v : w \in \mathcal{K} \text{ and } v \subseteq w\},$$

where $\Rightarrow_i$ is defined as in (2). In [8], the authors proved that $\langle \mathrm{H}(X), \Rightarrow_i, X \rangle$ is an Hilbert algebra. Actually, they showed that $\langle \mathrm{H}(X), \Rightarrow_i, X \rangle$ is a subalgebra of the Hilbert algebra $\langle \mathcal{P}_i(X), \Rightarrow_i, X \rangle$. One of the main results of [8] is that every finite Hilbert algebra $\langle A, \rightarrow, 1 \rangle$ is isomorphic to a Hilbert algebra $\langle \mathrm{H}(X), \Rightarrow_i, X \rangle$ for some finite H-set $\langle X, \leq, \mathcal{K} \rangle$. Let us sketch this. Let A be a finite Hilbert algebra. Let $\mathcal{L}_A = \{K_A^{-1}(a) : a \in A\}$. Then:

Theorem 1.6 ([8])**.** *The structure $\langle \mathrm{CIrr}(A), \subseteq, \mathcal{L}_A \rangle$ is a finite H-set, and the map $\varphi \colon A \to \mathrm{H}(\mathrm{CIrr}(A))$ is an isomorphism.*

2 Characterization

In this section, we present the algebraic conditions characterizing the dual Hilbert algebras coming from the H-sets. As we mentioned above, we will work with irreducible elements instead of completely irreducible implicative filters, thus we need to consider the algebra of decreasing subsets $\mathcal{P}_d(X)$ for posets instead of the algebra of increasing subsets. From now on, given an H-set $\langle X, \leq, \mathcal{K} \rangle$, we define the Hilbert algebra $\langle \mathrm{A}(X), \Rightarrow, X \rangle$ as follows:

$$\mathrm{A}(X) = \{w \Rightarrow v : w \in \mathcal{K} \text{ and } v \subseteq w\}$$

where $w \Rightarrow v$ is defined as in (1). Notice that if $\langle X, \leq, \mathcal{K} \rangle$ is an H-set, then $\langle X, \geq, \mathcal{K} \rangle$ is also an H-set. Hence, the Hilbert algebra $A(X)$ given by $\langle X, \leq, \mathcal{K} \rangle$ coincide with the Hilbert algebra $H(X)$ given by $\langle X, \geq, \mathcal{K} \rangle$.

We begin presenting the basic notion with what we shall work: irreducible element.

Definition 2.1. Let $\langle A, \rightarrow, 1 \rangle$ be a Hilbert algebra. An element $p \neq 1$ of A is called *irreducible* if for each $a \in A$, $a \leq p$ or $a \rightarrow p = p$. We denote by $\mathrm{Irr}(A)$ the set of all irreducible elements of A.

Proposition 2.2 ([8])**.** *Let A be a Hilbert algebra and $p \in A$. Then, the following are equivalent.*

1. *p is irreducible.*

2. *$(p]^c$ is an implicative filter.*

Actually, if p is an irreducible element, then $(p]^c$ is a completely irreducible implicative filter of A. In the finite case, the completely irreducible implicative filters are exactly of the form $(p]^c$, for $p \in A$. Hence, considering the order in A restricted to $\mathrm{Irr}(A)$, it follows that $\langle \mathrm{Irr}(A), \leq \rangle$ is dual order-isomorphic with $\langle \mathrm{CIrr}(A), \subseteq \rangle$.

Let $\langle A, \to, 1 \rangle$ be a Hilbert algebra. Let us consider the set $\mathrm{Irr}(A)$ ordered under the partial order of A. For every $a \in A$, let $\mathrm{M}_A(a)$ be the set of minimals elements of the subset $\{p \in \mathrm{Irr}(A) : a \leq p\}$ (ordered under the natural partial order given by A). That is,

$$\mathrm{M}_A(a) = \mathrm{Min}\{p \in \mathrm{Irr}(A) : a \leq p\}.$$

Definition 2.3. A *perfect Hilbert algebra* is a Hilbert algebra $\langle A, \to, 1 \rangle$ satisfying the following conditions:

(S) For all $a, b \in A$ y $p \in \mathrm{Irr}(A)$, if $a \to b \leq p$, then there exists $q \in \mathrm{Irr}(A)$ such that $q \leq p$, $a \nleq q$ and $b \leq q$.

(D) For every $a \in A$, $a = \bigwedge \{p \in \mathrm{Irr}(A) : a \leq p\}$.

(M) For all $a \in A$ and $p \in \mathrm{Irr}(A)$, if $a \leq p$, then there exists $q \in \mathrm{M}_A(a)$ such that $q \leq p$.

(C) For each $a \in A$ and $v \subseteq \mathrm{M}_A(a)$, there exists $\bigwedge v$ in A.

(I) For each $a \in A$ and $v \subseteq \mathrm{M}_A(a)$, if $p \in \mathrm{Irr}(A)$ and $\bigwedge v \leq p$, then there is $q \in v$ such that $q \leq p$.

Proposition 2.4. *If $\langle A, \to, 1 \rangle$ is a Hilbert algebra satisfying conditions (D) and (M), then $a = \bigwedge \mathrm{M}_A(a)$, for every $a \in A$.*

Proof. Let $a \in A$. It is clear that a is a lower bound of $\mathrm{M}_A(a)$. Let b be a lower bound of $\mathrm{M}_A(a)$. Let $p \in \mathrm{Irr}(A)$ be such that $a \leq p$. By (M), there is $q \in \mathrm{M}_A(a)$ such that $q \leq p$. Thus $b \leq q \leq p$. That is, $b \leq p$ for all $p \in \mathrm{Irr}(A)$ such that $a \leq p$. Then, by (D), $b \leq \bigwedge \{p \in \mathrm{Irr}(A) : a \leq p\} = a$. Hence, $a = \bigwedge \mathrm{M}_A(a)$. □

Example 2.5. We know that the implicative reduct of a Boolean algebra B is a Hilbert algebra (actually, it is a Tarski algebra, see Definition 3.11) with the implication given by $a \to b = \neg a \vee b$. Moreover, an element $b \in B$ is irreducible (Def. 2.1) if and only if b is a co-atom (equivalently meet-irreducible) of B, see [17]. Hence, any atomless Boolean algebra is an example of a Hilbert algebra which is not perfect, because it does not satisfy condition (D). Thus, the class of perfect Hilbert algebras is a proper subclass of the variety of Hilbert algebras. Moreover, the class of perfect Hilbert algebras is not a quasivariety, because it is not closed under subalgebra. See Example 3.20.

Now we proceed to prove that $\langle \mathrm{A}(X), \Rightarrow, X \rangle$ is a perfect Hilbert algebra for every H-set $\langle X, \leq, \mathcal{K} \rangle$. To attain this, we show that there is a nice characterization of the irreducible elements of the Hilbert algebras $\mathrm{A}(X)$ for H-sets $\langle X, \leq, \mathcal{K} \rangle$.

Proposition 2.6. *Let $\langle X, \leq, \mathcal{K}\rangle$ be an H-set. Then* $\mathrm{Irr}(\mathrm{A}(X)) = \{[x)^c : x \in X\}$.

Proof. Let $x \in X$. Notice that $[x)^c = \{x\} \Rightarrow \emptyset$. Thus, by (HS2), we obtain that $[x)^c \in \mathrm{A}(X)$. Now let us see that $[x)^c$ is an irreducible element of the Hilbert algebra $\mathrm{A}(X)$. Let $u \in \mathrm{A}(X)$. Suppose that $u \nsubseteq [x)^c$. Then, there is $y \in u$ such that $x \leq y$. Since u is a decreasing subset of X, we have that $x \in u$. We need to show that $u \Rightarrow [x)^c = [x)^c$. Since $\mathrm{A}(X)$ is a Hilbert algebra, we know that $[x)^c \subseteq u \Rightarrow [x)^c$. Let now $z \in u \Rightarrow [x)^c$, and suppose that $z \notin [x)^c$. Then $u \cap (z] \subseteq [x)^c$ and $x \leq z$. Thus $x \in u \cap (z]$. Hence $x \in [x)^c$, which is a contradiction. Hence $u \Rightarrow [x)^c \subseteq [x)^c$. Then $u \Rightarrow [x)^c = [x)^c$. Therefore, we have proved that $\{[x)^c : x \in X\} \subseteq \mathrm{Irr}(\mathrm{A}(X))$.

Let $u \in \mathrm{Irr}(\mathrm{A}(X))$. Thus $u = w \Rightarrow v$ for some $w \in \mathcal{K}$ and $v \subseteq w$. Since $u \neq X$, it follows that $v \neq w$. Let $x \in w \setminus v$. Thus $x \notin u$. Given that u is decreasing subset, we have that $u \subseteq [x)^c$. Suppose that $[x)^c \nsubseteq u$. Since u is an irreducible element of $\mathrm{A}(X)$, it follows that $[x)^c \Rightarrow u = u$. Then $x \notin [x)^c \Rightarrow u$. Thus $[x)^c \cap (x] \nsubseteq u$. Let $y \leq x$ such that $x \nleq y$ and $y \notin u$. So $y \notin w \Rightarrow v$. Then $w \cap (y] \nsubseteq v$. Let $z \in w$ be such that $z \leq y$ and $z \notin v$. We have that $z, x \in w$ and $z \leq y < x$, but this is a contradiction because w is an antichain of X. Hence, we obtain that $[x)^c \subseteq u$. Therefore $u = [x)^c$. Then $\mathrm{Irr}(\mathrm{A}(X)) \subseteq \{[x)^c : x \in X\}$. □

Proposition 2.7. *Let $\langle X, \leq, \mathcal{K}\rangle$ be an H-set. Let $u = w \Rightarrow v \in \mathrm{A}(X)$ with $w \in \mathcal{K}$ and $v \subseteq w$. Then*

$$\mathrm{M}_{\mathrm{A}(X)}(u) = \{[x)^c : x \in w \setminus u\}.$$

Proof. Let $y \in X$. Assume first that $[y)^c \in \mathrm{M}_{\mathrm{A}(X)}(u)$. Since $u \subseteq [y)^c$, we have that $y \notin u$. Thus $y \notin w \Rightarrow v$, that is, $w \cap (y] \nsubseteq v$. Let $z \in w$ such that $z \leq y$ and $z \notin v$. Then $[z)^c \subseteq [y)^c$. Since $z \in w \setminus v$, it follows that $z \notin u$. Thus $u \subseteq [z)^c$. We thus have that $u \subseteq [z)^c$ and $[z)^c \subseteq [y)^c$. The minimality of $[y)^c$ implies that $[z)^c = [y)^c$. Then $y = z \in w$. Hence $y \in w \setminus u$. Therefore, $\mathrm{M}_{\mathrm{A}(X)}(u) \subseteq \{[x)^c : x \in w \setminus u\}$.

Let now $y \in w \setminus u$. We need to show that $[y)^c$ is a minimal element of the set $\{[x)^c : u \subseteq [x)^c\}$. Since $y \notin u$ and u is a decreasing subset of X, we have that $u \subseteq [y)^c$. Let $z \in X$ be such that $u \subseteq [z)^c$ and $[z)^c \subseteq [y)^c$. We need to prove that $[z)^c = [y)^c$. Given that $[z)^c \subseteq [y)^c$, we have $z \leq y$. Since $u \subseteq [z)^c$, it follows that $z \notin u = w \Rightarrow v$. Thus $w \cap (z] \nsubseteq v$. Let $x' \in w$ be such that $x' \leq z$ and $x' \notin v$. Thus $x' \leq z \leq y$ and $x', y \in w$. By (HS3), it follows that $x' = z = y$. Then $[z)^c = [y)^c$. Hence $[y)^c \in \mathrm{M}_{\mathrm{A}(X)}(u)$. Therefore, $\{[x)^c : x \in w \setminus u\} \subseteq \mathrm{M}_{\mathrm{A}(X)}(u)$. □

Proposition 2.8. *Let $\langle X, \leq, \mathcal{K}\rangle$ be an H-set. Then $\langle \mathrm{A}(X), \Rightarrow, X\rangle$ is a perfect Hilbert algebra.*

Proof. We already know that $\langle \mathrm{A}(X), \Rightarrow, X\rangle$ is a Hilbert algebra.

Recall that $\mathrm{Irr}(\mathrm{A}(X)) = \{[x)^c : x \in X\}$.

(S) Let $u_1, u_2 \in \mathrm{A}(X)$ and $x \in X$. Assume that $u_1 \Rightarrow u_2 \subseteq [x)^c$. Thus $x \notin u_1 \Rightarrow u_2$. That is, $u_1 \cap (x] \nsubseteq u_2$. Let $y \in u_1$ such that $y \leq x$ and $y \notin u_2$. Then it follows, respectively, that $u_1 \nsubseteq [y)^c$, $[y)^c \subseteq [x)^c$, and $u_2 \subseteq [y)^c$. Hence, condition (S) holds.

(D) Let $u \in \mathrm{A}(X)$. Since u is a decreasing subset of X, it follows straightforwardly that $u = \bigcap\{[x)^c : u \subseteq [x)^c\}$.

(M) Let $u \in \mathrm{A}(X)$ and $x \in X$ be such that $u \subseteq [x)^c$. Let $w \in \mathcal{K}$ and $v \subseteq w$ be such that $u = w \Rightarrow v$. Since $x \notin u = w \Rightarrow v$, there is $y \in w$ such that $y \leq x$ and $y \notin v$. Thus $y \in w \setminus v$. This implies that $y \notin u$. Then $y \in w \setminus u$. By Proposition 2.7, we have that $[y)^c \in \mathrm{M}_{\mathrm{A}(X)}(u)$. Since $y \leq x$, it follows that $[y)^c \subseteq [x)^c$.

(C) It is enough to show that, for every $w \in \mathcal{K}$ and $v \subseteq w$, $\bigcap\{[x)^c : x \in v\} \in \mathrm{A}(X)$. Let $w \in \mathcal{K}$ and $v \subseteq w$. Let us prove that $\bigcap\{[x)^c : x \in v\} = w \Rightarrow (w \setminus v)$.

Let $y \in \bigcap\{[x)^c : x \in v\}$. Thus $x \nleq y$, for all $x \in v$. We need to prove that $w \cap (y] \subseteq w \setminus v$. Suppose that $w \cap (y] \nsubseteq w \setminus v$. Thus, there is $z \in w$ such that $z \leq y$ and $z \notin w \setminus v$. Then $z \in v$. Thus $z \nleq y$, which is a contradiction. Then $w \cap (y] \subseteq w \setminus v$, which implies that $y \in w \Rightarrow (w \setminus v)$.

Let $y \in w \Rightarrow (w \setminus v)$. Thus $w \cap (y] \subseteq w \setminus v$. Suppose that $y \notin \bigcap\{[x)^c : x \in v\}$. Then, there is $x \in v$ such that $x \leq y$. Thus $x \in w$ and $x \in (y]$. Hence $x \in w \setminus v$. Thus $x \notin v$, which is a contradiction. Then $y \in \bigcap\{[x)^c : x \in v\}$. Hence $w \Rightarrow (w \setminus v) \subseteq \bigcap\{[x)^c : x \in v\}$.

(I) It is enough to show that for all $\{x_i : i \in I\} \cup \{x\} \subseteq X$, if $\bigcap_{i\in I}[x_i)^c \subseteq [x)^c$, then there is $i_0 \in I$ such that $[x_{i_0})^c \subseteq [x)^c$. This follows straightforwardly. □

Proposition 2.9. *Let $\langle A, \to, 1\rangle$ be a Hilbert algebra satisfying conditions (D) and (S). Then, the map $\alpha\colon A \to \mathcal{P}_d(\mathrm{Irr}(A))$ defined by*

$$\alpha(a) = \{p \in \mathrm{Irr}(A) : a \nleq p\}$$

is an embedding from $\langle A \to, 1\rangle$ to the Hilbert algebra $\langle \mathcal{P}_d(\mathrm{Irr}(A)), \Rightarrow, \mathrm{Irr}(A)\rangle$.

Proof. It is clear that for every $a \in A$, $\alpha(a) \in \mathcal{P}_d(\mathrm{Irr}(A))$ and $\alpha(1) = \mathrm{Irr}(A)$. From condition (D) it follows that α is a one-to-one map. Let $a, b \in A$. We now show that $\alpha(a \to b) = \alpha(a) \Rightarrow \alpha(b)$. Let $p \in \alpha(a \to b)$. Thus $a \to b \nleq p$. We need to prove that $\alpha(a) \cap (p]_{\mathrm{Irr}(A)} \subseteq \alpha(b)$. Let $q \in \alpha(a) \cap (p]_{\mathrm{Irr}(A)}$. This is, $a \nleq q$ and $q \leq p$. Then $a \in (q]^c_A$ $a \to b \nleq q$. Thus $a \to b \in (q]^c_A$. Since $(q]^c_A$ is a deductive system, it follows that $b \in (q]^c_A$.

Then $q \in \alpha(b)$. Hence $p \in \alpha(a) \Rightarrow \alpha(b)$. Therefore, $\alpha(a \to b) \subseteq \alpha(a) \Rightarrow \alpha(b)$. Let $p \in \alpha(a) \Rightarrow (b)$. Thus $\alpha(p) \cap (p]_{\mathrm{Irr}(A)} \subseteq \alpha(b)$. Suppose that $a \to b \leq p$. By condition (S), there is $q \in \mathrm{Irr}(A)$ such that $q \leq p$, $a \nleq q$ and $b \leq q$. Thus $q \in \alpha(a) \cap (p]_{\mathrm{Irr}(A)}$ and $q \notin \alpha(b)$, which is a contradiction. Then $q \in \alpha(a \to b)$. Hence $\alpha(a) \Rightarrow \alpha(b) \subseteq \alpha(a \to b)$. □

Let us show how to obtain an H-set from an arbitrary Hilbert algebra. Let A be a Hilbert algebra. Let $\mathcal{K}_A = \{\mathrm{M}_A(a) : a \in A\}$, where $\mathrm{M}_A(a) = \mathrm{Min}(\{p \in \mathrm{Irr}(A) : a \leq p\})$, for every $a \in A$.

Proposition 2.10. *Let A be a Hilbert algebra. Then, the triple $\langle \mathrm{Irr}(A), \leq, \mathcal{K}_A \rangle$ is an H-set.*

Proof. Recall that $\langle \mathrm{Irr}(A), \leq \rangle$ is the subposet of $\langle A, \leq \rangle$, where $\leq$ is the natural order on A. Notice that $\mathrm{M}_A(1) \in \mathcal{K}_A$, thus $\mathcal{K}_A \neq \emptyset$. Let $p \in \mathrm{Irr}(A)$. It is straightforward to see that $\mathrm{M}_A(p) = \{p\}$. Then, (HS2) holds. And it is straightforward that $\mathrm{M}_A(a)$ is antichain of $\mathrm{Irr}(A)$, for every $a \in A$. Hence (HS3) holds. □

From Proposition 2.8 and the following theorem, we obtain the algebraic representation promised.

Theorem 2.11. *Let $\langle A, \to, 1 \rangle$ be a perfect Hilbert algebra. Then*

$$\langle A, \to, 1 \rangle \cong \langle \mathrm{A}(\mathrm{Irr}(A)), \Rightarrow, \mathrm{Irr}(A) \rangle,$$

where $\mathrm{A}(\mathrm{Irr}(A))$ is the dual Hilbert algebra of the H-set $\langle \mathrm{Irr}(A), \leq, \mathcal{K}_A \rangle$.

Proof. By Proposition 2.9 we know that $\langle A, \to, 1 \rangle \cong \langle \alpha(A), \Rightarrow, \mathrm{Irr}(A) \rangle$. We show that $\mathrm{A}(\mathrm{Irr}(A)) = \alpha(A)$.

Let $a \in A$. We show that $\alpha(a) = \mathrm{M}_A(a) \Rightarrow \emptyset$. Let $p \in \alpha(a)$. Thus $a \nleq p$. We need to prove that $\mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)} = \emptyset$. Suppose that there is $q \in \mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)}$. Thus $a \leq q$ and $q \leq p$. Then $a \leq p$, which is a contradiction. Hence $\alpha(a) \subseteq \mathrm{M}_A(a) \Rightarrow \emptyset$. Let now $p \in \mathrm{M}_A(a) \Rightarrow \emptyset$. Thus $\mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)} = \emptyset$. Suppose that $a \leq p$. By (M), there is $q \in \mathrm{M}_A(a)$ such that $q \leq p$. Thus $q \in \mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)}$, which is a contradiction. Then $p \in \alpha(a)$. Hence $\mathrm{M}_A(a) \Rightarrow \emptyset \subseteq \alpha(a)$. That is, $\alpha(a) = \mathrm{M}_A(a) \Rightarrow \emptyset \in \mathrm{A}(\mathrm{Irr}(A))$.

Let now $u \in \mathrm{A}(\mathrm{Irr}(A))$. Thus there is $a \in A$ and $v \subseteq \mathrm{M}_A(a)$ such that $u = \mathrm{M}_A(a) \Rightarrow v$. By (C), $\bigwedge v$ there exists in A. Let us prove that $\mathrm{M}_A(a) \Rightarrow v = \alpha\left(\bigwedge v \to a\right)$. Let $p \in \mathrm{M}_A(a) \Rightarrow v$. Thus $\mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)} \subseteq v$. Suppose that $\bigwedge v \to a \leq p$. By (S), there is $q \in \mathrm{Irr}(A)$ such that $q \leq p$, $\bigwedge v \nleq q$ and $a \leq q$. Since $\bigwedge v \nleq q$, it follows that $r \nleq q$, for all $r \in v$. On the other hand, since $a \leq q$, it follows by (M) that there is $q' \in \mathrm{M}_A(a)$ such that $q' \leq q$. Then $q' \leq p$. Thus $q' \in \mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)}$. Then $q' \in v$. It follows that $q' \nleq q$, which is a contradiction. Hence $p \in \alpha\left(\bigwedge v \to a\right)$. Therefore, $\mathrm{M}_A(a) \Rightarrow v \subseteq \alpha\left(\bigwedge v \to a\right)$.

Let now $p \in \alpha(\bigwedge v \to a)$. Thus $\bigwedge v \to a \nleq p$. We need to show that $p \in \mathrm{M}_A(a) \Rightarrow v$. Let $q \in \mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)}$. Thus $a \leq q$ and $q \leq p$. Given that $\bigwedge v \to a \nleq q$, we have that $\bigwedge v \to a \in (q]^c_A$. Since $(q]^c_A$ is a deductive system and $a \notin (q]^c_A$, it follows that $\bigwedge v \notin (q]^c_A$. Thus $\bigwedge v \leq q$. By (I), there is $q' \in v$ such that $q' \leq q$. Since $q, q' \in \mathrm{M}_A(a)$ (because $v \subseteq \mathrm{M}_A(a)$), we obtain that $q = q' \in v$. Thus, we have proved that $\mathrm{M}_A(a) \cap (p]_{\mathrm{Irr}(A)} \subseteq v$. Then, $p \in \mathrm{M}_A(a) \Rightarrow v$. Hence $\alpha(\bigwedge v \to a) \subseteq \mathrm{M}_A(a) \Rightarrow v$. This completes the proof. □

We summarize the previous results in the following corollary.

Corollary 2.12. *The class of perfect Hilbert algebras coincides with the class of Hilbert algebras* $\mathrm{A}(X)$ *for H-sets* $\langle X, \leq, \mathcal{K}\rangle$.

3 Perfect Hilbert algebras are not so strange

In this section, we shall see that the class of perfect Hilbert algebras is not as strange as it is looks. Indeed, we shall show that the class of perfect Hilbert algebras contains certain well-known classes of Hilbert algebras.

3.1 Every finite Hilbert algebra is a perfect Hilbert algebra

We need the representation given in [8] and some further results. Let $\langle X, \leq, \mathcal{K}\rangle$ be an H-set. Notice that we have two Hilbert algebras associated with $\langle X, \leq, \mathcal{K}\rangle$: $\langle \mathrm{H}(X), \Rightarrow_i, X\rangle$ (see on page 31) as a subalgebra of $\mathcal{P}_i(X)$, and $\langle \mathrm{A}(X), \Rightarrow, X\rangle$ (see on page 31) as a subalgebra of $\mathcal{P}_d(X)$.

Proposition 3.1. *Let* $\langle X, \leq, \mathcal{K}\rangle$ *be an H-set. Let* $w_1, w_2 \in \mathcal{K}$. *Then:*

(1) If $w_1 \Rightarrow_i \emptyset = w_2 \Rightarrow_i \emptyset$, *then* $w_1 = w_2$.

(2) If $w_1 \Rightarrow \emptyset = w_2 \Rightarrow \emptyset$, *then* $w_1 = w_2$.

Proof. They are straightforward. □

Let $\langle A, \to, 1\rangle$ be a finite Hilbert algebra. We already know that

$$\langle A, \to, 1\rangle \cong \langle \mathrm{H}(\mathrm{CIrr}(A), \Rightarrow_i, \mathrm{CIrr}(A)\rangle$$

and $\langle \mathrm{A}(\mathrm{Irr}(A)), \Rightarrow, \mathrm{Irr}(A)\rangle$ is a perfect Hilbert algebra. In order to show that A is a perfect Hilbert algebra, let us to prove that $\mathrm{H}(\mathrm{CIrr}(A)) \cong \mathrm{A}(\mathrm{Irr}(A))$. The following result is the key.

Proposition 3.2 ([8])**.** *Let A be a finite Hilbert algebra. Then,* $\mathrm{CIrr}(A) = \{(p]^c : p \in \mathrm{Irr}(A)\}$.

Notice that for every completely irreducible implicative filter $(p]^c$ of a finite Hilbert algebra A and $a \in A$, it follows that

$$(p]^c \in K_A^{-1}(a) \text{ if and only if } (p]^c \text{ is maximal relative to } a \text{ if and only if } p \in \mathrm{M}_A(a).$$

Moreover, notice that the mapping $K_A^{-1}(a) \mapsto \mathrm{M}_A(a)$ is one-to-one correspondence from $\mathcal{L}_A$ to $\mathcal{K}_A$.

Proposition 3.3 ([8])**.** *Let A be a finite Hilbert algebra. Then* $\mathrm{H}(\mathrm{CIrr}(A)) = \{K_A^{-1}(a) \Rightarrow_i \emptyset : a \in A\}$.

Proposition 3.4. *Let A be a finite Hilbert algebra. Then* $\mathrm{A}(\mathrm{Irr}(A)) = \{\mathrm{M}_A(a) \Rightarrow \emptyset : a \in A\}$.

Proof. By definition, it is clear that $\mathrm{M}_A(a) \Rightarrow \emptyset \in \mathrm{A}(\mathrm{Irr}(A))$, for all $a \in A$. Let now $u \in \mathrm{A}(\mathrm{Irr}(A))$. Then, there is $a \in A$ such that $u = \mathrm{M}_A(a) \Rightarrow v$ for some $v \subseteq \mathrm{M}_A(a)$. Assume that $v \neq \emptyset$. Say $v = \{p_1, \ldots, p_k\}$. Let $b = p_1 \to (p_2 \to (\ldots (p_k \to a) \ldots))$. Let us prove that $\mathrm{M}_A(a) \Rightarrow v = \mathrm{M}_A(b) \Rightarrow \emptyset$. Let $p \in \mathrm{M}_A(a) \Rightarrow v$. Thus $\mathrm{M}_A(a) \cap (p] \subseteq v$. Suppose that $p \notin \mathrm{M}_A(b) \Rightarrow \emptyset$. Thus, let $q \in \mathrm{M}_A(b)$ such that $q \leq p$. Then $b \notin (q]^c$. Since $(q]^c$ is an implicative filter, it follows that $a \notin \langle (q]^c, p_1, \ldots, p_k \rangle$. There exists a completely irreducible implicative filter $(s]^c$ such that $(s]^c$ is maximal relative to a and $\langle (q]^c, p_1, \ldots, p_k \rangle \subseteq (s]^c$. Hence, $s \in \mathrm{M}_A(a)$ and $s \leq q \leq p$. Then $s \in v$. It follows that there is i such that $s = p_i \in (s]^c$, which is a contradiction. Thus $p \in \mathrm{M}_A(b) \Rightarrow \emptyset$. Hence $\mathrm{M}_A(a) \Rightarrow v \subseteq \mathrm{M}_A(b) \Rightarrow \emptyset$. On the other hand, let now $p \in \mathrm{M}_A(b) \Rightarrow \emptyset$. Thus $\mathrm{M}_A(b) \cap (p] = \emptyset$. Since A is finite, we have that $b \nleq p$. Let $q \in \mathrm{M}_A(a) \cap (p]$. Then, $a \notin (q]^c$ and $b \in (q]^c$. Thus, there is $p_i \in v$ such that $p_i \notin (q]^c$. That is, $p_i \leq q$. Since $p_i, q \in v$, it follows that $q = p_i \in v$. Hence $\mathrm{M}_A(a) \cap (p] \subseteq v$. Therefore, $\mathrm{M}_A(b) \Rightarrow \emptyset \subseteq \mathrm{M}_A(a) \Rightarrow v$. ☐

We need a further result before the main theorem of this subsection.

Proposition 3.5. *Let A be a finite Hilbert algebra. Let $a, b \in A$. Then:*

(1) $K_A^{-1}(a \to b) \Rightarrow_i \emptyset = (K_A^{-1}(a) \Rightarrow_i \emptyset) \Rightarrow_i (K_A^{-1}(b) \Rightarrow_i \emptyset)$.

(2) $\mathrm{M}_A(a \to b) \Rightarrow \emptyset = (\mathrm{M}_A(a) \Rightarrow \emptyset) \Rightarrow (\mathrm{M}_A(b) \Rightarrow \emptyset)$.

Proof. (1) It follows from [8].

(2) ($\subseteq$) Let $p \in \mathrm{M}_A(a \to b) \Rightarrow \emptyset$. Thus $\mathrm{M}_A(a \to b) \cap (p] = \emptyset$. It follows that $a \to b \nleq p$, and thus $a \to b \in (p]^c$. Let $q \in (\mathrm{M}_A(a) \Rightarrow \emptyset) \cap (p]$. Thus $\mathrm{M}_A(a) \cap (q] = \emptyset$ and $q \leq p$. Then, $a \nleq q$ and $(p]^c \subseteq (q]^c$. We need to show that $q \in \mathrm{M}_A(b) \Rightarrow \emptyset$. Suppose that it is not satisfied. Then there is $s \in \mathrm{M}_A(b) \cap (q]$. Thus $b \leq s$ and $s \leq q$. Then $b \notin (s]^c$ and $(q]^c \subseteq (s]^c$. Hence $b \notin (q]^c$. On the other hand, notice that $a, a \to b \in (q]^c$. Then $b \in (q]^c$, which is a contradiction. Hence $q \in \mathrm{M}_A(b) \Rightarrow \emptyset$. Therefore, $\mathrm{M}_A(a \to b) \Rightarrow \emptyset \subseteq (\mathrm{M}_A(a) \Rightarrow \emptyset) \Rightarrow (\mathrm{M}_A(b) \Rightarrow \emptyset)$.

($\supseteq$) Let $p \in (\mathrm{M}_A(a) \Rightarrow \emptyset) \Rightarrow (\mathrm{M}_A(b) \Rightarrow \emptyset)$. Thus $(\mathrm{M}_A(a) \Rightarrow \emptyset) \cap (p] \subseteq \mathrm{M}_A(b) \Rightarrow \emptyset$. We need to show that $\mathrm{M}_A(a \to b) \cap (p] = \emptyset$. Suppose that there is $q \in \mathrm{M}_A(a \to b) \cap (p]$. Thus $a \to b \notin (q]^c$ and $q \leq p$. Since $(q]^c$ is an implicative filter, there is a $(s]^c \in \mathrm{CIrr}(A)$ such that $(s]^c$ is maximal relative to b, $a \in (s]^c$ and $(q]^c \subseteq (s]^c$. Thus $s \in \mathrm{M}_A(b)$. Then $s \notin \mathrm{M}_A(b) \Rightarrow \emptyset$. Let us show that $s \in \mathrm{M}_A(a) \Rightarrow \emptyset$. Suppose that there is $t \in \mathrm{M}_A(a) \cap (s]$. Thus $a \notin (t]^c$ and $(s]^c \subseteq (t]^c$. Then $a \notin (s]^c$, which is a contradiction. Hence $s \in (\mathrm{M}_A(a) \Rightarrow \emptyset) \cap (p]$. Thus $s \in \mathrm{M}_A(b) \Rightarrow \emptyset$, which is a contradiction. Then $\mathrm{M}_A(a \to b) \cap (p] = \emptyset$, and thus $p \in \mathrm{M}_A(a \to b) \Rightarrow \emptyset$. Therefore, $(\mathrm{M}_A(a) \Rightarrow \emptyset) \Rightarrow (\mathrm{M}_A(b) \Rightarrow \emptyset) \subseteq \mathrm{M}_A(a \to b) \Rightarrow \emptyset$. □

Now we are ready to prove the main result of this subsection.

Theorem 3.6. *Every finite Hilbert algebra is a perfect Hilbert algebra.*

Proof. Let A be a finite Hilbert algebra. Let $\Psi\colon \mathrm{H}(\mathrm{CIrr}(A)) \to \mathrm{A}(\mathrm{Irr}(A))$ be defined as follows: $\Psi(K_A^{-1}(a) \Rightarrow_i \emptyset) = \mathrm{M}_A(a) \Rightarrow \emptyset$. By Propositions 3.1, 3.3, and 3.4, it follows that Ψ is a well-defined one-to-one correspondence. Now we check that Ψ is a Hilbert homomorphism. Let $a, b \in A$. By Proposition 3.5, we have

$$\begin{aligned}\Psi\left((K_A^{-1}(a) \Rightarrow_i \emptyset) \Rightarrow_i (K_A^{-1}(b) \Rightarrow_i \emptyset)\right) &= \Psi\left(K_A^{-1}(a \to b) \Rightarrow_i \emptyset\right)\\ &= \mathrm{M}_A(a \to b) \Rightarrow \emptyset = (\mathrm{M}_A(a) \Rightarrow \emptyset) \Rightarrow (\mathrm{M}_A(b) \Rightarrow \emptyset)\\ &= \Psi\left(K_A^{-1}(a) \Rightarrow_i \emptyset\right) \Rightarrow \Psi\left(K_A^{-1}(b) \Rightarrow_i \emptyset\right).\end{aligned}$$

Hence Ψ is an isomorphism from $\mathrm{H}(\mathrm{CIrr}(A))$ onto $\mathrm{A}(\mathrm{Irr}(A))$. Thus, $A \cong \mathrm{H}(\mathrm{CIrr}(A)) \cong \mathrm{A}(\mathrm{Irr}(A))$. Therefore, A is a perfect Hilbert algebra. □

3.2 Hilbert algebras given by the order are perfect Hilbert algebras

Definition 3.7. A Hilbert algebra A is said to be *given by the order* if for every $a, b \in A$,

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a \nleq b. \end{cases}$$

The class of Hilbert algebras given by the order was studied by several authors, see for instance [15, 10, 4, 5].

Proposition 3.8 ([8])**.** *Let A be a Hilbert algebra. Then the following are equivalent.*

(1) *A is a Hilbert algebra given by the order.*

(2) $\mathcal{P}_i(A) \setminus \{\emptyset\} = \mathrm{ImFi}(A)$.

(3) *If* $p \in A \setminus \{1\}$*, then* $(p]^c \in \mathrm{CIrr}(A)$*.*

(4) *If* $p \in A \setminus \{1\}$*, then* $p \in \mathrm{Irr}(A)$*.*

(5) *For every* $p \in A \setminus \{1\}$*,* $K_A^{-1}(p) = \{P\}$*.*

(6) *For every* $p \in A \setminus \{1\}$*,* $\mathrm{M}_A(p) = \{p\}$*.*

Theorem 3.9. *Every Hilbert algebra given by the order is a perfect Hilbert algebra.*

Proof. Let A be a Hilbert algebra given by the order. Thus, we have that for every $p \in A \setminus \{1\}$, $\mathrm{M}_A(p) = \{p\}$. Then, it is straightforward to verify that A satisfies conditions (S), (D), (M), (C) and (I). Therefore, the class of perfect Hilbert algebras contains the class of all Hilbert algebras given by the order. □

Corollary 3.10. *Let* A *be a Hilbert algebra and* $\langle X, \leq, \mathcal{K}\rangle$ *its dual H-set. Then,* A *is given by the order if and only if* $\mathcal{K} = \{\{x\} : x \in X\}$*.*

3.3 Every atomic and complete Tarski algebra is a perfect Hilbert algebra

Definition 3.11. A *Tarski algebra* (also known as *implication algebras*) is a Hilbert algebra $\langle A, \to, 1\rangle$ satisfying the identity $(a \to b) \to b = (b \to a) \to a$.

It is well-known that the class of Tarski algebras is the algebraic semantics of the implicative fragment of the classical logic. Moreover, A with its associated order is a join-semilattice, where $a \vee b := (a \to b) \to b$ is the supremum of all a and b in A. For more details on Tarski algebras we refer the reader to [1, 16].

Definition 3.12. Let A be a Tarski algebra. An element $a \in A \setminus \{1\}$ is called *dual atom* if for any $x \in A$, $a \leq x$ implies that $a = x$ or $x = 1$.

Proposition 3.13 ([7])**.** *Let* A *be a Tarski algebra and* $a \in A$*. Then,* a *is a dual atom if and only if* a *is irreducible.*

Definition 3.14 ([7])**.** A Tarski algebra A is said to be *complete* if for every non-empty subset $D \subseteq A$ there exists the supremum $\bigvee D$. A complete Tarski algebra A is said to be *atomic* if for every element $a \neq 1$, there exists a subset $G \subseteq \mathrm{Irr}(A)$ such that $a = \bigwedge G$.

Definition 3.15. An implicative filter P of a complete Tarski algebra A is called *completely prime* if for every non-empty $D \subseteq A$, if $\bigvee D \in P$, then $D \cap P \neq \emptyset$.

Proposition 3.16 ([7])**.** *Let* A *be a complete and atomic Tarski algebra. Then, an implicative filter* P *is completely prime if and only if there exists* $p \in \mathrm{Irr}(A)$ *such that* $P = (p]^c$*.*

Proposition 3.17 ([7]). *Let A be a complete and atomic Tarski algebra and P an implicative filter of A. Then, P is completely prime if and only if P is maximal and closed under existing infimum.*

Theorem 3.18. *If A is a complete and atomic Tarski algebra, then A is a perfect Hilbert algebra.*

Proof. Let A be a complete and atomic Tarski algebra.

(S) Let $a, b \in A$ and $p \in \mathrm{Irr}(A)$. Suppose that $a \to b \leq p$. Thus $a \to b \notin (p]^c$. By Propositions 3.16 and 3.17, we have that $(p]^c$ is a maximal implicative filter. Then, by Theorem 1.3 (3), it follows that $a \in (p]^c$ and $b \notin (p]^c$. Thus, $p \in \mathrm{Irr}(A)$ such that $a \nleq p$ and $b \leq p$.

(D) It follows by the atomic condition.

(M) Since for every $p \in \mathrm{Irr}(A)$, $(p]^c$ is maximal, it follows that $\mathrm{M}_A(a) = \{p \in \mathrm{Irr}(A) : a \leq p\}$, for all $a \in A$. Then, condition (M) is straightforward.

(C) Since there exists the supremum of all non-empty subsets of A, it follows that there exists the infimum of any subset of A bounded below. Hence, (C) holds.

(I) Let $a \in A$ and $v \subseteq \mathrm{M}_A(a)$. Let $p \in \mathrm{Irr}(A)$ and assume that $\bigwedge v \leq p$. Thus $\bigwedge \notin (p]^c$. By Proposition 3.17, there is $q \in v$ such that $q \notin (p]^c$. Hence $q \leq p$.

Therefore, A is a perfect Hilbert algebra. □

Proposition 3.19. *Let A be a perfect Hilbert algebra and $\langle X, \leq \mathcal{K} \rangle$ its dual H-set. Then, A is a complete and atomic Tarski algebra if and only if $\leq$ is the equality, that is, $a \leq b \iff a = b$.*

Proof. Let A be a complete and atomic Tarski algebra. Since $\mathrm{Irr}(A)$ is exactly the set of all dual atoms of A, it follows that the order $\leq$ of A restricted to A is the equality. Now suppose that the H-set $\langle X, \leq, \mathcal{K} \rangle$ is such that $\leq$ is the equality. It follows, for every $u, v \subseteq X$, that $u \Rightarrow v = u^c \cup v$. Then, $A \cong \mathrm{A}(X) = \{w^c \cup v : w \in \mathcal{K} \text{ and } v \subseteq w\}$ is a complete and atomic Tarski algebra, see [7, Thm. 15]. □

Example 3.20. As we mentioned before, the class of perfect Hilbert algebras is not a quasivariety because it is not closed under subalgebra. For instance, the power set of real numbers $\mathcal{P}(\mathbb{R})$ is a complete atomic Boolean algebra, and thus it is in particular a complete and atomic Tarski algebra with the Boolean implication. Hence $\mathcal{P}(\mathbb{R})$ is a perfect Hilbert algebra. Now the interval algebra of $\mathbb{R}$ (see for instance [13, pp. 118]) is an atomless Boolean subalgebra of $\mathcal{P}(\mathbb{R})$. Since co-atoms and irreducible elements are equivalent (see Example 2.5), the interval algebra is a subalgebra of the perfect Hilbert algebra $\mathcal{P}(\mathbb{R})$ but without irreducible elements. Hence, interval algebra is a Hilbert algebra but not perfect.

4 Morphisms and dual equivalence

In this section, we extend the representation of perfect Hilbert algebras through H-sets to a full categorical dual equivalence. Thus, we need to establish who will be the morphisms between H-sets and between perfect Hilbert algebras. For H-sets, we shall consider the morphisms used in [8] to establish their duality in the finite case. The next definition is not exactly as in [8], in fact, it is dual, and this is due that we are using irreducible elements instead of completely irreducible implicative filters as in [8].

Definition 4.1 ([8])**.** Let $\langle X_1, \leq_1, \mathcal{K}_1\rangle$ and $\langle X_2, \leq_2, \mathcal{K}_2\rangle$ be two H-sets. A relation $R \subseteq X_1 \times X_2$ is called *H-functional* if the following conditions are satisfied:

(HF1) If $(x, y) \in R$, then there is $x' \in X_1$ such that $x' \leq x$ and $R(x') = (y]$.

(HF2) If $x \leq x'$, then $R(x) \subseteq R(x')$. If $y \leq y'$, then $R^{-1}(y') \subseteq R^{-1}(y)$.

(HF3) For all $u \in H(X_2)$, $h_R(u) := \{x \in X_1 : R(x) \subseteq u\} \in H(X_1)$.

By [8], we have that the class of H-sets with H-functional relations is a category, where the composition $\circ$ is the usual composition of relations and the dual order $\geq$ in each H-set $\langle X, \leq, \mathcal{K}\rangle$ plays the role of the identity morphism. We denote by $\mathbb{HS}$ the category of H-sets and H-functional relations. To be clear and avoid misunderstood, we consider the composition of relations as follows: for relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, $(x, z) \in R \circ S$ if and only if there exists $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$.

We now introduce the morphisms between perfect Hilbert algebras which will be dual to the H-functional relations.

Definition 4.2. Let A and B be perfect Hilbert algebras. A homomorphism $h\colon A \to B$ is called *perfect* if the following conditions holds:

(WH1) For all $a \in A$ and $q \in \mathrm{Irr}(B)$, if $h(a) \leq q$, then there is $p \in \mathrm{Irr}(A)$ such that $a \leq p$ and $h(p) \leq q$.

(WH2) For all $(p, q) \in \mathrm{Irr}(A) \times \mathrm{Irr}(B)$, if $h(p) \leq q$, then there is $q' \in \mathrm{Irr}(B)$ such that $q' \leq q$ and $h^{-1}[(q']] = (p]$.

Example 4.3. Let A and B be finite Hilbert algebras. Then, every homomorphism $h\colon A \to B$ is perfect. In fact:

(WH1) Suppose that $h(a) \leq q$. Then $a \notin h^{-1}[(q]^c]$. Since B is finite and $q \in \mathrm{Irr}(B)$, it follows that $(q]^c$ is an implicative filter of B. Then, $h^{-1}[(q]^c]$ is an implicative filter of A. Hence, there is a completely irreducible implicative filter P of A such that $a \notin P$ and $h^{-1}[(q]^c] \subseteq P$. Since A is finite, it follows that there is $p \in \mathrm{Irr}(A)$ such that $P = (p]^c$. Therefore, $a \leq p$ and $h(p) \leq q$.

(WH2) Suppose that $h(p) \leq q$. Thus $h^{-1}[(q]^c] \subseteq (p]^c$. Since $(p]^c$ and $(q]^c$ are completely irreducible implicative filters of A and B, respectively, it follows by [6, Thm. 3.3] that there is an irreducible implicative filter Q' of B such that $(q]^c \subseteq Q'$ and $h^{-1}[Q'] = (p]^c$. Since B is finite, $Q' = (q']^c$. Hence $q' \leq q$ and $h^{-1}[(q']] = (p]$.

Example 4.4. Every homomorphism between Hilbert algebras given by the order is perfect. Indeed, let A and B be Hilbert algebras given by the order and $h\colon A \to B$ be a homomorphism. Recall that $\mathrm{Irr}(A) = A \setminus \{1\}$ and $\mathrm{Irr}(B) = B \setminus \{1\}$. Condition (WH1) is straightforward taking $p := a$. Let us prove (WH2). Let $(p, q) \in \mathrm{Irr}(A) \times \mathrm{Irr}(B)$ be such that $h(p) \leq q$. Let $q' := h(p) \in \mathrm{Irr}(A)$. Clearly $q' \leq q$. It is straightforward that $(p] \subseteq h^{-1}[(q']]$. Let now $a \in h^{-1}[(q']]$. Thus $h(a) \leq q' = h(p)$. Then $h(a) \to h(p) = 1$. If $a \nleq p$, then $a \to p = p$. It follows that $1 = h(a) \to h(p) = h(p) \leq q$, which is a contradiction. Hence $a \in (p]$. Therefore, $h^{-1}[(q']] = (p]$.

Example 4.5. A Hilbert homomorphism $h\colon A \to B$ between perfect Hilbert algebras is not necessarily perfect. Let X be an infinity set and $\mathbf{2} = \{0, 1\}$ the two-elements chain. We consider the atomic and complete Boolean algebras $\mathcal{P}(X)$ and $\mathbf{2}$. Hence, they are perfect Hilbert algebras. Recall that the irreducible elements of $\mathcal{P}(X)$ are exactly its co-atoms. Then $\mathrm{Irr}(\mathcal{P}(X)) = \{X \setminus \{x\} : x \in X\}$ and $\mathrm{Irr}(\mathbf{2}) = \{0\}$. Since X is infinity, there exists an ultrafilter U of $\mathcal{P}(X)$ such that $\mathrm{Irr}(\mathcal{P}(X)) \subseteq U$. Let $h\colon \mathcal{P}(X) \to \mathbf{2}$ be given by $h(A) = 1$ if and only if $A \in U$. It is clear that h is a Boolean homomorphism, and thus h is a Hilbert homomorphism. But h is not perfect, because it does not satisfy (WH1). Indeed, let $A \notin U$. Thus $h(A) = 0$. However there is not irreducible element $p = X \setminus \{x\}$ of $\mathcal{P}(X)$ such that $h(p) \leq 0$, because $\mathrm{Irr}(\mathcal{P}(X)) \subseteq U$.

Proposition 4.6. *Let $h\colon A \to B$ and $g\colon B \to C$ be perfect homomorphisms between perfect Hilbert algebras. Then, $g \circ h\colon A \to C$ is a perfect homomorphism.*

Proof. (WH1) Let $a \in A$ and $s \in \mathrm{Irr}(C)$ be such that $(g \circ h)(a) \leq s$. By (WH1) for g, there is $q \in \mathrm{Irr}(B)$ such that $h(a) \leq q$ and $g(q) \leq s$. Now by (WH1) for h, there is $p \in \mathrm{Irr}(A)$ such that $a \leq p$ and $h(p) \leq q$. Hence, there is $p \in \mathrm{Irr}(A)$ such that $a \leq p$ and $(g \circ h)(p) \leq s$.

(WH2) Let $p \in \mathrm{Irr}(A)$ and $s \in \mathrm{Irr}(C)$, and assume that $(g \circ h)(p) \leq s$. Then, by (WH1) for g, there is $q \in \mathrm{Irr}(B)$ such that $h(p) \leq q$ and $g(q) \leq s$. Now by (WH2) for h, there is $q' \in \mathrm{Irr}(B)$ such that $q' \leq q$ and $h^{-1}[(q']] = (p]$. Thus $g(q') \leq s$. By (HW2) for g, there is $s' \in \mathrm{Irr}(C)$ such that $s' \leq s$ and $g^{-1}[(s']] = (q']$. Hence, $s' \in \mathrm{Irr}(C)$, $s' \leq s$, and $(g \circ h)^{-1}[(s']] = (p]$. □

It is easy to verify that the identity function is always a perfect homomorphism for every Hilbert algebra. Hence, the class of perfect Hilbert algebras and perfect homomorphisms is a category. We denote this category by $\mathbb{PHA}$.

The next results are needed to define the corresponding functors between the categories $\mathbb{PHA}$ and $\mathbb{HS}$.

Recall that, for every relation $R \subseteq X_1 \times X_2$, we define the map $h_R \colon \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ as follows:

$$h_R(u) = \{x \in X_1 : R(x) \subseteq u\}$$

for every $u \in \mathcal{P}(X_2)$.

Proposition 4.7. *Let $\langle X_1, \leq_1, \mathcal{K}_1\rangle$ and $\langle X_2, \leq_2, \mathcal{K}_2\rangle$ be H-sets and let $R \subseteq X_1 \times X_2$ be an H-functional relation. Then $h_R \colon \mathrm{A}(X_2) \to \mathrm{A}(X_1)$ is a perfect homomorphism.*

Proof. By [8, Thm. 26], it follows that h_R is a homomorphism.

(WH1) Let $u \in \mathrm{A}(X_2)$ and $x \in X_1$. Assume that $h_R(u) \subseteq [x)^c$. Thus $x \notin h_R(u)$. That is, $R(x) \nsubseteq u$. Let $y \in R(x)$ such that $y \notin u$. Then, since u is a decreasing subset, we have that $u \subseteq [y]^c$. Let us show that $h_R([y)^c) \subseteq [x)^c$. Let $x' \in h_R([y)^c)$. Thus $R(x') \subseteq [y)^c$. Suppose, by contradiction, that $x' \in [x)^c$. Thus $x \leq x'$. By (HF2), it follows that $y \in R(x) \subseteq R(x') \subseteq [y)^c$, which is a contradiction. Hence $x' \in [x)^c$. Therefore, there is $[y)^c \in \mathrm{Irr}(\mathrm{A}(X_2))$ such that $u \subseteq [y)^c$ and $h_R([y)^c) \subseteq [x)^c$.

(WH2) Let $x \in X_1$ and $y \in X_2$ be such that $h_R([y)^c) \subseteq [x)^c$. Thus $x \notin h_R([y)^c)$. That is, $R(x) \nsubseteq [y)^c$. There is $y' \in R(x)$ such that $y \leq y'$. By (HF2), it follows that $x \in R^{-1}(y') \subseteq R^{-1}(y)$. Since $(x, y) \in R$, it follows by (HF1) that there is $x' \in X_1$ such that $x' \leq x$ and $R(x') = (y]$. Now let us prove that for every $u \in \mathrm{A}(X_2)$,

$$h_R(u) \subseteq [x') \iff u \subseteq [y)^c.$$

($\Rightarrow$) Assume that $h_R(u) \subseteq [x')^c$. Suppose $u \nsubseteq [y)^c$. Then, there is $z \in u$ such that $y \leq z$. Thus $y \in u$. Then $R(x') = (y] \subseteq u$. It follows that $x' \in h_R(u) \subseteq [x')^c$, which is a contradiction. Hence $u \subseteq [y)^c$.
($\Leftarrow$) Assume that $u \subseteq [y)^c$. Suppose that $h_R(u) \nsubseteq [x')$. Let $x'' \in h_R(u)$ such that $x'' \notin [x')^c$. Thus $R(x'') \subseteq u \subseteq [y)^c$ and $x' \leq x''$. It follows by (FH2) that $(y] = R(x') \subseteq R(x'')$. Then $y \in R(x'')$, which is a contradiction. Therefore, $h_R(u) \subseteq [x')^c$.

Hence, we have proved that there exists $[x')^c \in \mathrm{Irr}(\mathrm{A}(X_1))$ such that $[x')^c \subseteq [x)^c$ and $h_R^{-1}(([x')^c]) = ([y)^c]$. ☐

Let A and B be Hilbert algebras. For every map $h \colon A \to B$, we define the relation $R_h \subseteq \mathrm{Irr}(B) \times \mathrm{Irr}(A)$ as follows:

$$(q, p) \in R_h \iff h(p) \leq q.$$

Proposition 4.8. *Let A and B be perfect Hilbert algebras. If $h\colon A \to B$ is a perfect homomorphism, then $R_h\colon \mathrm{Irr}(B) \times \mathrm{Irr}(A)$ is an H-functional relation.*

Proof. (HF1) follows by condition (WH2).

(HF2) is straightforward by definition of R_h.

(HF3) We need to prove that $h_{R_h}(u) \in \mathrm{A}(\mathrm{Irr}(B))$, for all $u \in \mathrm{A}(\mathrm{Irr}(A))$. First, recall that $\mathrm{A}(\mathrm{Irr}(A)) = \{\alpha_A(a) : a \in A\}$ and $\alpha_A(a) = \{p \in \mathrm{Irr}(A) : a \nleq p\}$ (analogously for B). Thus, let us prove that for every $a \in A$,

$$h_{R_h}(\alpha_A(a)) = \alpha_B(h(a)).$$

Let $a \in A$. Suppose that $h_{R_h}(\alpha_A(a)) \nsubseteq \alpha_b(h(a))$. Thus, there is $q \in h_{R_h}(\alpha_A(a))$ such that $q \notin \alpha_B(h(a))$. Then, $R_h(q) \subseteq \alpha_A(a)$ and $h(a) \leq q$. By (WH1), there is $p \in \mathrm{Irr}(A)$ such that $a \leq p$ and $h(p) \leq q$. It follows that $p \in R_h(q)$ and $p \notin \alpha_A(a)$, which is a contradiction. Hence $h_{R_h}(\alpha_A(a)) \subseteq \alpha_b(h(a))$. Now let $q \in \alpha_B(h(a))$. Thus $h(a) \nleq q$. We need to show that $R_h(q) \subseteq \alpha_A(a)$. Suppose it is not. Thus, there is $p \in R_h(q)$ such that $p \notin \alpha_A(a)$. Then, $h(p) \leq q$ and $a \leq p$. It follows that $h(a) \leq h(p) \leq q$, which is a contradiction. Hence $\alpha_B(h(a)) \subseteq h_{R_h}(\alpha_A(a))$. □

From we have proved in the previous proposition, we obtain the following.

Corollary 4.9. *Let $h\colon A \to B$ be a perfect homomorphism between perfect Hilbert algebras. Then, $h_{R_h} \circ \alpha_A = \alpha_B \circ h$.*

Proposition 4.10. *Let A, B and C perfect Hilbert algebras and X_1, X_2 and X_3 H-sets.*

(1) *If $\mathrm{id}\colon A \to A$ is de identity map, then $R_{\mathrm{id}} = \geq$.*

(2) *If $\leq_1$ is the order relation of the H-set X_1, then $h_{\geq_1}$ is the identity map of $\mathrm{A}(X_1)$.*

(3) *If $h\colon A \to B$ and $g\colon B \to C$ are perfect homomorphism, then $R_{g\circ h} = R_g \circ R_h$.*

(4) *If $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ are H-functional relations, then $h_{R\circ T} = h_R \circ h_T$.*

Proof. Conditions (1) and (2) are straightforward.

(3) The inclusion $R_{g\circ h} \subseteq R_g \circ R_h$ follows by the definitions of R_h and R_g, and by condition (WH1). On the other hand, the inclusion $R_g \circ R_h \subseteq R_{g\circ h}$ follows just by the definitions of R_h and R_g.

(4) It is straightforward taking into account that

$$h_{R\circ T}(u_3) = \{x_1 \in X_1 : (R \circ T)(x_1) \subseteq u_3\}$$

and

$$(h_R \circ h_T)(u_3) = \{x_1 \in X_1 : R(x_1) \subseteq h_T(u_3)\}.$$

□

Now we can define the corresponding functors. Let $\mathbf{A}\colon \mathbb{HS} \to \mathbb{PHA}$ be defined as follows:

- For every $X \in \mathbb{HS}$, let $\mathbf{A}(X) = \langle \mathrm{A}(X), \Rightarrow, X \rangle$;
- For every $R \in \mathbb{HS}(X_1, X_2)$, let $\mathbf{A}(R) = h_R$

Let $\mathbf{X}\colon \mathbb{PHA} \to \mathbb{HS}$ be defined as follows:

- For every $A \in \mathbb{PHA}$, let $\mathbf{X}(A) = \langle \mathrm{Irr}(A), \leq, \mathcal{K}_A \rangle$;
- For every $h \in \mathbb{PHA}(A, B)$, let $\mathbf{X}(h) = R_h$.

By the results of Section 2 and by the results this section, we have the following.

Corollary 4.11. $\mathbf{A}\colon \mathbb{HS} \to \mathbb{PHA}$ *and* $\mathbf{X}\colon \mathbb{PHA} \to \mathbb{HS}$ *are contravariant functors.*

To attain the main goal of this section, we need three further auxiliary results.

Proposition 4.12 ([8, Example 25])**.** *Let* $\langle X_1, \leq_1, \mathcal{K}_1 \rangle$ *and* $\langle X_2, \leq_2, \mathcal{K}_2 \rangle$ *be H-sets. Let* $f\colon X_1 \to X_2$ *be an order-isomorphism such that*

(1) *for every* $w_2 \in \mathcal{K}_2$*, there is* $w_1 \in \mathcal{K}_1$ *such that* $f^{-1}[w_2] \subseteq w_1$*;*

(2) *for every* $w_1 \in \mathcal{K}_1$*, there is* $w_2 \in \mathcal{K}_2$ *such that* $f[w_1] \subseteq w_2$*.*

Then, the relations $f^* \subseteq X_1 \times X_2$ *and* $(f^{-1})^* \subseteq X_2 \times X_1$ *defined by:*

$$(x_1, x_2) \in f^* \iff x_2 \leq f(x_1), \text{ and } (x_2, x_1) \in (f^{-1})^* \iff x_1 \leq f^{-1}(x_2)$$

are H-functional relations. Moreover, $f^* \circ (f^{-1})^* = \geq_1$ *and* $(f^{-1})^* \circ f^* = \geq_2$*.*

Let $\langle X, \leq, \mathcal{K} \rangle$ be an H-set. By Proposition 2.6, we have that $\mathrm{Irr}(\mathrm{A}(X)) = \{[x)^c : x \in X\}$. Thus, we consider the map $\epsilon\colon X \to \mathrm{Irr}(\mathrm{A}(X))$ defined as follows:

$$\epsilon(x) = [x)^c.$$

Proposition 4.13. *Let* $\langle X, \leq, \mathcal{K} \rangle$ *be an H-set. Then, the map* $\epsilon\colon X \to \mathrm{Irr}(\mathrm{A}(X))$ *is an order-isomorphism satisfying conditions (1) and (2) of Proposition 4.12. Therefore,* $\epsilon^* \subseteq X \times \mathrm{Irr}(\mathrm{A}(X))$ *is an isomorphism of the category* $\mathbb{HS}$ *between the H-sets* $\langle X, \leq, \mathcal{K} \rangle$ *and* $\langle \mathrm{Irr}(\mathrm{A}(X)), \subseteq, \mathcal{K}_{\mathrm{A}(X)} \rangle$.

Proof. It is clear that ϵ is an order-isomorphism. In order to prove conditions (1) and (2) of Proposition 4.12, recall that $\mathcal{K}_{\mathrm{A}(X)} = \{\mathrm{M}_{\mathrm{A}(X)}(u) : u \in \mathrm{A}(X)\}$, and by Proposition 2.7 $\mathrm{M}_{\mathrm{A}(X)}(u) = \{[x)^c \in \mathrm{Irr}(\mathrm{A}(X)) : x \in w \setminus u\}$, where $u = w \Rightarrow v$ with $w \in \mathcal{K}$ and $v \subseteq w$.

(1) For every $u = w \Rightarrow v \in \mathrm{A}(X)$, with $w \in \mathcal{K}$ and $v \subseteq w$, it follows that $\epsilon^{-1}\left[\mathrm{M}_{\mathrm{A}(X)}(u)\right] \subseteq w$.

(2) Let $w \in \mathcal{K}$. Thus $w \Rightarrow \emptyset \in \mathrm{A}(X)$. It is straightforward to show that $\epsilon[w] \subseteq \mathrm{M}_{\mathrm{A}(X)}(w \Rightarrow \emptyset)$.

Hence, the map ϵ satisfies conditions (1) and (2) of Proposition 4.12. Therefore, ϵ^* is an isomorphism of the category $\mathbb{HS}$, where its inverse is given by $(\epsilon^{-1})^*$. □

Proposition 4.14. *Let $\langle X_1, \leq_1, \mathcal{K}_1\rangle$ and let $\langle X_2, \leq_2, \mathcal{K}_2\rangle$ be H-sets and $R \subseteq X_1 \times X_2$ an H-functional relation. Then $\epsilon_1^* \circ R_{h_R} = R \circ \epsilon_2^*$.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. On the one hand, we have

$$\begin{aligned}
(x_1, [x_2)^c) \in \epsilon_1^* \circ R_{h_R} &\iff \exists x_1' \in X_1 \text{ such that } (x_1, [x_1')^c) \in \epsilon_1^* \text{ and } ([x_1')^c, [x_2)^c) \in R_{h_R} \\
&\iff \exists x_1' \in X_1 \text{ such that } [x_1')^c \subseteq \epsilon_1(x_1) = [x_1)^c \text{ and } h_r\left([x_2)^c\right) \subseteq [x_1')^c \\
&\iff \exists x_1' \in X_1 \text{ such that } x_1' \leq x_1 \text{ and } h_R\left([x_2)^c\right) \subseteq [x_1')^c
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(x_1, [x_2)^c) \in R \circ \epsilon_2^* &\iff \exists x_2' \in X_2 \text{ such that } (x_1, x_2') \in R \text{ and } (x_2', [x_2)^c) \in \epsilon_2^* \\
&\iff \exists x_2' \in X_2 \text{ such that } (x_1, x_2') \in R \text{ and } [x_2)^c \subseteq \epsilon_2(x_2') = [x_2')^c \\
&\iff \exists x_2' \in X_2 \text{ such that } (x_1, x_2') \in R \text{ and } x_2 \leq x_2'.
\end{aligned}$$

If $(x_1, [x_2)^c) \in \epsilon_1^* \circ R_{h_R}$, then there is $x_1' \in X_1$ such that $x_1' \leq x_1$ and $h_R\left([x_2)^c\right) \subseteq [x_1')^c$. Then $R(x_1) \nsubseteq [x_2)^c$. Thus, there is $x_2' \in X_2$ such that $(x_1, x_2) \in R$ and $x_2 \leq x_2'$. Hence $(x_1, [x_2)^c) \in R \circ \epsilon_2^*$.

Now suppose that $(x_1, [x_2)^c) \in R \circ \epsilon_2^*$. Thus, there is $x_2' \in X_2$ such that $(x_1, x_2') \in R$ and $x_2 \leq x_2'$. By (HF2), it follows that $(x_1, x_2) \in R$. By the definition of h_R and using again (HF2), it is straightforward to show that $h_R\left([x_2)^c\right) \subseteq [x_1)^c$. Hence $(x_1, [x_2)^c) \in \epsilon_1^* \circ R_{h_R}$. □

By Corollary 4.9, we have that α is a natural isomorphism between the identity functor in $\mathbb{PHA}$ and the functor $\mathbf{A} \circ \mathbf{X}$. By Proposition 4.14, we obtain that ϵ^* is a natural isomorphism between the identity functor in $\mathbb{HS}$ and the functor $\mathbf{X} \circ \mathbf{A}$. Hence, putting all these results together, we obtain the main result of this section.

Theorem 4.15. *The categories* $\mathbb{PHA}$ *and* $\mathbb{HS}$ *are dually equivalent given by the contravariant functor* $\mathbf{X}\colon \mathbb{PHA} \to \mathbb{HS}$ *and* $\mathbf{A}\colon \mathbb{HS} \to \mathbb{PHA}$.

5 Final summary and future work

We have obtained an algebraic characterization of the Hilbert algebras $\mathrm{A}(X)$ that rise from the H-set $\langle X, \leq, \mathcal{K}\rangle$. From the inverse point of view, we have developed a representation and a categorical discrete duality for the class of perfect Hilbert algebras through the structures $\langle X, \leq, \mathcal{K}\rangle$.

Despite the conditions that define perfect Hilbert algebra are something like strange, we showed that this class contains some well-known classes of algebras: it contains the finite Hilbert algebras, the Hilbert algebras given by the order, and the class of atomic and complete Tarski algebras. We also have proved that the class of perfect Hilbert algebras is not a quasivariety.

There are some questions or problems which might be considered for future work:

• The variety generated by the class of Hilbert algebras given by the order (see for instance [5]) is the algebraic counterpart of the order implicative calculus axiomatized by Bull [2], which is an extension of the implicative fragment of intuitionistic logic. As we mentioned, the class of perfect Hilbert algebras $\mathbb{PHA}$ contains the class of Hilbert algebras given by the order. Thus, we might ask ourselves, does the class $\mathbb{PHA}$ contain the variety generated by the class of Hilbert algebras generated by the order?

• If the answer to the previous question is negative, we can restrict the problem to the following. Another important variety of Hilbert algebras is that generated by the Hilbert algebras which are totally ordered (regarding its natural order). This variety was mainly studied in [15] (here these Hilbert algebras are called linear) and in [4] (here these Hilbert algebras are called prelinear). By [5, Lemma 12], it follows that every Hilbert algebra which is totally ordered is a Hilbert algebra given by the order. Thus, we might ask ourselves, does the class $\mathbb{PHA}$ contain the variety generated by the class of Hilbert algebras which are totally ordered?

• If at least one of the two previous questions is answered affirmatively, it might be interesting to develop a relational semantic, through the H-sets $\langle X, \leq, \mathcal{K}\rangle$, for the logic defined by the corresponding subvariety of Hilbert algebras.

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