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Priestley-style duality for DN-algebras

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Abstract. The aim of this article is to develop a Priestley-style duality for the variety of DN-algebras. In order to achieve this, we use the concept of free distributive lattice extension of a DN-algebra. We establish a connection with the Priestley duality for distributive lattices. Finally, we present topological descriptions for the lattice of filters, for the lattice of congruences, and for certain kinds of subalgebras of a DN-algebra.

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1. Introduction

Distributive nearlattices arose as a natural generalization of distributive lattices, and they were studied by several authors under different names, see for instances [4,15,27,30,32,12,13,8,22,6,23,7]. A *distributive nearlattice* is a join-semilattice $\langle A, \vee\rangle$ such that for every element $a \in A$, the principal upset $\uparrow a = \{x \in A : a \leq x\}$ is a distributive lattice. Equivalently, a distributive nearlattice can be defined as a join-semilattice $\langle A, \vee\rangle$ satisfying the following conditions: (i) if $a, b \in A$ are bounded below, then there exists the infimum of a and b; (ii) if $a_1 \wedge \cdots \wedge a_n$ there exists in A, then $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ exists for all $b \in A$ and equals $(a_1 \wedge \cdots \wedge a_n) \vee b$. Thus, it is clear that the notion of distributive nearlattice is a natural generalization of distributive lattice and it is an ordered algebraic structure much richer than join-semilattice. Moreover, it is worth noting that an interesting subclass of distributive nearlattice is provided by those join-semilattice in which each principal upset is a boolean algebra. These semilattices were studied by Abbott [2,1] under the

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name *semi-boolean algebras* and mainly from the point of view of implication algebras. The class of implication algebras is the algebraic semantics of the $\{\to\}$-fragment of classical propositional logic.

A drawback of the class of distributive nearlattices is that not a variety. However, several authors defined a variety of ternary algebras, which we shall call *DN-algebras*, through certain identities in such a way that the class of distributive nearlattices is in a one-to-one correspondence with this variety, see [27,13,3]. Moreover, in [21], it was proposed a propositional logic $\mathcal{S}_{\mathbb{DN}}$ associated with the variety $\mathbb{DN}$ of DN-algebras. There, it was proved among other things that the algebraic counterpart (in the sense of Abstract Algebraic Logic, see [17]) of this logic is the variety of DN-algebras, that is, $\mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) = \mathbb{DN}$. Furthermore, in [24] it was shown that the logic $\mathcal{S}_{\mathbb{DN}}$ can be interpreted inside the $\{\wedge, \vee\}$-fragment of classical logic. These results show us that there is a deep connection between the logic $\mathcal{S}_{\mathbb{DN}}$ and the $\{\wedge, \vee\}$-fragment of classical logic (see [18,19]), and thus there is also a deep connection between the variety of DN-algebras and the variety of distributive lattices. For these reasons, we consider that is convenient to work with DN-algebras instead of distributive nearlattices.

A very useful tool for studying distributive lattices and those logics which have an algebraic semantic based on distributive lattices is the Priestley duality [28,29]. Furthermore, there is a vast literature of papers using and generalizing the Priestley duality to different classes of algebras and applying it to the study of logic. The Priestley duality and its variants are central in making the link between syntactical and semantic approaches to logic. Hence, since DN-algebras are a natural generalization of distributive lattices, we think it will be important to try to establish a Priestley-style duality for the class of DN-algebras. On the other hand, it is worth mentioning that in [8] (see also [9,10]) is developed a topological duality for the class of distributive nearlattices with a greatest element in the spirit of Stone's duality for distributive lattices.

The paper is organized as follows. In Section 2, we provide the required notations and preliminaries on partially ordered sets and lattices, DN-algebras, and Priestley duality. Section 3 is devoted to an overview of the notion of free distributive lattice extension of a DN-algebra (see [15,9]). We show that the algebraic categories of DN-algebras and distributive lattices are related by an adjunction. In [9], it is proved that the lattice of filters of a DN-algebra and the lattice of filters of the free distributive lattice extension are isomorphic. Then, we show that this isomorphism can be restricted to an order-isomorphisms between the set of prime filters of the DN-algebra and the set of prime filters of its free distributive lattice extension, both sets ordered by set-theoretical inclusion. This is important to develop our duality. In Section 4, we introduce the definition of Priestley DN-space and show, at the object level, that the class of DN-algebras is categorically equivalent to the class of Priestley DN-spaces. In Section 5, we extend the equivalence of the previous section to a full categorical duality between the category of DN-algebras and homomorphisms and the category of Priestley DN-spaces and certain morphisms. The section is ended by establishing a connection between the four categories mentioned so far: the categories of DN-algebras, distributive lattices, Priestley DN-spaces, and Priestley spaces. In Section 6, we put the duality for DN-algebras to work to obtain topological descriptions of some important algebraic concepts as filters, congruences, and subalgebras of DN-algebras. These results provide a wider context and generalize the Priestley duality for distributive lattices.

2. Preliminaries

2.1. Order and lattices

We assume that the reader is familiar with order and lattice theory. Our main references for these are [25, 16, 5].

Let $\langle P, \leq \rangle$ be a poset. For every subset $U \subseteq P$, we define the set $\uparrow U = \{x \in P : x \geq y, \text{ for some } y \in U\}$. Dually, we have the definition of $\downarrow U$. We shall write $\uparrow x$ instead of $\uparrow\{x\}$. A subset $U \subseteq P$ is called *upset* if $U = \uparrow U$. Dually, $V \subseteq P$ is a *downset* if $\downarrow V = V$. We denote by $\mathrm{Up}(P)$ the collection of all upsets of P. We shall consider the complete distributive lattice $\langle \mathrm{Up}(P), \cap, \cup, \emptyset, P\rangle$.

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice. We will say that a subset $F \subseteq L$ is a *filter* of $\mathbf{L}$ if it is an upset and closed under $\wedge$. We denote the collection of all filters of $\mathbf{L}$ by $\mathrm{Fi}(\mathbf{L})$. A filter F is said to be *prime* if $a \in F$ or $b \in F$ whenever $a \vee b \in F$. Let $\mathrm{Fi}_{\mathrm{pr}}(\mathbf{L})$ be the collection of all prime filters of $\mathbf{L}$. (See Remark 2.5).

2.2. DN-algebras

As we mentioned in the introduction, a *distributive nearlattice* is a join-semilattice $\langle A, \vee \rangle$ such that for every element $a \in A$, the principal upset $\uparrow a = \{x \in A : a \leq x\}$ is a distributive lattice. We encourage the reader unfamiliar with distributive nearlattices to consult [13, 11, 27].

Definition 2.1 [3]. An algebra $\mathbf{A} = \langle A, m\rangle$ of type (3) is called a *DN-algebra* if the following identities hold:

(P1) $m(x, y, x) = x$;
(P2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$;
(P3) $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

We denote by $\mathbb{DN}$ the variety of DN-algebras. At first glance, the above identities appear to be strange. However, the next theorem shows us the real meaning of these identities.

Theorem 2.2 [13]. (1) *If $\mathbf{A} = \langle A, m\rangle$ is a DN-algebra, then the algebra $\mathbf{A}_* = \langle A, \vee \rangle$, where*

$$x \vee y := m(x, x, y), \tag{2.1}$$

is a distributive nearlattice.

(2) *If $\mathbf{S} = \langle S, \vee \rangle$ is a distributive nearlattice, then the algebra $\mathbf{S}^* = \langle S, m\rangle$ with*

$$m(x, y, z) := (x \vee z) \wedge_z (y \vee z)$$

(where $\wedge_z$ is the meet of $x \vee z$ and $y \vee z$ in $[z)$) is a DN-algebra.

(3) *If* $\mathbf{A}$ *is a DN-algebra and* $\mathbf{S}$ *is a distributive nearlattice, then* $(\mathbf{A}_*)^* = \mathbf{A}$ *and* $(\mathbf{S}^*)_* = \mathbf{S}$.

The previous theorem shows us that there is a one-to-one correspondence between DN-algebras and distributive nearlattices. Actually, this can be easily extended to a categorical equivalence between the algebraic category of DN-algebras and homomorphisms and the category of distributive nearlattices whose morphisms are join-homomorphisms preserving existent finite meets.

Example 2.3. Let $\langle L, \wedge, \vee\rangle$ be a distributive lattice. Then, it is clear that $\langle L, \vee\rangle$ is a distributive nearlattice. Hence, $\langle L, m\rangle$ is a DN-algebra, where the operation m is defined by $m(x, y, z) = (x \vee z) \wedge (y \vee z)$.

Example 2.4. Let $A = \{X \subseteq \mathbb{N} : \#(X) = \aleph_0\}$ and m is the ternary map on A defined as follows: $m(X, Y, Z) = (X \cup Z) \cap (Y \cup Z)$, for all $X, Y, Z \in A$. It is straightforward to show that $\langle A, \cup\rangle$ is a distributive nearlattice, where the meet in the principal upsets of $\langle A, \cup\rangle$ is $\cap$. Hence, by Theorem 2.2, $\langle A, m\rangle$ is a DN-algebra.

Given a DN-algebra $\mathbf{A} = \langle A, m\rangle$, we consider the join operation $\vee$ on A defined as in (2.1) and the partial order $\leq$ on A is determined by $\vee$, i.e., $x \leq y$ if and only if $y = x \vee y = m(x, x, y)$. For every element $a \in A$, we denote the meet in $[a)$ by $\wedge_a$. It should be noted that the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A. Thus, the meet of x and y in $[a)$ coincides with their meet in A, that is, $x \wedge_a y = x \wedge y$ for all $x, y \in [a)$. Moreover, whenever we write $a_1 \wedge \cdots \wedge a_n$ will means that the meet of $a_1, \ldots, a_n$ there exists in A and is $a_1 \wedge \cdots \wedge a_n$. On the other hand, notice that if $a_1 \wedge \cdots \wedge a_n$ there exists in $\mathbf{A}$, then $(a_1 \wedge \cdots \wedge a_n) \vee b = (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$, for all $b \in A$. All these considerations should be kept in mind since we will use them without mention.

Let $\mathbf{A}$ be a DN-algebra. A subset $F \subseteq A$ is said to be a *filter* of $\mathbf{A}$ if it is an upset of $\mathbf{A}$ and closed under existing finite meets. We denote by $\mathrm{Fi}(\mathbf{A})$ the collection of all filters of $\mathbf{A}$. It is clear that $\mathrm{Fi}(\mathbf{A})$ is an algebraic closure system, and thus it is a complete lattice. Given a nonempty subset $X \subseteq A$, let us denote by $\mathrm{Fig}_{\mathbf{A}}(X)$ the filter of $\mathbf{A}$ generated by X. It is known that

$$\mathrm{Fig}_{\mathbf{A}}(X) = \{a \in A : a = a_1 \wedge \cdots \wedge a_n, \text{ for some } a_1, \ldots, a_n \in {\uparrow}X\}.$$

This can be proved by the results given in [15]. It is also known that

$$\langle \mathrm{Fi}(\mathbf{A}), \cap, \veebar\rangle$$

is a distributive lattice, where $F \veebar G = \mathrm{Fig}_{\mathbf{A}}(F \cup G)$ (see for instance [15, 27, 11]). Moreover, we notice that $\mathrm{Fig}_{\mathbf{A}}(X)$ can be written in terms of the ternary operation m as follows:

$$\mathrm{Fig}_{\mathbf{A}}(X) = \{a \in A : a = m(x_1, \ldots, x_n, a), \text{ for some } x_1, \ldots, x_n \in X\}$$

where $m(x_1, \ldots, x_n, y) = (x_1 \vee y) \wedge \cdots \wedge (x_n \vee y)$. We leave the details to the reader.

A filter F is said to be *prime* if $a \vee b \in F$, then $a \in F$ or $b \in F$, for all $a, b \in A$. Let us denote by $\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ the collection of all prime filters of $\mathbf{A}$.

Remark 2.5. Notice that the empty set is always a filter of a DN-algebra (lattice). Moreover, for every DN-algebra (lattice) $\mathbf{A}$, from the definition, we have also that the empty set and the whole set A are prime filters. These are unusual facts. It is standard to restrict the filters to the nonempty sets and the prime filters must be proper. Our considerations made here are the key to developing our duality.

Let $\mathbf{A}$ be DN-algebra. A subset $I \subseteq A$ is said to be an *ideal* of $\mathbf{A}$ if it is a downset closed under $\vee$. Given a subset $X \subseteq A$,

$$\mathrm{Idg}_{\mathbf{A}}(X) = \{a \in A : a \leq x_1 \vee \cdots \vee x_n, \text{ for some } x_1, \ldots, x_n \in X\}$$

is the least ideal of $\mathbf{A}$ containing X. An ideal I is called *prime* if $a \in I$ or $b \in I$, whenever $a \wedge b$ exists and $a \wedge b \in I$.

Theorem 2.6 [26]. *Let $\mathbf{A}$ be a DN-algebra, I an ideal of $\mathbf{A}$ and F a filter of $\mathbf{A}$. If $F \cap I = \emptyset$, then there exists a prime filter P of $\mathbf{A}$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let I be an ideal and F a filter of $\mathbf{A}$ such that $F \cap I = \emptyset$. Then, by [26, Theorem 1], there is a prime ideal P of $\mathbf{A}$ such that $I \subseteq P$ and $F \cap P = \emptyset$. It is straightforward show that P^c is a prime filter. Hence $F \subseteq P^c$ and $P^c \cap I = \emptyset$. □

Corollary 2.7. *Let $\mathbf{A}$ be a DN-algebra and $a, b \in A$. If $a \nleq b$, then there is a prime filter P of $\mathbf{A}$ such that $a \in P$ and $b \notin P$.*

Let $\mathbf{A} = \langle A, m \rangle$ be a DN-algebra and $\mathbf{L} = \langle L, \wedge, \vee \rangle$ a distributive lattice. Given that $\mathbf{L}$ can be considered as a DN-algebra with the operation $m_{\mathbf{L}}(a, b, c) = (a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c$, we shall say that a map $f \colon A \to L$ is a *homomorphism* from $\mathbf{A}$ into $\mathbf{L}$ if $f(m(a, b, c)) = m_{\mathbf{L}}(f(a), f(b), f(c)) = (f(a) \vee f(c)) \wedge (f(b), \vee f(c))$, for all $a, b, c \in A$. It is straightforward to check that a map $f \colon A \to L$ is a homomorphism if and only if satisfies the following: for all $a, b \in A$,

(1) $f(a \vee b) = f(a) \vee f(b)$, and
(2) $f(a \wedge b) = f(a) \wedge f(b)$, whenever $a \wedge b$ exists in $\mathbf{A}$.

A *embedding* is an injective homomorphism.

2.3. Bounded Priestley spaces

In this subsection, we recall the Priestley duality for distributive lattices (not necessarily bounded). For more details, we refer the reader to [14,16].

Definition 2.8. A structure $\langle X, \tau, \leq, 0, 1 \rangle$ is said to be a *bounded Priestley space* if:

(1) $\langle X, \tau \rangle$ is a compact space.
(2) $\langle X, \leq \rangle$ is a poset with least element 0 and greatest element 1, and $0 \neq 1$.
(3) $\langle X, \tau, \leq \rangle$ is *totally order-disconnected*: for any $x, y \in X$ where $x \nleq y$, there is a clopen upset U such that $x \in U$ and $y \notin U$.

Notice that $\langle X, \tau, \leq, 0, 1\rangle$ is a bounded Priestley space if and only if $\langle X, \tau, \leq\rangle$ is a Priestley space [16] such that 0 and 1 are, respectively, the least and greatest elements of $\langle X, \leq\rangle$.

Let $\mathbf{L}$ be a distributive lattice (not necessarily bounded). Recall that $\mathrm{Fi}_{\mathrm{pr}}(\mathbf{L})$ denotes the collection of all prime filters of $\mathbf{L}$ and it is ordered by the set-theoretical inclusion. Recall also that $\emptyset, L \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{L})$. Let $\varphi \colon L \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{L}))$ defined as follows: for every $u \in L$,

$$\varphi(u) = \{F \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{L}) : u \in F\}.$$

It follows that φ is a lattice-embedding. Let

$$\mathcal{P}(\mathbf{L}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{L}), \tau_{\mathbf{L}}, \subseteq, \emptyset, L\rangle$$

be defined as follows: $\tau_{\mathbf{L}}$ is the topology on $\mathrm{Fi}_{\mathrm{pr}}(\mathbf{L})$ having as subbasis the collection

$$\{\varphi(u) : u \in L\} \cup \{\varphi(v)^c : v \in L\}.$$

Proposition 2.9. *Let* $\mathbf{L}$ *be a distributive lattice. Then, the structure* $\mathcal{P}(\mathbf{L}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{L}), \tau_{\mathbf{L}}, \subseteq, \emptyset, L\rangle$ *is a bounded Priestley space.*

Given a distributive lattice $\mathbf{L}$, we will say that $\mathcal{P}(\mathbf{L})$ is the *dual bounded Priestley space of* $\mathbf{L}$. Let $\langle X, \tau, \leq\rangle$ be an ordered topological space. We denote by $\mathrm{ClUp}^*(X)$ the collection of all proper, nonempty clopen upsets of X.

Proposition 2.10. *Let* $\mathbf{L}$ *be a distributive lattice. Then,*

$$\mathrm{ClUp}^*(\mathcal{P}(\mathbf{L})) = \{\varphi(u) : u \in L\}.$$

Corollary 2.11 (Representation). *Every distributive lattice is isomorphic to the lattice of all proper, nonempty clopen upsets of some bounded Priestley space.*

Let $\langle X, \tau, \leq, 0, 1\rangle$ be a bounded Priestley space. Consider the distributive lattice $\mathcal{D}(X) = \langle \mathrm{ClUp}^*(X), \cap, \cup\rangle$. Then, we can consider the dual bounded Priestley space of $\mathcal{D}(X)$:

$$\mathcal{P}(\mathcal{D}(X)) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathcal{D}(X)), \tau_{\mathcal{D}(X)}, \subseteq, \emptyset, \mathcal{D}(X)\rangle.$$

Let $\theta \colon X \to \mathrm{Fi}_{\mathrm{pr}}(\mathcal{D}(X))$ be defined as follows: for every $x \in X$,

$$\theta(x) = \{U \in \mathrm{ClUp}^*(X) : x \in U\}.$$

It is straightforward to show that $\theta(x)$ is a prime filter of the lattice $\mathcal{D}(X)$. Thus, θ is well-defined.

Let $\langle X, \tau_X, \leq_X\rangle$ and $\langle Y, \tau_Y, \leq_Y\rangle$ be ordered topological spaces. A map $f \colon X \to Y$ is said to be an *order-homeomorphism* if it is a homeomorphism and an order-isomorphism.

Proposition 2.12. *Let* $\langle X, \tau, \leq, 0, 1\rangle$ *be a bounded Priestley space. Then,* θ *is an order-homeomorphism.*

Let $\mathcal{DL}$ be the category of distributive lattices as objects and lattice-homomorphisms as morphisms. Let X and Y be two bounded Priestley spaces. A map $f\colon X \to Y$ is said to be *order-continuous* if it is a continuous order-preserving map such that $f(0) = 0$ and $f(1) = 1$. Let $\mathcal{BPS}$ be the category of bounded Priestley spaces and order-continuous maps.

Let us define the following functors. Let $\mathcal{P}\colon \mathcal{DL} \to \mathcal{BPS}$ be defined as follows: for every $\mathbf{L} \in \mathcal{DL}$, $\mathcal{P}(\mathbf{L}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{L}), \tau_{\mathbf{L}}, \subseteq, \emptyset, L\rangle$. For a lattice-homomorphism $h\colon L \to M$, $\mathcal{P}(h) = h^{-1}\colon \mathcal{P}(\mathbf{M}) \to \mathcal{P}(\mathbf{L})$. On the other hand, let $\mathcal{D}\colon \mathcal{BPS} \to \mathcal{DL}$ be defined by: for every $X \in \mathcal{BPS}$, $\mathcal{D}(X) = \langle \mathrm{ClUp}^*(X), \cap, \cup\rangle$, and for every order-continuous map $f\colon X \to Y$, $\mathcal{D}(f) = f^{-1}\colon \mathcal{D}(Y) \to \mathcal{D}(X)$.

Theorem 2.13 (Priestley duality). *The categories $\mathcal{DL}$ and $\mathcal{BPS}$ are dually equivalents under the corresponding functors $\mathcal{P}\colon \mathcal{DL} \to \mathcal{BPS}$ and $\mathcal{D}\colon \mathcal{BPS} \to \mathcal{DL}$.*

3. The free distributive lattice extension

The following definition can be found in [15] and [9].

Definition 3.1. Let $\mathbf{A}$ be a DN-algebra. A distributive lattice $\mathbf{L}$ is called a *free distributive lattice extension of* $\mathbf{A}$ if there exists a embedding $e\colon A \to L$ such that for each distributive lattice $\mathbf{M}$ and a embedding $h\colon A \to M$, there is a unique lattice-embedding $\widehat{h}\colon L \to M$ with $h = \widehat{h} \circ e$.

By an argument from category theory, it follows that if a free distributive lattice extension exists for a DN-algebra, then it is unique, up to isomorphism. In [9], it is proved, by a topological approach, the existence of the free distributive lattice extension of a DN-algebra. By the construction itself given in [9] (see Theorem 3.2), it follows that every DN-algebra is finitely meet-dense in its free distributive lattice extension. Next, we shall present a useful characterization of the free distributive lattice extensions.

Proposition 3.2. *Let $\mathbf{A}$ be a DN-algebra. Let $\mathbf{L}$ be a distributive lattice and $e\colon A \to L$ an embedding. The following are equivalent*:

(1) *$\langle \mathbf{L}, e\rangle$ is the free distributive lattice extension of $\mathbf{A}$.*
(2) *$e[A]$ is finitely meet-dense in L, that is,*

$$L = \{e(a_1) \wedge \cdots \wedge e(a_n) : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}.$$

As we mentioned in the above paragraph, the implication (1) ⇒ (2) is consequence of the results in [9]. Here we present an alternative proof, which does not depend on the specific construction of the free distributive lattice extension.

Proof. (1) ⇒ (2) Let $\mathbf{M}$ be the sublattice of $\mathbf{L}$ generated by $e[A]$. Since $e[A]$ is closed under joins in L, it follows that for every $u \in M$, there are $a_1, \ldots, a_n \in A$ such that $u = e(a_1) \wedge \cdots \wedge e(a_n)$. Let $h\colon A \to M$ be the map defined by $h(a) = e(a)$, for all $a \in A$. It is clear that h is an embedding. By (1), there is a unique lattice-embedding $\widehat{h}\colon L \to M$ with $h = \widehat{h} \circ e$. Let $u \in L$. Then, there are $a_1, \ldots, a_n \in A$ such that $\widehat{h}(u) = e(a_1) \wedge \cdots \wedge e(a_n)$. Notice that for each $a \in A$, $e(a) = h(a) = \widehat{h}(e(a))$. Then, it follows that

$$\widehat{h}(u) = \widehat{h}(e(a_1)) \wedge \cdots \wedge \widehat{h}(e(a_n)) = \widehat{h}(e(a_1) \wedge \cdots \wedge e(a_n)).$$

Since $\widehat{h}$ is an embedding, we obtain that $u = e(a_1) \wedge \cdots \wedge e(a_n)$. Hence, $e[A]$ is finitely meet-dense in L.

(2) $\Rightarrow$ (1) Let $\mathbf{M}$ be a distributive lattice and $h\colon A \to M$ an embedding. We define $\widehat{h}\colon L \to M$ as follows: for $u \in L$,

$$\widehat{h}(u) = h(a_1) \wedge \cdots \wedge h(a_n)$$

whenever $u = e(a_1) \wedge \cdots \wedge e(a_n)$ with $a_1, \ldots, a_n \in A$. We show that $\widehat{h}$ is well-defined. Let $u \in L$ be such that $u = e(a_1) \wedge \cdots \wedge e(a_n) = e(b_1) \wedge \cdots \wedge e(b_m)$. We have that $e(a_1) \wedge \cdots \wedge e(a_n) \leq e(b_j)$, for all $j \in \{1, \ldots, m\}$. Then

$$e(b_j) = (e(a_1) \vee e(b_j)) \wedge \cdots \wedge (e(a_n) \vee e(b_j)) = e((a_1 \vee b_j) \wedge \cdots \wedge (a_n \vee b_j)).$$

Since e is injective, we obtain that $b_j = (a_1 \vee b_j) \wedge \cdots \wedge (a_n \vee b_j)$, for all $j \in \{1, \ldots, m\}$. Thus, applying the embedding h, it follows that $h(a_1) \wedge \cdots \wedge h(a_n) \leq h(b_j)$, for all $j \in \{1, \ldots, m\}$. Then $h(a_1) \wedge \cdots \wedge h(a_n) \leq h(b_1) \wedge \cdots \wedge h(b_m)$. Similarly, we obtain that $h(b_1) \wedge \cdots \wedge h(b_m) \leq h(a_1) \wedge \cdots \wedge h(a_n)$. Hence, $h(a_1) \wedge \cdots \wedge h(a_n) = h(b_1) \wedge \cdots \wedge h(b_m)$. Therefore, h is well-defined. By a very similar argument, it can be proved that $\widehat{h}$ is injective. Moreover, it is straightforward to show that $\widehat{h}$ is a lattice-homomorphism, $h = \widehat{h} \circ e$ and it is the unique lattice-embedding with this property. □

Now we show that there exists a free distributive lattice extension for each DN-algebra (cf. Theorem 1.3 in [15] and Theorem 3.2 in [9]). Let $\mathbf{A}$ be a DN-algebra. Consider the set $\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ ordered by the set-theoretical inclusion. We define the map $\alpha_A\colon A \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}))$ as follows: for every $a \in A$

$$\alpha_A(a) = \{F \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) : a \in F\}.$$

It is clear that α_A is well-defined. When there is no danger of confusion, we omit the subscript the α_A.

Proposition 3.3. *Let* $\mathbf{A}$ *be a DN-algebra. Then, the map* $\alpha\colon A \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}))$ *is an embedding.*

Proof. It is straightforward by Corollary 2.7 and the definition of α itself. □

Let $\mathbf{A}$ be a DN-algebra. Let $L(\mathbf{A})$ be the sublattice of $\mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}))$ generated by $\alpha[A]$. From the above proposition, we have $\alpha[A]$ is closed under finite unions. Thus, it follows that

$$L(\mathbf{A}) = \{\alpha(a_1) \cap \cdots \cap \alpha(a_n) : n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A\}.$$

Therefore, the following proposition follows directly from the previous proposition and by the characterization given in Proposition 3.2.

Proposition 3.4. *For every DN-algebra* $\mathbf{A}$, $\langle L(\mathbf{A}), \alpha\rangle$ *is the free distributive lattice extension of* $\mathbf{A}$.

From now on, we will denote by $\langle L(\mathbf{A}), e_A\rangle$ the free distributive lattice extension of a DN-algebra $\mathbf{A}$.

Now let us establish a categorical relation between the variety of DN-algebras and the variety of distributive lattices. Let $\mathcal{DN}$ be the algebraic category of DN-algebras and homomorphisms and $\mathcal{DL}$ the algebraic category of distributive lattices and homomorphisms.

We define $\mathcal{U}\colon \mathcal{DL} \to \mathcal{DN}$ as follows. For every distributive lattice $\mathbf{L}$, let $\mathcal{U}(\mathbf{L}) = \langle L, m\rangle$, where $m(x,y,z) = (x \vee z) \wedge (y \vee z)$. We know that $\mathcal{U}(\mathbf{L})$ is a DN-algebra. Let $\mathbf{L}$ and $\mathbf{M}$ be distributive lattices. For every homomorphism $h\colon L \to M$, we define $\mathcal{U}(h) = h\colon \mathcal{U}(\mathbf{L}) \to \mathcal{U}(\mathbf{M})$. Then, $\mathcal{U}(h)$ is a homomorphism between DN-algebras. It is straightforward to show directly that $\mathcal{U}$ is a functor. Next we want a functor from $\mathcal{DN}$ to $\mathcal{DL}$. We need the following.

Proposition 3.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be DN-algebras. If* $h\colon A \to B$ *is a homomorphism, then there is a unique lattice-homomorphism* $L(h)\colon L(\mathbf{A}) \to L(\mathbf{B})$ *such that* $L(h) \circ e_A = e_B \circ h$. *Moreover, if* h *is injective, then so is* $L(h)$.

Proof. Let $h\colon A \to B$ be a homomorphism of DN-algebras. We define the map $L(h)\colon L(\mathbf{A}) \to L(\mathbf{B})$ as follows: for every $u \in L(\mathbf{A})$,

$$L(h)(u) = \bigwedge_{1\le i\le n} e_B(h(a_i))$$

whenever $u = e_A(a_1) \wedge \cdots \wedge e_A(a_n)$ for some $a_1, \ldots, a_n \in A$. First, we need to show that $L(h)$ is well-defined. Let $u \in L(\mathbf{A})$ and suppose that $u = \bigwedge_{1\le i\le n} e_A(a_i) = \bigwedge_{1\le j\le m} e_A(b_j)$, for some $a_i, b_j \in A$. Then, for every $j = 1, \ldots, m$, we have

$$\bigwedge_{1\le i\le n} e_A(a_i) \le e_A(b_j)$$

$$\left(\bigwedge_{1\le i\le n} e_A(a_i)\right) \vee e_A(b_j) = e_A(b_j)$$

$$\bigwedge_{1\le i\le n} e_A(a_i \vee b_j) = e_A(b_j)$$

$$e\left(\bigwedge_{1\le i\le n} (a_i \vee b_j)\right) = e_A(b_j).$$

Since e_A is an embedding, we obtain that $\bigwedge_{1\le i\le n} (a_i \vee b_j) = b_j$. Then,

$$\bigwedge_{1\le i\le n} (h(a_i) \vee h(b_j)) = h(b_j)$$

$$e_B\left(\bigwedge_{1\le i\le n} (h(a_i) \vee h(b_j))\right) = e_B(h(b_j))$$

$$\bigwedge_{1\leq i\leq n} (e_B(h(a_i)) \vee e_B(h(b_j))) = e_B(h(b_j))$$

$$\bigwedge_{1\leq i\leq n} e_B(h(a_i)) \leq e_B(h(b_j)).$$

Hence, $\bigwedge_{1\leq i\leq n} e_B(h(a_i)) \leq \bigwedge_{1\leq j\leq m} e_B(h(b_j))$. By analogous argument, we obtain the reverse inequality. Then, $\bigwedge_{1\leq i\leq n} e_B(h(a_i)) = \bigwedge_{1\leq j\leq m} e_B(h(b_j))$. Hence, $L(h)$ is well-defined. Now, it is easy to show that $L(h)$ is a lattice-homomorphism. Moreover, by definition of $L(h)$, it is clear that $L(h) \circ e_A = e_B \circ h$ and $L(h)$ is the unique lattice-homomorphism with this property. Now, if h is injective, then following a similar argumentation showing that $L(h)$ is well-defined it can be proved that $L(h)$ is also injective. □

Proposition 3.6. *Let $h\colon A \to B$ and $g\colon B \to C$ be homomorphisms between DN-algebras $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Then, $L(g \circ h) = L(g) \circ L(h)$.*

Proof. Let $u \in L(\mathbf{A})$. Let $a_1, \dots, a_n \in A$ be such that $u = e_A(a_1) \wedge \cdots \wedge e_A(a_n)$. Then

$$L(g \circ h)(u) = \bigwedge_{1\leq i\leq n} e_C((g \circ h)(a_i)) = \bigwedge_{1\leq i\leq n} e_C(g(h(a_i))) = L(g)(L(h)(u)).$$

□

We are ready to define $\mathcal{L}\colon \mathcal{DN} \to \mathcal{DL}$ as follows. For every DN-algebra $\mathbf{A}$, let $\mathcal{L}(\mathbf{A}) = L(\mathbf{A})$, and for every homomorphism of DN-algebras $h\colon A \to B$, let $\mathcal{L}(h) = L(h)$. By the above propositions, it follows that $\mathcal{L}$ is a functor.

Theorem 3.7. *The pair $\langle \mathcal{L}, \mathcal{U} \rangle$ is an adjoint of functors.*

Proof. Notice that for every DN-algebra $\mathbf{A}$, $(\mathcal{U} \circ \mathcal{L})(\mathbf{A}) = L(\mathbf{A})$. Then, for every DN-algebra $\mathbf{A}$, we have the morphism $e_A\colon A \to (\mathcal{U} \circ \mathcal{L})(\mathbf{A})$ of the category $\mathcal{DN}$. We need to show that for each DN-algebra $\mathbf{A}$ and each homomorphism $h\colon A \to \mathcal{U}(\mathbf{L})$, there exists a unique homomorphism $\widehat{h}\colon \mathcal{L}(\mathbf{A}) \to L$ such that $\mathcal{U}(\widehat{h}) \circ e_A = h$.

Let $\mathbf{A}$ be a DN-algebra and $h\colon A \to \mathcal{U}(\mathbf{L})$ a homomorphism. By Proposition 3.5, there exists a unique homomorphism $\widehat{h} := L(h)\colon \mathcal{L}(\mathbf{A}) \to \mathcal{L}(\mathcal{U}(\mathbf{L}))$ such that $\widehat{h} \circ e_A = e_{\mathcal{U}(\mathbf{L})} \circ h$. Notice that $\mathcal{L}(\mathcal{U}(\mathbf{L})) = \mathbf{L}$ and $e_{\mathcal{U}(\mathbf{L})} = \mathrm{id}_L$ is the identity map. Let $a \in A$. Then,

$$(\mathcal{U}(\widehat{h}) \circ e_A)(a) = (\widehat{h} \circ e_A)(a) = (e_{\mathcal{U}(\mathcal{L})} \circ h)(a) = h(a).$$

□

Now let us establish the relation between the filters of a DN-algebra $\mathbf{A}$ and the filters of the free distributive lattice extension $L(\mathbf{A})$.

Let $\mathbf{A}$ be a DN-algebra and $\langle L(\mathbf{A}), e \rangle$ its free distributive lattice extension. We define $\Psi\colon \mathrm{Fi}(L(\mathbf{A})) \to \mathrm{Fi}(\mathbf{A})$ and $\Phi\colon \mathrm{Fi}(\mathbf{A}) \to \mathrm{Fi}(L(\mathbf{A}))$ as follows:

$$\Psi(G) = e^{-1}[G] \quad \text{and} \quad \Phi(F) = \mathrm{Fig}_{L(\mathbf{A})}(e[F]) \tag{3.1}$$

for each $G \in \mathrm{Fi}(L(\mathbf{A})$ and $F \in \mathrm{Fi}(\mathbf{A})$, where $\mathrm{Fig}_{L(\mathbf{A})}(e[F])$ is the filter of $L(\mathbf{A})$ generated by $e[F]$. Since $e\colon A \to L(\mathbf{A})$ is an embedding, it follows that $e^{-1}[G] \in \mathrm{Fi}(\mathbf{A})$. Thus, Ψ is well-defined.

Proposition 3.8 [9]. *Let* $\mathbf{A}$ *be a DN-algebra and* $\langle L(\mathbf{A}), e\rangle$ *its free distributive lattice extension of* $\mathbf{A}$. *Then, the maps* $\Psi\colon \mathrm{Fi}(L(\mathbf{A})) \to \mathrm{Fi}(\mathbf{A})$ *and* $\Phi\colon \mathrm{Fi}(\mathbf{A}) \to \mathrm{Fi}(L(\mathbf{A}))$ *are isomorphisms, one inverse of the other. Moreover, the corresponding restriction* $\Psi\colon \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A})) \to \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ *and* $\Phi\colon \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) \to \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A}))$ *are order-isomorphisms.*

Proof. The proof that the maps $\Psi\colon \mathrm{Fi}(L(\mathbf{A})) \to \mathrm{Fi}(\mathbf{A})$ and $\Phi\colon \mathrm{Fi}(\mathbf{A}) \to \mathrm{Fi}(L(\mathbf{A}))$ are isomorphisms can be found in [9]. It is well-known that a proper filter F of a distributive lattice $\mathbf{L}$ is prime if and only if F is a meet-prime element of the lattice $\mathrm{Fi}(\mathbf{L})$. The same statement is true in the context of distributive nearlattices. Moreover, it is clear that if two lattices are isomorphic, then the ordered sets of meet-prime elements are order-isomorphic. Hence, we have that the corresponding restriction $\Psi\colon \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A})) \to \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ and $\Phi\colon \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) \to \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A}))$ are order-isomorphisms. □

Let $\langle L(\mathbf{A}), e\rangle$ be the free distributive lattice extension of a DN-algebra $\mathbf{A}$. Recall that the map $\varphi\colon L(\mathbf{A}) \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A})))$ is defined as follows

$$\varphi(u) = \{G \in \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A})) : u \in G\}$$

and it is a lattice-embedding. We have the following.

Lemma 3.9. *Let* $a \in A$ *and* $u \in L(\mathbf{A})$. *Let* $a_1, \ldots, a_n \in A$ *be such that* $u = e(a_1) \wedge \cdots \wedge e(a_n)$. *Then, we have:*

(1) $\Psi[\varphi(e(a))] = \alpha(a)$.
(2) $\Psi[\varphi(u)] = \alpha(a_1) \cap \cdots \cap \alpha(a_n)$.
(3) $\Psi[\varphi(u)^c] = \alpha(a_1)^c \cup \cdots \cup \alpha(a_n)^c$.

Here $\varphi(u)^c = \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A}))\backslash\varphi(u)$ *and* $\alpha(a)^c = \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})\backslash\alpha(a)$.

Proof. (1) Let $a \in A$. Then,

$$\Psi[\varphi(e(a))] = \{\Psi(G) : G \in \varphi(e(a))\} = \{e^{-1}[G] : a \in e^{-1}[G]\}.$$

It follows that $\Psi[\varphi(e(a))] \subseteq \alpha(a)$. Let $F \in \alpha(a)$. Since $F \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$, it follows that there is $G \in \mathrm{Fi}_{\mathrm{pr}}(L(\mathbf{A}))$ such that $F = \Psi(G) = e^{-1}[G]$. Thus $a \in e^{-1}[G]$. Then, $F = e^{-1}[G] \in \Psi[\varphi(e(a))]$. Hence $\alpha(a) \subseteq \Psi[\varphi(e(a))]$.

(2) and (3) are consequence of (1) using that φ is a lattice-embedding and Ψ is a bijection. □

4. Representation

Let X be a subset and $\mathcal{A}$ a collection of subsets of X. We define the binary relation $\leq_{\mathcal{A}}$ on X as follows:

$$x \leq_{\mathcal{A}} y \iff \forall U \in \mathcal{A}(x \in U \implies y \in U).$$

It is clear that $\leq_{\mathcal{A}}$ is a quasiorder (a reflexive and transitive relation) on X.

Definition 4.1. A tuple $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ is called a *Priestley DN-space* if:

(1) $\langle X, \tau\rangle$ is a compact space and $0, 1 \in X$.
(2) $\mathcal{A}$ is a collection of subsets of X such that $\mathcal{A} \cup \{U^c : U \in \mathcal{A}\}$ is a subbasis for τ.

(3) $0 \notin \bigcup \mathcal{A}$ and $1 \in \bigcap \mathcal{A}$.
(4) For all $U, V, W \in \mathcal{A}$, $(U \cup W) \cap (V \cup W) \in \mathcal{A}$.
(5) For each pair of distinct points $x, y \in X$, there exists $U \in \mathcal{A}$ containing exactly one of these points.

Let us point out several facts and properties about Priestley DN-spaces. By (S2), we can see that the members in $\mathcal{A}$ are clopen subsets of the space $\langle X, \tau \rangle$, and by (S3) we have that every $U \in \mathcal{A}$ is proper and nonempty. From condition (S5), it follows that the relation $\leq_{\mathcal{A}}$ is a partial order on X. Moreover, each $U \in \mathcal{A}$ is an upset of the poset $\langle X, \leq_{\mathcal{A}} \rangle$. Condition (S3) implies that 0 and 1 are, respectively, the least element and the greatest element of $\langle X, \leq_{\mathcal{A}} \rangle$. Condition (S4) is equivalent to the following two conditions: (i) if $U, V \in \mathcal{A}$, then $U \cup V \in \mathcal{A}$; (ii) for any $U, V, W \in \mathcal{A}$, if $W \subseteq U \cap V$, then $U \cap V \in \mathcal{A}$.

Proposition 4.2. *Let $\langle X, \tau, \mathcal{A}, 0, 1 \rangle$ by a Priestley DN-space. Then, the structure $\langle X, \tau, \leq_{\mathcal{A}}, 0, 1 \rangle$ is a bounded Priestley space.*

Proof. It only remains to prove that $\langle X, \tau, \leq_{\mathcal{A}} \rangle$ is totally order-disconnected. As we saw above, we know that $\mathcal{A} \subseteq \mathrm{ClUp}^*(X)$. Now, by definition of the order $\leq_{\mathcal{A}}$, we have that

$$x \nleq_{\mathcal{A}} y \implies \exists U \in \mathcal{A}(x \in U \text{ and } y \notin U).$$

Hence, $\langle X, \tau, \leq_{\mathcal{A}} \rangle$ is totally order-disconnected. □

Proposition 4.3. *Let $\langle X, \tau, \mathcal{A}, 0, 1 \rangle$ be a Priestley DN-space. Then,*

$$\mathrm{ClUp}^*(X) = \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N},\ U_1, \ldots, U_n \in \mathcal{A}\}.$$

Proof. Since $\mathcal{A} \subseteq \mathrm{ClUp}^*(X)$, it is clear that

$$\{U_1 \cap \cdots \cap U_n : n \in \mathbb{N},\ U_1, \ldots, U_n \in \mathcal{A}\} \subseteq \mathrm{ClUp}^*(X).$$

Let $U \in \mathrm{ClUp}^*(X)$. Let $y \notin U$. For every $x \in U$, we have $x \nleq_{\mathcal{A}} y$. Thus, for every $x \in U$, there is $U_x \in \mathcal{A}$ such that $x \in U_x$ and $y \notin U_x$. Then $U \subseteq \bigcup \{U_x : x \in U\}$. Since U is closed, it follows that U is compact. Thus, there are $U_{x_1}, \ldots, U_{x_n} \in \mathcal{A}$ such that $U \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. By (S4), $U_y := U_{x_1} \cup \cdots \cup U_{x_n} \in \mathcal{A}$ and $y \notin U_y$. Hence, we have proved that for every $y \notin U$, there is $U_y \in \mathcal{A}$ such that $U \subseteq U_y$ and $y \notin U_y$. Then, $U = \bigcap \{V \in \mathcal{A} : U \subseteq V\}$. Since U^c is also compact, it follows that $U = V_1 \cap \cdots \cap V_n$ for some $V_1, \ldots, V_n \in \mathcal{A}$. This completes the proof. □

Proposition 4.4. *Let $\langle X, \tau, \mathcal{A}, 0, 1 \rangle$ be a Priestley DN-space. Then $\langle \mathcal{A}, \widetilde{m} \rangle$ is a DN-algebra, where $\widetilde{m}$ is defined as follows: for all $U, V, W \in \mathcal{A}$,*

$$\widetilde{m}(U, V, W) = (U \cup W) \cap (V \cup W). \tag{4.1}$$

Proof. Notice that $\widetilde{m}(U, U, V) = U \cup V$, for all $U, V \in \mathcal{A}$. Moreover, from condition (S4) and the paragraph after Definition 4.1, it follows that $\langle \mathcal{A}, \cup \rangle$ is a distributive nearlattice. Hence, by Theorem 2.2, we obtain that $\langle \mathcal{A}, \widetilde{m} \rangle$ is a DN-algebra. □

Given a Priestley DN-space $\langle X, \tau, \mathcal{A}, 0, 1 \rangle$, we will say that $\langle \mathcal{A}, \widetilde{m} \rangle$ is the *dual DN-algebra of X*.

Theorem 4.5. *Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space. Then, the distributive lattice $\langle \mathrm{ClUp}^*(X), \cap, \cup\rangle$ is the free distributive lattice extension of the DN-algebra $\langle \mathcal{A}, \widetilde{m}\rangle$.*

Proof. We know that $\mathcal{A} \subseteq \mathrm{ClUp}^*(X)$. Then the identity map $\mathrm{id}\colon \mathcal{A} \to \mathrm{ClUp}^*(X)$ is an embedding. Now, by Proposition 4.3, $\mathcal{A}$ is finitely meet-dense in $\mathrm{ClUp}^*(X)$. Hence, from Proposition 3.2, it follows that $\mathrm{ClUp}^*(X)$ is the free distributive lattice extension of the DN-algebra $\mathcal{A}$. □

Let now $\mathbf{A} = \langle A, m\rangle$ be a DN-algebra. Recall that $\alpha\colon A \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}))$ is the map defined by $\alpha(a) = \{F \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) : a \in F\}$, for all $a \in A$. Let τ_A be the topology generated by the subbasis

$$\{\alpha(a) : a \in A\} \cup \{\alpha(b)^c : b \in A\}.$$

Proposition 4.6. *Let* $\mathbf{A}$ *be a DN-algebra. Then,*

$$\mathcal{X}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}}, \alpha[A], \emptyset, A\rangle$$

is a Priestley DN-space.

Proof. (S1) Let $A_0, B_0 \subseteq A$ be such that

$$\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) = \bigcup\{\alpha(a) : a \in A_0\} \cup \bigcup\{\alpha(b)^c : b \in B_0\}.$$

Let $F := \mathrm{Fig}_{\mathbf{A}}(B_0)$ and $I := \mathrm{Idg}_{\mathbf{A}}(A_0)$. Suppose that $F \cap I = \emptyset$. Thus, by Theorem 2.6, there is $P \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ such that $F \subseteq P$ and $P \cap I = \emptyset$. In particular, $B_0 \subseteq P$ and $P \cap A_0 = \emptyset$. Then $P \notin \bigcup\{\alpha(a) : a \in A_0\} \cup \bigcup\{\alpha(b)^c : b \in B_0\}$, which is a contradiction. Hence, $F \cap I \neq \emptyset$. Let $c \in F \cap I$. Then, there are $y_1, \ldots, y_n \in {\uparrow}B_0$ and $a_1, \ldots, a_m \in A_0$ such that $y_1 \wedge \cdots \wedge y_n = c \leq a_1 \vee \cdots \vee a_n$. Thus, since α is an embedding, we have $\alpha(y_1) \cap \cdots \cap \alpha(y_n) = \alpha(c) \subseteq \alpha(a_1) \cup \cdots \cup \alpha(a_m)$. Since $y_1, \ldots, y_n \in {\uparrow}B_0$, there are $b_1, \ldots, b_n \in B_0$ such that $b_i \leq y_i$ for all $i = 1, \ldots, n$. Hence

$$\alpha(b_1) \cap \cdots \cap \alpha(b_n) \subseteq \alpha(a_1) \cup \cdots \cup \alpha(a_m).$$

That is,

$$\emptyset = \alpha(b_1) \cap \cdots \cap \alpha(b_n) \cap \alpha(a_1)^c \cap \cdots \cap \alpha(a_m)^c.$$

Then,

$$\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) = \alpha(a_1) \cup \cdots \cup \alpha(a_n) \cup \alpha(b_1)^c \cup \cdots \cup \alpha(b_m)^c.$$

Therefore, $\langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$ is a compact space.

(S2) and (S3) are straightforward by definition.

(S4) Let $a, b, c \in A$. Since $\alpha\colon A \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}))$ is an embedding, it follows that

$$(\alpha(a) \cup \alpha(c)) \cap (\alpha(b) \cup \alpha(c)) = \alpha(m(a, b, c)) \in \alpha[A].$$

(S5) is straightforward. □

Given a DN-algebra $\mathbf{A}$, we say that $\mathcal{X}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}}, \alpha[A], \emptyset, A\rangle$ is the *dual Priestley DN-space* of $\mathbf{A}$. Notice that the partial order $\leq_{\alpha[A]}$ is actually the set-theoretical inclusion. Indeed, let $F, G \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A})$,

$$F \leq_{\alpha[A]} G \iff \forall a \in A(F \in \alpha(a) \implies G \in \alpha(a))$$
$$\iff \forall a \in A(a \in F \implies a \in G) \iff F \subseteq G.$$

Let $\mathbf{A} = \langle A, m\rangle$ be a DN-algebra. Since $\alpha\colon A \to \mathrm{Up}(\mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}))$ is an embedding, it follows that for all $a, b, c \in A$,

$$\alpha(m(a, b, c)) = (\alpha(a) \cup \alpha(c)) \cap (\alpha(b) \cup \alpha(c)) = \widetilde{m}(\alpha(a), \alpha(b), \alpha(c)).$$

Hence, we obtain that $\langle A, m\rangle \cong \langle \alpha[A], \widetilde{m}\rangle$, where $\widetilde{m}$ is defined by (4.1). Therefore, we have the following representation.

Corollary 4.7 (Representation). *Every DN-algebra* $\mathbf{A}$ *is isomorphic to the dual DN-algebra* $\mathcal{A}$ *of some Priestley DN-space* $\langle X, \tau, \mathcal{A}, 0, 1\rangle$.

Now, let us consider the opposite direction. Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space. Let $\mathcal{A}(X) = \langle \mathcal{A}, \widetilde{m}\rangle$ its dual DN-algebra. Hence, we can consider the dual Priestley DN-space of $\mathcal{A}(X)$:

$$\mathcal{X}(\mathcal{A}(X)) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X)), \tau_{\mathcal{A}(X)}, \alpha_{\mathcal{A}(X)}[\mathcal{A}], \emptyset, \mathcal{A}\rangle.$$

Let us define $\theta\colon X \to \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X))$ as follows: for every $x \in X$,

$$\theta(x) = \{u \in \mathcal{A} : x \in u\}.$$

It is easy to show that $\theta(x) \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X))$, for all $x \in X$. Thus, θ is well-defined.

Proposition 4.8. *Let* $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ *be a Priestley DN-space. Then the map* $\theta\colon X \to \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X))$ *is a homeomorphism from the space* $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ *onto the space* $\mathcal{X}(\mathcal{A}(X)) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X)), \tau_{\mathcal{A}(X)}, \alpha_{\mathcal{A}(X)}[\mathcal{A}], \emptyset, \mathcal{A}\rangle$. *Moreover,* $\{\theta[u] : u \in \mathcal{A}\} = \alpha_{\mathcal{A}(X)}[\mathcal{A}]$.

Proof. We prove this proposition in several steps.

• θ is injective. It is a direct consequence from condition (S5)

• θ is onto. Let $\mathcal{F} \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X))$. Since $\mathrm{ClUp}^*(X)$ is the free distributive lattice extension of the DN-algebra $\mathcal{A}(X)$, it follows by Proposition 3.8 that $\mathcal{G} = \mathrm{Fig}_{\mathrm{ClUp}^*(X)}(\mathcal{F})$ is a prime filter of the distributive lattice $\mathrm{ClUp}^*(X)$. Since $\mathrm{ClUp}^*(X)$ is the dual distributive lattice of the bounded Priestley space $\langle X, \tau, \leq_{\mathcal{A}}, 0, 1\rangle$, it follows by Proposition 2.12 that there is $x \in X$ such that $\mathcal{G} = \{U \in \mathrm{ClUp}^*(X) : x \in U\}$. Then, it is clear that $\mathcal{G} \cap \mathcal{A} = \theta(x)$. Now we show that $\mathcal{F} = \mathcal{G} \cap \mathcal{A}$. It straightforward that $\mathcal{F} \subseteq \mathcal{G} \cap \mathcal{A}$. Let $W \in \mathcal{G} \cap \mathcal{A}$. Thus, there are $U_1, \ldots, U_n \in \mathcal{F}$ such that $U_1 \cap \cdots \cap U_n \subseteq W$. Then $W = (U_1 \cup W) \cap \cdots \cap (U_n \cup W)$. Notice that each $U_i \cup W$ is in $\mathcal{F}$. Hence $W \in \mathcal{F}$. Thus, we have $\mathcal{G} \cap \mathcal{A} \subseteq \mathcal{F}$. Hence $\mathcal{F} = \theta(x)$.

• θ is continuous. Notice that the corresponding subbasic opens of the space $\mathcal{X}(\mathcal{A}(X))$ are the form:

$$\alpha_{\mathcal{A}(X)}(u) = \{\mathcal{F} \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X)) : u \in \mathcal{F}\}$$

and

$$\alpha_{\mathcal{A}(X)}(v)^c = \{\mathcal{F} \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{A}(X)) : v \notin \mathcal{F}\}$$

with $u, v \in \mathcal{A}$. It is enough to show that for each $u, v \in \mathcal{A}$, $\theta^{-1}[\alpha_{\mathcal{A}(X)}(u)]$ and $\theta^{-1}[\alpha_{\mathcal{A}(X)}(v)^c]$ are opens of X. Let $u, v \in \mathcal{A}$ and $x \in X$. Then

$$x \in \theta^{-1}[\alpha_{\mathcal{A}(X)}(u)] \iff \theta(x) \in \alpha_{\mathcal{A}(X)}(u) \iff u \in \theta(x) \iff x \in u.$$

Hence $\theta^{-1}[\alpha_{\mathcal{A}(X)}(u)] = u$. By a similar argument, $\theta^{-1}[\alpha_{\mathcal{A}(X)}(v)^c] = v^c$. Therefore, we have proved that θ is a homeomorphism.

• From the previous point, it follows that $\{\theta[u] : u \in \mathcal{A}\} = \{\alpha_{\mathcal{A}}(u) : u \in \mathcal{A}\} = \alpha[\mathcal{A}]$. □

5. Duality for the category of DN-algebras

Recall that $\mathcal{DN}$ is the algebraic category of DN-algebras and homomorphisms. In the previous section, we show that the class of DN-algebras is categorically equivalent (at object level) to the class of Priestley DN-spaces. In order to extend this equivalence to a full dual categorical equivalence between the category of DN-algebras and a certain category of Priestley DN-space, we need to introduce the corresponding morphisms between Priestley DN-spaces.

Definition 5.1. Let $X = \langle X, \tau, \mathcal{A}, 0_X, 1_X \rangle$ and $Y = \langle Y, \eta, \mathcal{B}, 0_Y, 1_Y \rangle$ be Priestley DN-spaces. A map $f \colon X \to Y$ is said to be a *Priestley DN-morphism* from X to Y if $f^{-1}[V] \in \mathcal{A}$, for all $V \in \mathcal{B}$.

Let $f \colon X \to Y$ be a Priestley DN-morphism. By (S2), it follows that f is continuous. From conditions (S3) and (S5), we obtain that $f(0_X) = 0_Y$ and $f(1_X) = 1_Y$. Moreover, from the definition of Priestley DN-morphism, it follows that $x_1 \leq_{\mathcal{A}} x_2 \implies f(x_1) \leq_{\mathcal{B}} f(x_2)$, for all $x_1, x_2 \in X$. That is, f is order-preserving.

Notice that the usual composition of two Priestley DN-morphisms is a Priestley DN-morphism. Hence, we can define the category $\mathcal{PDNS}$ of Priestley DN-spaces and Priestley DN-morphisms.

Proposition 5.2. (1) *Let $X = \langle X, \tau, \mathcal{A}, 0_X, 1_X \rangle$ and $Y = \langle Y, \eta, \mathcal{B}, 0_Y, 1_Y \rangle$ be Priestley DN-spaces. If $f \colon X \to Y$ is a Priestley DN-morphism, then $f^{-1} \colon \mathcal{B} \to \mathcal{A}$ is a homomorphism of DN-algebras from $\langle \mathcal{A}, \widetilde{m} \rangle$ to $\langle \mathcal{B}, \widetilde{m} \rangle$.*

(2) *Let $\mathbf{A}$ and $\mathbf{B}$ be DN-algebras. If $h \colon A \to B$ is a homomorphism, then $h^{-1} \colon \mathcal{X}(\mathbf{B}) \to \mathcal{X}(\mathbf{A})$ is a Priestley DN-morphism.*

Proof. (1) Let $v_1, v_2, v_3 \in \mathcal{B}$. Then,

$$\begin{aligned} f^{-1}(\widetilde{m}(v_1, v_2, v_3)) &= f^{-1}((v_1 \cup v_3) \cap (v_2 \cup v_3)) \\ &= (f^{-1}(v_1) \cup f^{-1}(v_3)) \cap (f^{-1}(v_2) \cup f^{-1}(v_3)) \\ &= \widetilde{m}(f^{-1}(v_1), f^{-1}(v_2), f^{-1}(v_3)). \end{aligned}$$

(2) First, notice that $h^{-1} \colon \mathrm{Fi}_{\mathrm{pr}}(\mathbf{B}) \to \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ is a well-defined function because $h \colon A \to B$ is a homomorphism. Now, let $a \in A$. We need to show that $(h^{-1})^{-1}[\alpha_{\mathbf{A}}(a)] \in \alpha_{\mathbf{B}}[B]$. Let $G \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{B})$. Then,

$$\begin{aligned} G \in (h^{-1})^{-1}[\alpha_{\mathbf{A}}(a)] &\iff h^{-1}(G) \in \alpha_{\mathbf{A}}(a) \iff a \in h^{-1}(G) \\ &\iff h(a) \in G \iff G \in \alpha_{\mathbf{B}}(h(a)). \end{aligned}$$

Hence $(h^{-1})^{-1}[\alpha_{\mathbf{A}}(a)] = \alpha_{\mathbf{B}}(h(a)) \in \alpha_{\mathbf{B}}[B]$. Therefore, $h^{-1}\colon \mathcal{X}(\mathbf{B}) \to \mathcal{X}(\mathbf{A})$ is a Priestley DN-morphism. $\square$

Now we can define the corresponding functors. Let

(1) $\mathcal{X}\colon \mathcal{DN} \to \mathcal{PDNS}$ be defined as follows:
- $\mathcal{X}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}}, \alpha[A], \emptyset, A\rangle$, for each DN-algebra $\mathbf{A}$;
- $\mathcal{X}(h)\colon \mathcal{X}(\mathbf{B}) \to \mathcal{X}(\mathbf{A})$ is given by $\mathcal{X}(h) = h^{-1}$, for each homomorphism $h\colon A \to B$ of DN-algebras.

(2) $\mathcal{A}\colon \mathcal{PDNS} \to \mathcal{DN}$ be defined as follows:
- $\mathcal{A}(X) = \langle \mathcal{A}, \widetilde{m}\rangle$, for each Priestley DN-space $X = \langle X, \tau, \mathcal{A}, 0, 1\rangle$;
- $\mathcal{A}(f)\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ is given by $\mathcal{A}(f) = f^{-1}$, for each Priestley DN-morphism $f\colon X \to Y$ between Priestley DN-spaces.

Proposition 5.3. *Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between DN-algebras and let $f\colon X \to Y$ be a Priestley DN-morphism between Priestley DN-spaces. Then, we have $\mathcal{A}(\mathcal{X}(h)) \circ \alpha_{\mathbf{A}} = \alpha_{\mathbf{B}} \circ h$ and $\mathcal{X}(\mathcal{A}(f)) \circ \theta_X = \theta_Y \circ f$.*

Proof. Let $a \in A$ and $G \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{B})$. Then,

$$\begin{aligned} G \in \mathcal{A}(\mathcal{X}(h))(\alpha_{\mathbf{A}}(a)) &\iff G \in \mathcal{X}(h)^{-1}(\alpha_{\mathbf{A}}(a)) \iff \mathcal{X}(h)(G) \in \alpha_{\mathbf{A}}(a) \\ &\iff a \in h^{-1}[G] \iff h(a) \in G \iff G \in \alpha_{\mathbf{B}}(h(a)). \end{aligned}$$

Hence, $\mathcal{A}(\mathcal{X}(h))(\alpha_{\mathbf{A}}(a))) = \alpha_{\mathbf{B}}(h(a))$. Let now $x \in X$ and $V \in \mathcal{B}$. Then,

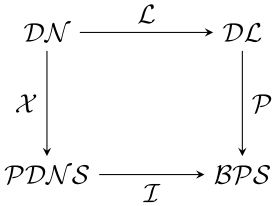
$$\begin{aligned} V \in \mathcal{X}(\mathcal{A}(f))(\theta_X(x)) &\iff V \in \mathcal{A}(f)^{-1}(\theta_X(x)) \iff f^{-1}(V) \in \theta_X(x) \\ &\iff f(x) \in V \iff V \in \theta_Y(f(x)). \end{aligned}$$

Hence, $\mathcal{X}(\mathcal{A}(f))(\theta_X(x)) = \theta_Y(f(x))$. $\square$

Now we are ready to establish the main result. The proof of this theorem is a matter of putting together what we have developed and proved so far, and thus we leave the details to the reader.

Theorem 5.4. *The functors $\mathcal{X}\colon \mathcal{DN} \to \mathcal{PDNS}$ and $\mathcal{A}\colon \mathcal{PDNS} \to \mathcal{DN}$ are dual equivalences of categories, and hence the categories $\mathcal{DN}$ and $\mathcal{PDNS}$ are dually equivalent.*

We end this section by establishing a connection between the categories mentioned so far. First, let us define $\mathcal{I}\colon \mathcal{PDNS} \to \mathcal{BPS}$ as follows. For every Priestley DN-space $X = \langle X, \tau, \mathcal{A}, 0, 1\rangle$, $\mathcal{I}(X) = \langle X, \tau, \leq_{\mathcal{A}}, 0, 1\rangle$. For every Priestley DN-morphism $f\colon X \to Y$, let $\mathcal{I}(f) = f$. By Proposition 4.2 and from Definition 5.1, it follows that $\mathcal{I}$ is a functor. Now we consider the following diagram.



We will show that the above diagram commutes. In order to accomplish this, notice first that for all $\mathbf{A} \in \mathcal{DN}$,

$$(\mathcal{P} \circ \mathcal{L})(\mathbf{A}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathcal{L}(\mathbf{A})), \tau_{\mathcal{L}(\mathbf{A})}, \subseteq, \emptyset, \mathcal{L}(\mathbf{A}) \rangle$$

and

$$(\mathcal{I} \circ \mathcal{X})(\mathbf{A}) = \langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \emptyset, A \rangle.$$

Now we define $\Psi\colon (\mathcal{P} \circ \mathcal{L}) \to (\mathcal{I} \circ \mathcal{X})$ as follows: for every $\mathbf{A} \in \mathcal{DN}$,

$$\Psi_{\mathbf{A}}\colon (\mathcal{P} \circ \mathcal{L})(\mathbf{A}) \to (\mathcal{I} \circ \mathcal{X})(\mathbf{A})$$

is defined by

$$\Psi_{\mathbf{A}}(G) = e_A^{-1}[G],$$

for each $G \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{L}(\mathbf{A}))$.

Proposition 5.5. $\Psi\colon (\mathcal{P} \circ \mathcal{L}) \to (\mathcal{I} \circ \mathcal{X})$ *is a natural isomorphism.*

Proof. We need to prove that: (1) $\Psi_{\mathbf{A}}$ is an isomorphism of the category $\mathcal{BPS}$, for all $\mathbf{A} \in \mathcal{DN}$; and (2) for each Priestley DN-morphism $h\colon \mathbf{A} \to \mathbf{B}$ of the category $\mathcal{DN}$, the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{P} \circ \mathcal{L})(\mathbf{A}) & \xrightarrow{\Psi_{\mathbf{A}}} & (\mathcal{I} \circ \mathcal{X})(\mathbf{A}) \\
{\scriptstyle (\mathcal{P} \circ \mathcal{L})(h)} \uparrow & & \uparrow {\scriptstyle (\mathcal{I} \circ \mathcal{X})(h)} \\
(\mathcal{P} \circ \mathcal{L})(\mathbf{B}) & \xrightarrow[\Psi_{\mathbf{B}}]{} & (\mathcal{I} \circ \mathcal{X})(\mathbf{B})
\end{array}$$

(1) Let $\mathbf{A} \in \mathcal{DN}$. By Proposition 3.8, the map $\Psi_{\mathbf{A}}\colon \mathrm{Fi}_{\mathrm{pr}}(\mathcal{L}(\mathbf{A})) \to \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ is an order-isomorphism. By Lemma 3.9, it follows that $\Psi_{\mathbf{A}}$ is a continuous map from the space $\langle \mathrm{Fi}_{\mathrm{pr}}(\mathcal{L}(\mathbf{A})), \tau_{\mathcal{L}(\mathbf{A})} \rangle$ onto the space $\langle \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}), \tau_{\mathbf{A}} \rangle$. Hence, $\Psi_{\mathbf{A}}$ is an isomorphism of the category $\mathcal{BPS}$ from space $(\mathcal{P} \circ \mathcal{L})(\mathbf{A})$ onto the space $(\mathcal{I} \circ \mathcal{X})(\mathbf{A})$.

(2) Let $h\colon \mathbf{A} \to \mathbf{B}$ be Priestley DN-morphism. Let $H \in \mathrm{Fi}_{\mathrm{pr}}(\mathcal{L}(\mathbf{B}))$ and $a \in A$. Then, by Proposition 3.5, we obtain that

$$\begin{aligned}
a \in (\Psi_{\mathbf{A}} \circ (\mathcal{P} \circ \mathcal{L})(h))\,(H) &\iff a \in \Psi_{\mathbf{A}}\,(\mathcal{P}(\mathcal{L}(h))(H)) \\
&\iff a \in \Psi_{\mathbf{A}}\,\big(\mathcal{L}(h)^{-1}[H]\big) \\
&\iff a \in e_{\mathbf{A}}^{-1}\,\big[\mathcal{L}(h)^{-1}[H]\big] \\
&\iff \mathcal{L}(h)\,(e_{\mathbf{A}}(a)) \in H \\
&\iff e_{\mathbf{B}}(h(a)) \in H \\
&\iff a \in h^{-1}\,\big[e_{\mathbf{B}}^{-1}[H]\big] \\
&\iff a \in \mathcal{X}(h)\,\big(e_{\mathbf{B}}^{-1}[H]\big) \\
&\iff a \in \mathcal{I}(\mathcal{X}(h))\,\big(e_{\mathbf{B}}^{-1}[H]\big) \\
&\iff a \in (\mathcal{I} \circ \mathcal{X})(h)(\Psi_{\mathbf{B}}(H))
\end{aligned}$$

$$\iff a \in ((\mathcal{I} \circ \mathcal{X})(h) \circ \Psi_{\mathbf{B}})(H).$$

Hence, $\Psi_{\mathbf{A}} \circ (\mathcal{P} \circ \mathcal{L})(h) = (\mathcal{I} \circ \mathcal{X})(h) \circ \Psi_{\mathbf{B}}$. □

6. Dual descriptions

6.1. Dual description for filters

Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space. Let $\mathrm{CUp}^*(X)$ be the lattice of all nonempty closed upsets of X. Let $\mathbf{A}$ be a DN-algebra and $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ its dual Priestley DN-space. We define the map $C\colon \mathrm{Fi}(\mathbf{A}) \to \mathrm{CUp}^*(X)$ as follows:

$$C(F) = \bigcap \{\alpha(a) : a \in F\},$$

for every $F \in \mathrm{Fi}(\mathbf{A})$. Notice that $C(F) = \{P \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}) : F \subseteq P\}$.

Proposition 6.1. *The lattice* $\mathrm{Fi}(\mathbf{A})$ (*ordered by* $\supseteq$) *of* $\mathbf{A}$ *is isomorphic to the lattice* $\mathrm{CUp}^*(X)$.

Proof. We will show that the map $C\colon \mathrm{Fi}(\mathbf{A}) \to \mathrm{CUp}^*(X)$ is an order-isomorphism. It is clear that for all $F, G \in \mathrm{Fi}(\mathbf{A})$, $F \subseteq G$ implies that $C(G) \subseteq C(F)$. Now let $F, G \in \mathrm{Fi}(\mathbf{A})$ and assume that $C(G) \subseteq C(F)$. Suppose by contradiction that $F \nsubseteq G$. Then, there is $a \in F$ and $a \notin G$. By Theorem 2.6, there exists $P \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$ such that $G \subseteq P$ and $a \notin P$. Since $a \in F$ and $P \notin \alpha(a)$, it follows that $P \notin C(F)$. On the other hand, $G \subseteq P$ implies that $P \in C(G)$ and thus, $P \in C(F)$. A contradiction. Hence $F \subseteq G$. We have proved that C is an order-embedding. Now we show that C is onto. Let $\mathcal{F} \in \mathrm{CUp}^*(X)$. Let $F := \{a \in A : \mathcal{F} \subseteq \alpha(a)\}$. It is straightforward that F is a filter of $\mathbf{A}$. Let $P \in C(F)$. Thus $F \subseteq P$. Suppose that $P \notin \mathcal{F}$. Since $\mathcal{F}$ is an upset, it follows that for every $Q \in \mathcal{F}$, $Q \nsubseteq P$. Then, for every $Q \in \mathcal{F}$, there is $a_Q \in A$ such that $a_Q \in Q \backslash P$. It follows that $\mathcal{F} \subseteq \bigcup \{\alpha(a_Q) : Q \in \mathcal{F}\}$. Since $\mathcal{F}$ is compact, there are $a_{Q_1}, \dots, a_{Q_n} \in A$ with $Q_1, \dots, Q_n \in \mathcal{F}$ such that $\mathcal{F} \subseteq \alpha(a_{Q_1}) \cup \dots \cup \alpha(a_{Q_n})$ and $a_{Q_1}, \dots, a_{Q_n} \notin P$. Let $a = a_{Q_1} \vee \dots \vee a_{Q_n}$. Then, $a \notin P$ and $\mathcal{F} \subseteq \alpha(a)$. Thus, $a \notin P$ implies that $a \notin F$, and $\mathcal{F} \subseteq \alpha(a)$ implies $a \in F$. A contradiction. Hence, we obtain that $C(F) \subseteq \mathcal{F}$. Now let $P \in \mathcal{F}$. Let $a \in F$. By definition of F, $\mathcal{F} \subseteq \alpha(a)$. Then $P \in \alpha(a)$. It follows that, $P \in \bigcap \{\alpha(a) : a \in F\} = C(F)$. Hence $\mathcal{F} \subseteq C(F)$. Therefore, C is onto. This completes the proof. □

6.2. Dual description for congruences

Let $\mathbf{A} = \langle A, m\rangle$ be a DN-algebra and $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ its dual Priestley DN-space. Let $\mathrm{Con}(\mathbf{A})$ be the lattice of congruences of $\mathbf{A}$ and $\mathrm{C}(X)$ the lattice of closed subsets of X. Our aim here is to show that the lattices $\mathrm{Con}(\mathbf{A})$ and $\mathrm{C}(X)$ are dually isomorphic.

It is known that a relation $\theta \subseteq A \times A$ is a congruence of $\mathbf{A}$ if and only if it is a congruence with respect to $\vee$, and for all $a_1, a_2, b_1, b_2 \in A$, if $a_1 \wedge b_1$ and $a_2 \wedge b_2$ exist in A and $(a_1, a_2), (b_1, b_2) \in \theta$, then $(a_1 \wedge b_1, a_2 \wedge b_2) \in \theta$.

For every subset $Y \subseteq X$, we define the relation $\theta_Y \subseteq A \times A$ as follows:

$$\theta_Y = \{(a, b) \in A \times A : \forall P \in Y (a \in P \iff b \in P)\}.$$

Notice that $\theta_Y = \{(a,b) \in A \times A : \alpha(a) \cap Y = \alpha(b) \cap Y\}$. It is straightforward to show that $\theta_Y \in \mathrm{Con}(\mathbf{A})$. We need the following lemma. For every $Y \subseteq X$, let $\mathrm{cl}_X(Y)$ be the topological closure of Y in the space X.

Lemma 6.2. *Let $Y, Z \subseteq X$. Then, $\theta_Z \subseteq \theta_Y$ if and only if $Y \subseteq \mathrm{cl}_X(Z)$.*

Proof. ($\Rightarrow$) Assume that $\theta_Z \subseteq \theta_Y$. Suppose that $Y \nsubseteq \mathrm{cl}_X(Z)$. So, there is $P \in Y \backslash \mathrm{cl}_X(Z)$. Then, $P \in \mathrm{cl}_X(Z)^c$. Thus, there are $a_1, \dots, a_n, b \in A$ such that $P \in \alpha(a_1) \cap \dots \alpha(a_n) \cap \alpha(b)^c \subseteq \mathrm{cl}_X(Z)^c$. Thus

$$\alpha(a_1) \cap \dots \cap \alpha(a_n) \cap \alpha(b)^c \cap Z = \emptyset. \tag{6.1}$$

Let us show that $((a_1 \vee b) \wedge \dots \wedge (a_n \vee b), b) \in \theta_Z$. Let $Q \in Z$. If $(a_1 \vee b) \wedge \dots \wedge (a_n \vee b) \in Q$, then $a_i \vee b \in Q$, for all $i = 1, \dots, n$. Since Q is a prime filter and by (6.1), it follows that $b \in Q$. On the other hand, if $b \in Q$, then $a_i \vee b \in Q$, for all $i = 1, \dots, n$. Then $(a_1 \vee b) \wedge \dots \wedge (a_n \vee b) \in Q$. Hence, $((a_1 \vee b) \wedge \dots \wedge (a_b \vee b), b) \in \theta_Z$. Now, since $P \in \alpha(a_1) \cap \dots \cap \alpha(a_n) \cap \alpha(b)^c$, we obtain that $a_1, \dots, a_n \in P$ and $b \notin P$. Thus, $a_i \vee b \in P$, for all $i = 1, \dots, n$. Then, we have that $(a_1 \vee b) \wedge \dots \wedge (a_n \vee b) \in P$ and $b \notin P$. Hence $((a_1 \vee b) \wedge \dots \wedge (a_n \vee b), b) \notin \theta_Y$. A contradiction. Therefore, $Y \subseteq \mathrm{cl}_X(Z)$.

($\Leftarrow$) Assume that $Y \subseteq \mathrm{cl}_X(Z)$. Let $(a,b) \notin \theta_Y$. So, there is $P \in Y$ such that $a \in P$ and $b \notin P$ (or $a \notin P$ and $b \in P$). Then $\alpha(a) \cap \alpha(b)^c \cap \mathrm{cl}_X(Z) \neq \emptyset$. Thus, $\alpha(a) \cap \alpha(b)^c \cap Z \neq \emptyset$. Let $Q \in \alpha(a) \cap \alpha(b)^c \cap Z$. Then, we have that $Q \in Z$, $a \in Q$ and $b \notin Q$. Hence $(a,b) \notin \theta_Z$. Therefore, $\theta_Z \subseteq \theta_Y$. □

Proposition 6.3. *The lattices $\mathrm{Con}(\mathbf{A})$ and $\mathrm{C}(X)$ are dually isomorphic.*

Proof. We define $R\colon \mathrm{C}(X) \to \mathrm{Con}(\mathbf{A})$ as follows: for every $Y \in \mathrm{C}(X)$, $R(Y) = \theta_Y$. Let $Y, Z \in \mathrm{C}(X)$. By Lemma 6.2, it follows that $Y \subseteq Z$ if and only if $\theta_Z \subseteq \theta_Y$. Then, R is a dual order-embedding. Now we show that R is onto. Let $\theta \in \mathrm{Con}(\mathbf{A})$. Consider the quotient DN-algebra $\mathbf{A}/\theta$ and the natural projection $\pi\colon A \to A/\theta$. Let $\mathcal{X}(\mathbf{A}/\theta)$ be the dual Priestley DN-space of the DN-algebra $\mathbf{A}/\theta$. We have that the map $\mathcal{X}(\pi) = \pi^{-1}\colon \mathcal{X}(\mathbf{A}/\theta) \to X$ is a Priestley DN-morphism. In particular, $\mathcal{X}(\pi)$ is continuous. Since the spaces X and $\mathcal{X}(\mathbf{A}/\theta)$ are Hausdorff and compact, it follows that $Y := \mathcal{X}(\pi)[\mathcal{X}(\mathbf{A}/\theta)] \in \mathrm{C}(X)$. Notice that

$$Y = \{\pi^{-1}[G] : G \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}/\theta)\}.$$

Now let us show that $\theta = \theta_Y$. Let $(a,b) \in \theta$ and $Q \in Y$. Thus, there is $G \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}/\theta)$ such that $Q = \pi^{-1}[G]$. Then, $a \in Q \iff \pi(a) \in G \iff \pi(b) \in G \iff b \in Q$. Hence $(a,b) \in \theta_Y$. Let $(a,b) \notin \theta$. Without loss of generality we can assume that $\pi(a) \nleq_{\mathbf{A}/\theta} \pi(b)$. Thus, by Corollary 2.7, there is $G \in \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A}/\theta)$ such that $\pi(a) \in G$ and $\pi(b) \notin G$. Then $a \in \pi^{-1}[G]$, $b \notin \pi^{-1}[G]$ and $\pi^{-1}[G] \in Y$. It follows that $(a,b) \notin \theta_Y$. Hence, we have proved that $\theta = \theta_Y$. Therefore, R is onto. □

6.3. Dual description for subalgebras

The aim here is to obtain a topological description of certain kinds of subalgebras of a DN-algebra. This topological description is a generalization of what happens in the context of distributive lattices, where there is a correspondence between sublattices of a distributive lattice and the Priestley quotients of the dual Priestley space of the lattice, see [31, 20].

We start with one of the main notions of this subsection.

Definition 6.4. Let $\mathbf{A} = \langle A, m\rangle$ be a DN-algebra. We will said that $B \subseteq A$ is a *strong-subalgebra* of $\mathbf{A}$ if for all $b_1, b_2 \in B$,

(1) $b_1 \vee b_2 \in B$;
(2) If $b_1 \wedge b_2$ exists in A, then $b_1 \wedge b_2 \in B$.

It is straightforward to show that every strong-subalgebra of a DN-algebra $\mathbf{A}$ is in fact a subalgebra of $\mathbf{A}$. But the converse is not true.

From now on, let $\mathbf{A} = \langle A, m\rangle$ be a DN-algebra and $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ its dual Priestley DN-space. Recall that $X = \mathrm{Fi}_{\mathrm{pr}}(\mathbf{A})$, $\tau = \tau_{\mathbf{A}}$, $\mathcal{A} = \alpha[A]$, and for each $a \in A$, $\alpha(a) = \{x \in X : a \in x\}$. Moreover, $\leq_{\mathcal{A}} = \subseteq$.

Let $S \subseteq A$. We define the binary relation $\preceq_S \subseteq X \times X$ as follows: for all $x, y \in X$,

$$x \preceq_S y \iff \forall a \in S(a \in x \implies a \in y).$$

Notice that

$$x \preceq_S y \iff S \cap x \subseteq S \cap y \iff \forall a \in S(x \in \alpha(a) \implies y \in \alpha(a)).$$

It is straightforward to show that $\preceq_S$ is a quasiorder on X. Moreover, it is important to note that $\leq_{\mathcal{A}} \subseteq \preceq_S$.

Now we introduce the second main notion.

Definition 6.5. Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space. A quasiorder $\preceq$ on X is said to be $\mathcal{A}$-*compatible* when for all $x, y \in X$, if $x \npreceq y$, then there is a $\preceq$-upset $U \in \mathcal{A}$ such that $x \in U$ and $y \notin U$.

Proposition 6.6. *For each $S \subseteq A$, the quasiorder $\preceq_S$ is $\mathcal{A}$-compatible.*

Proof. Let $x, y \in X$ and suppose that $x \npreceq_S y$. By definition of $\preceq_S$, there is $a \in S$ such that $a \in x$ and $a \notin y$. Thus $x \in \alpha(a)$ and $y \notin \alpha(a)$. We know that $\alpha(a) \in \mathcal{A}$. Let us see that $\alpha(a)$ is $\preceq_S$-upset. Let $z, w \in X$ be such that $z \preceq_S w$ and $z \in \alpha(a)$. Then, $S \cap z \subseteq S \cap w$ and $a \in z$. Thus $a \in w$. That is $w \in \alpha(a)$. Hence $\alpha(a) \in \mathcal{A}$ is $\preceq_S$-upset. □

It is easy to check that for all $S, T \subseteq A$, $S \subseteq T$ implies that $\preceq_T \subseteq \preceq_S$.

Now let $R \subseteq X \times X$. We define the following subset of A:

$$B_R = \{a \in A : \forall (x, y) \in R(a \in x \implies a \in y)\}.$$

Proposition 6.7. *For each $R \subseteq X \times X$, B_R is a strong-subalgebra of* $\mathbf{A}$.

Proof. Let $a, b \in B_R$.

• Let $(x, y) \in R$ and suppose that $a \vee b \in x$. Since x is a prime filter of $\mathbf{A}$, it follows that $a \in x$ or $b \in x$. Suppose that $a \in x$ (analogously if $b \in x$). Since $a \in B_R$ and $(x, y) \in R$, we have that $a \in y$. Then $a \vee b \in y$. Hence $a \vee b \in B_R$.

• Suppose $a \wedge b$ exists in A. Let $(x, y) \in R$ and suppose that $a \wedge b \in x$. Since x is a filter of A, we obtain that $a, b \in x$. Given that $a \in B_R$ and $(x, y) \in R$, we have $a \in y$. Similarly, $b \in y$. Then $a \wedge b \in y$ because y is a filter. Hence $a \wedge b \in B_R$. Therefore, B_R is a strong-subalgebra of $\mathbf{A}$. □

It is clear that for all $R_1, R_2 \subseteq X \times X$, $R_1 \subseteq R_2$ implies that $B_{R_1} \subseteq B_{R_2}$.

Proposition 6.8. *If B is a strong-subalgebra of $\mathbf{A}$, then $B_{\preceq_B} = B$.*

Proof. Let $b \in B$. Let $x, y \in X$ be such that $x \preceq_B y$. Suppose $b \in x$. By definition of $\preceq_B$, we have that $b \in y$. Then $b \in B_{\preceq_B}$. Hence $B \subseteq B_{\preceq_B}$. Let now $a \in B_{\preceq_B}$. Let $y \in X$ be such that $a \notin y$. For each $x \in X$ such that $a \in x$, we have that $x \not\preceq_B y$. Thus, by definition of $\preceq_B$, there is $b_x \in B$ such that $b_x \in x$ and $b_x \notin y$. Then,

$$\alpha(a) \subseteq \bigcup \{\alpha(b_x) : x \in X \text{ and } a \in x\}.$$

By compactness of $\alpha(a)$, we obtain that $\alpha(a) \subseteq \alpha(b_{x_1}) \cup \cdots \cup \alpha(b_{x_n})$ with $b_{x_1}, \ldots, b_{x_n} \in B$ and $b_{x_1}, \ldots, b_{x_n} \notin y$. Let $b_y := b_{x_1} \vee \cdots \vee b_{x_n}$. Since B is a strong-subalgebra, it follows that $b_y \in B$. Moreover $\alpha(a) \subseteq \alpha(b_y)$ and $b_y \notin y$. We have proved that for every $y \in X$ such that $a \notin y$, there is $b_y \in B$ such that $b_y \notin y$ and $\alpha(a) \subseteq \alpha(b_y)$ (which implies that $a \leq b_y$). Then,

$$\alpha(a) = \bigcap \{\alpha(b_y) : y \in X \text{ and } a \notin y\}.$$

By compactness of $\alpha(a)^c$, $\alpha(a) = \alpha(b_{y_1}) \cap \cdots \cap \alpha(b_{y_m})$. Since $a \leq b_{y_1}, \ldots, b_{y_m}$, it follows that $b_{y_1} \wedge \cdots \wedge b_{y_m}$ exists in A. Then, given that B is a strong-subalgebra, $b_{y_1} \wedge \cdots \wedge b_{y_m} \in B$. Hence, $\alpha(a) = \alpha(b_{y_1}) \cap \cdots \cap \alpha(b_{y_m}) = \alpha(b_{y_1} \wedge \cdots \wedge b_{y_m})$. This implies that $a = b_{y_1} \wedge \cdots \wedge b_{y_m} \in B$. Therefore, $B_{\preceq_B} \subseteq B$. □

Proposition 6.9. *If $\preceq$ is an $\mathcal{A}$-compatible quasiorder on X, then $\preceq_{B_\preceq} = \preceq$.*

Proof. Let $x \preceq y$. Let $b \in B_\preceq$ and suppose that $b \in x$. Then, it follows by definition of $B_\preceq$ that $b \in y$. Hence $x \preceq_{B_\preceq} y$. Conversely, suppose that $x \preceq_{B_\preceq} y$. This means that $\forall b \in B_\preceq (b \in x \implies b \in y)$. Suppose by contradiction that $x \not\preceq y$. Since $\preceq$ is $\mathcal{A}$-compatible, there exists an $\preceq$-upset $U \in \mathcal{A}$ such that $x \in U$ and $y \notin U$. Given that $U \in \mathcal{A}$, there is $a \in A$ such that $U = \alpha(a)$. Now we show that $a \in B_\preceq$. Let $z \preceq w$ and suppose that $a \in z$. Thus $z \in \alpha(a)$. Since $\alpha(a)$ is $\preceq$-upset, it follows that $w \in \alpha(a)$. That is, $a \in w$. Then $a \in B_\preceq$. Summarizing, we have that $a \in B_\preceq$, $a \in x$ and $a \notin y$. A contradiction. Hence $x \preceq y$. □

Now by the previous results, we have the following.

Theorem 6.10. *The set of all strong-subalgebras of $\mathbf{A}$ ordered by $\subseteq$ is isomorphic to the set of all $\mathcal{A}$-compatible quasiorders on X ordered by $\supseteq$ under the maps $B \mapsto \preceq_B$ and $\preceq \mapsto B_\preceq$.*

Remark 6.11. Notice that in the definition of strong-subalgebra we allow that the empty set be a strong-subalgebra of $\mathbf{A}$. The greatest $\mathcal{A}$-compatible quasiorder of the Priestley DN-space $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ is $X \times X$. Its corresponding strong-subalgebra is $B_{X \times X} = \emptyset$. On the other hand, the least $\mathcal{A}$-compatible quasiorder on X is $\leq_\mathcal{A} = \subseteq$ and its corresponding strong-subalgebra is $B_{\leq_\mathcal{A}} = A$.

Now let us show that the $\mathcal{A}$-compatible quasiorders on a Priestley DN-space X correspond to quotients spaces of X.

Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space and $\preceq$ an $\mathcal{A}$-compatible quasiorder on X. Let $\mathcal{A}_{\preceq} = \{U \in \mathcal{A} : U$ is an $\preceq$ -upset$\}$. Let $\equiv\ =\ \preceq \cap \succeq$. It is clear that $\equiv$ is equivalence relation on X. Let $q\colon X \to X/{\equiv}$ be the natural map. We consider the quotient space $\langle X/{\equiv}, \tau_{\equiv}\rangle$ of $\langle X, \tau\rangle$. That is, $\tau_{\equiv} = \{V \subseteq X/{\equiv} : q^{-1}[V] \in \tau\}$. Now we define

$$\mathcal{A}_{\equiv} = \{V \subseteq X/{\equiv} : q^{-1}[V] \in \mathcal{A}_{\preceq}\}.$$

We want to prove that the structure $\langle X/{\equiv}, \tau_{\equiv}, \mathcal{A}_{\equiv}, 0/{\equiv}, 1/{\equiv}\rangle$ is a Priestley DN-space. Before that, we need some auxiliaries results.

Proposition 6.12. *Let $U, U_1, U_2 \in \mathcal{A}_{\preceq}$.*

(1) *$q(z) \in q[U]$ if and only if $z \in U$.*
(2) *$q^{-1}[q[U]] = U$.*
(3) *$q[U_1 \cap U_2] = q[U_1] \cap q[U_2]$.*

Proof. (1) is straightforward. (2) and (3) are consequence of (1). □

Lemma 6.13. *Let $\langle X, \tau\rangle$ be a topological space and let $\mathcal{B} \subseteq \tau$ be such that $\mathcal{B} \cup \{U^c : U \in B\}$ is a subbasis for τ. If $\mathcal{A} \subseteq \tau$ is such that for each $U \in \mathcal{B}$, $U = V_1 \cap \cdots \cap V_n$ for some $V_1, \ldots, V_n \in \mathcal{A}$, then $\mathcal{A} \cup \{V^c : V \in \mathcal{A}\}$ is a subbasis for τ.*

Proof. We leave the details to the reader. □

Proposition 6.14. *$\mathcal{A}_{\equiv} \cup \{W^c : W \in \mathcal{A}_{\equiv}\}$ is a subbasis for the topology $\tau_{\equiv}$.*

Proof. Notice that $\preceq$ is a compatible quasiorder of the bounded Priestley space $\langle X, \tau, \leq_{\mathcal{A}}, 0, 1\rangle$ (see [20,31]). Then, the ordered quotient space $\langle X/{\equiv}, \tau_{\equiv}, \leq_{\equiv}, 0/{\equiv}, 1/{\equiv}\rangle$ is a bounded Priestley space (see [20,31]), where $q(x) \leq_{\equiv} q(y) \iff x \preceq y$. Thus, we know that $\mathrm{ClUp}^*(X/{\equiv}) \cup \{W^c : W \in \mathrm{ClUp}^*(X/{\equiv})\}$ is a subbasis for $\tau_{\equiv}$. Let us use Lemma 6.13 in order to prove this proposition.

Let $W \in \mathrm{ClUp}^*(X/{\equiv})$. Since $q\colon X \to X/{\equiv}$ is continuous and order-preserving from $\langle X, \tau, \preceq\rangle$ onto $\langle X/{\equiv}, \tau_{\equiv}, \leq_{\equiv}\rangle$, it follows that $q^{-1}[W]$ is a proper and nonempty clopen $\preceq$-upset of X. Let $y \notin q^{-1}[W]$. For each $x \in q^{-1}[W]$, we have $x \npreceq y$. Thus, for each $x \in q^{-1}[W]$, there is $U_x \in \mathcal{A}_{\preceq}$ such that $x \in U_x$ and $y \notin U_x$. Then,

$$q^{-1}[W] \subseteq \bigcup\{U_x : x \in q^{-1}[W]\}.$$

Since $q^{-1}[W]$ is compact, it follows that $q^{-1}[W] \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. Notice that $U_y := U_{x_1} \cup \cdots \cup U_{x_n} \in \mathcal{A}_{\preceq}$. Thus, we have proved that for every $y \notin q^{-1}[W]$, there is $U_y \in \mathcal{A}_{\preceq}$ such that $y \notin U_y$ and $q^{-1}[W] \subseteq U_y$. Then,

$$q^{-1}[W] = \bigcap\{U_y : y \notin q^{-1}[W]\}.$$

By compactness of $q^{-1}[W]^c$, it follows that $q^{-1}[W] = U_{y_1} \cap \cdots \cap U_{y_m}$. By Proposition 6.12, we obtain that

$$W = q\left[q^{-1}[W]\right] = q[U_{y_1} \cap \cdots \cap U_{y_m}] = q[U_{y_1}] \cap \cdots \cap q[U_{y_m}].$$

Notice, by Proposition 6.12, that $q^{-1}\left[q[U_{y_j}]\right] = U_{y_j} \in \mathcal{A}_{\preceq}$, for all $j = 1, \dots, m$. Then, $q[U_{y_j}] \in \mathcal{A}_{\equiv}$, for all $j = 1, \dots, m$. Hence, W is a finite intersection of elements of $\mathcal{A}_{\equiv}$. Therefore, by Lemma 6.13, $\mathcal{A}_{\equiv} \cup \{W^c : W \in \mathcal{A}_{\equiv}\}$ is a subbasis of $\tau_{\equiv}$. □

Proposition 6.15. *Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space and $\preceq$ an $\mathcal{A}$-compatible quasiorder on X. Then, $\langle X/{\equiv}, \tau_{\equiv}, \mathcal{A}_{\equiv}, 0/{\equiv}, 1/{\equiv}\rangle$ is a Priestley DN-space. Moreover, $q\colon X \to X/{\equiv}$ is an onto Priestley DN-space.*

Proof. (S1) Since $q\colon X \to X/{\equiv}$ is continuous and X is compact, it follows that $X/{\equiv} = q[X]$ is compact.

(S2) It was proved in Proposition 6.14.

(S3) Let $W \in \mathcal{A}_{\equiv}$. Thus $q^{-1}[W] \in \mathcal{A}_{\preceq}$. Then $0 \notin q^{-1}[W]$ and $1 \in q^{-1}[W]$. Thus $0/{\equiv} \notin W$ and $1/{\equiv} \in W$. Hence $0/{\equiv} \notin \bigcup \mathcal{A}_{\equiv}$ and $1/{\equiv} \in \bigcap \mathcal{A}_{\equiv}$.

(S4) $W_1, W_2, W_3 \in \mathcal{A}_{\equiv}$. Thus $q^{-1}[W_i] \in \mathcal{A}_{\preceq}$ for $i = 1, 2, 3$. Then

$$q^{-1}[(W_1 \cup W_3) \cap (W_2 \cup W_3)] = (q^{-1}[W_1] \cup q^{-1}[W_3]) \cap (q^{-1}[W_2] \cup q^{-1}[W_3]) \in \mathcal{A}_{\preceq}.$$

Hence, $(W_1 \cup W_3) \cap (W_2 \cup W_3) \in \mathcal{A}_{\equiv}$.

(S5) Let $x/{\equiv}, y/{\equiv} \in X/{\equiv}$ be distinct. Then $x \not\preceq y$ or $y \not\preceq x$. Without loss of generality assume that $x \not\preceq y$. Thus there is $U \in \mathcal{A}_{\preceq}$ such that $x \in U$ and $y \notin U$. By Proposition 6.12, it follows that $q[U] \in \mathcal{A}_{\equiv}$, $x/{\equiv} \in q[U]$ and $y \notin q[U]$. This completes the proof. □

Corollary 6.16. *Let $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ be a Priestley DN-space and $\preceq$ an $\mathcal{A}$-compatible quasiorder on X. Then,*

$$\langle B_{\preceq}, \widetilde{m}\rangle = \langle \mathcal{A}_{\preceq}, \widetilde{m}\rangle \cong \langle \mathcal{A}_{\equiv}, \widetilde{m}\rangle.$$

Proof. For each of the algebras above, the operation $\widetilde{m}$ is given by:

$$\widetilde{m}(U, V, W) = (U \cup W) \cap (V \cup W).$$

Recall that

$$B_{\preceq} = \{U \in \mathcal{A} : \forall (x, y) \in \preceq (x \in U \implies y \in U)\}$$

and

$$\mathcal{A}_{\preceq} = \{U \in \mathcal{A} : U \text{ is } \preceq\text{-upset}\}.$$

Hence, it is clear that $\langle B_{\preceq}, \widetilde{m}\rangle = \langle \mathcal{A}_{\preceq}, \widetilde{m}\rangle$. On the other hand, given that the map $q^{-1}\colon \mathcal{A}_{\equiv} \to \mathcal{A}_{\preceq}$ preserves unions and intersections, then it is a homomorphism of DN-algebras. Since q is onto and by Proposition 6.12, it follows that q^{-1} is bijective. Hence, q^{-1} is an isomorphism from $\langle \mathcal{A}_{\equiv}, \widetilde{m}\rangle$ onto $\langle \mathcal{A}_{\preceq}, \widetilde{m}\rangle$. □

Hence, notice that if $\mathbf{A}$ is a DN-algebra with dual Priestley DN-space $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ and B is a strong-subalgebra of $\mathbf{A}$, then the quotient structure $\langle X/{\equiv_B}, \tau_{\equiv_B}, \mathcal{A}_{\equiv_B}, 0/{\equiv_B}, 1/{\equiv_B}\rangle$ is the dual Priestley DN-space of B, where $\equiv_B = \preceq_B \cap \succeq_B$.

We close this section proving the converse of Proposition 6.15.

Proposition 6.17. *Let $h\colon X \to Y$ be an onto Priestley DN-morphism from $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ onto $\langle Y, \eta, \mathcal{B}, 0, 1\rangle$. Then, the relation $\preceq_h$ on X defined by $x \preceq_h x' \iff h(x) \leq_{\mathcal{B}} h(x')$ is an $\mathcal{A}$-compatible quasiorder on X. Moreover, $\langle Y, \eta, \mathcal{B}, 0, 1\rangle \cong \langle X/{\equiv_h}, \tau_{\equiv_h}, \mathcal{A}_{\equiv_h}, 0/{\equiv_h}, 1/{\equiv_h}\rangle$, where $\equiv_h = \preceq_h \cap \succeq_h$.*

Proof. First, we show that $\preceq_h$ is an $\mathcal{A}$-compatible quasiorder on X. It is clear that $\preceq_h$ is a quasiorder. Let $x_1, x_2 \in X$ be such that $x_1 \not\preceq_h x_2$. Thus $h(x_1) \not\leq_{\mathcal{B}} h(x_2)$. Then, there is $V \in \mathcal{B}$ such that $h(x_1) \in V$ and $h(x_2) \notin V$. That is, $x_1 \in h^{-1}[V]$, $x_2 \notin h^{-1}[V]$ and $h^{-1}[V] \in \mathcal{A}$. Now we prove that $h^{-1}[V]$ is an $\preceq_h$-upset. Let $x, x' \in X$ be such that $x \preceq_h x'$ and $x \in h^{-1}[V]$. Thus $h(x) \leq_{\mathcal{B}} h(x')$ and $h(x) \in V$. Since $V \in \mathcal{B}$, it follows that V is $\leq_{\mathcal{B}}$-upset. Then $x' \in h^{-1}[V]$. Hence $\preceq_h$ is $\mathcal{A}$-compatible.

Let $\widehat{h}\colon X/{\equiv_h} \to Y$ be defined by $\widehat{h}(x/{\equiv_h}) = h(x)$. It is a routine matter to show that $\widehat{h}$ is well-defined and bijective. Now we show that $\widehat{h}$ is a Priestley DN-morphism from $\langle X/{\equiv_h}, \tau_{\equiv_h}, \mathcal{A}_{\equiv_h}, 0/{\equiv_h}, 1/{\equiv_h}\rangle$ to $\langle Y, \eta, \mathcal{B}, 0, 1\rangle$. Let $V \in \mathcal{B}$. We need to show that $\widehat{h}^{-1}[V] \in \mathcal{A}_{\equiv_h} = \{W \subseteq X/{\equiv_h} : q^{-1}[W] \in \mathcal{A}_{\preceq_h}\}$, where $q\colon X \to X/{\equiv_h}$ is the natural map. Let $x \in X$. Then,

$$\begin{aligned} x \in q^{-1}\left[\widehat{h}^{-1}[V]\right] &\iff q(x) \in \widehat{h}^{-1}[V] \iff \widehat{h}(x/{\equiv_h}) \in V \\ &\iff h(x) \in V \\ &\iff x \in h^{-1}[V]. \end{aligned}$$

Hence, $q^{-1}\left[\widehat{h}^{-1}[V]\right] = h^{-1}[V] \in \mathcal{A}_{\preceq_h}$. Therefore, $\widehat{h}$ is an isomorphism of the category $\mathcal{PDNS}$. □

From what we have just proved, we can say that the $\mathcal{A}$-compatible quasiorders on a Priestley DN-space $\langle X, \tau, \mathcal{A}, 0, 1\rangle$ give an intrinsic description of the Priestley DN-spaces Y which are quotients of X, that is, for which exists an onto Priestley DN-morphism $h\colon X \to Y$.

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Declarations

Conflict of Interest The author declares that they have no conflict of interest.

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