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REMARKS ON NORMAL DISTRIBUTIVE NEARLATTICES [1]

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Abstract. In this paper, we present the notions of α-ideal and α-congruence on a distributive nearlattice. We prove that the α-ideals and α-congruences of a normal distributive lattice $\mathbf{A}$ are in a one-to-one correspondence with the ideals and congruences of the distributive nearlattice $\mathrm{R}(\mathbf{A})$ of the annihilators of $\mathbf{A}$, respectively.

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1. Introduction and preliminaries. The distributivity of a lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ can be characterized through some special subsets: if $a, b \in L$, the *annihilator of a relative to b* is defined as the set $\langle a, b\rangle = \{x \in L\colon x \wedge a \leq b\}$. A well known result given by Mandelker in [19] asserts that $\mathbf{L}$ is distributive if and only if $\langle a, b\rangle$ is an ideal of L, for all $a, b \in L$. Then, this result was extended by Varlet to the class of distributive semilattices ([20]), and by Chajda and Kolařík to the class of distributive nearlattices ([9]). The distributive nearlattices have been studied by several authors in [2, 6, 7, 10, 13, 14, 15, 16, 17, 18], and they are a natural generalization of implication algebras, in the sense of [1], and also of bounded distributive lattices. In [4], the authors presented an alternative definition of relative annihilator in distributive nearlattices different from that given in [9], and established new equivalences of the distributivity of a nearlattice. Later, using the results developed in [4], a particular class of filters and annihilator-preserving congruence relations were studied in [3, 5].

The class of normal distributive nearlattices was introduced in [4], which generalize the class of normal lattices given by Cornish in [11, 12]. There is a strong connection between normal distributive nearlattices and the set of its annihilators. For example, in [3] it is proved that there is a correspondence between a class of

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filters (called *α-filters*) of a normal distributive nearlattice and the filters of the distributive nearlattice of the annihilators. The aim of this note is to introduce and study the classes of *α-ideals* and *α-congruences* in a normal distributive nearlattice, and prove that they are in a one-to-one correspondence with the classes of ideals and congruences of the distributive nearlattice of the annihilators, respectively.

The paper is organized as follows. In this section we give some definitions and basic results about distributive nearlattices which are needed in the rest of the paper. In Section 2, we introduce the α-ideals in distributive nearlattices. We prove a separation theorem between ideals and α-filters by means of prime α-ideals and we see the relationship between α-ideals in a normal distributive nearlattice and ideals of the distributive nearlattice of the annihilators. In Section 3, we study α-congruences, and we also prove that there is a one-to-one correspondence between the α-congruences of a normal distributive nearlattice and the congruences of the distributive nearlattice of the annihilators.

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element. A *filter* is a subset F of A such that $1 \in F$, if $a \leq b$ and $a \in F$, then $b \in F$ and if $a, b \in F$, then $a \wedge b \in F$, whenever $a \wedge b$ exists. Denote by $\mathrm{Fi}(A)$ the set of all filters of A. If X is a non-empty subset of A, the smallest filter containing X is called the *filter generated by* X and will be denoted by $\mathrm{Fig}(X)$. A filter G is said to be *finitely generated* if $G = \mathrm{Fig}(X)$, for some finite non-empty subset X of A. If $X = \{a\}$, then $\mathrm{Fig}(\{a\}) = [a) = \{x \in A \colon a \leq x\}$ called the *principal filter of* a. A subset I of A is called an *ideal* if $a \leq b$ and $b \in I$, then $a \in I$ and if $a, b \in I$, then $a \vee b \in I$. If X is a non-empty set of A, the smallest ideal containing X is called the *ideal generated by* X and will be denoted by $\mathrm{Idg}(X)$. It follows that $\mathrm{Idg}(X) = \{a \in A \colon \exists x_1, \ldots, x_n \in X\ (a \leq x_1 \vee \ldots \vee x_n)\}$. A non-empty proper ideal P is *prime* if for every $a, b \in A$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$, whenever $a \wedge b$ exists. We denote by $\mathrm{Id}(A)$ and $\mathrm{X}(A)$ the set of all ideals and prime ideals of A, respectively. It is clear that $\mathrm{Id}(A)$ is an algebraic closure system, and thus $\mathrm{Id}(\mathbf{A}) = \langle \mathrm{Id}(A), \subseteq\rangle$ is a complete lattice. Finally, a non-empty ideal I of A is *maximal* if it is proper and for every $J \in \mathrm{Id}(A)$, if $I \subseteq J$, then $J = I$ or $J = A$. Denote by $\mathrm{X}_m(A)$ the set of all maximal ideals of A.

The class of distributive nearlattices can be presented in two equivalent ways: as join-semilattices with greatest element that satisfy some property or as algebras with only one ternary connective satisfying some identities. The two different ways to consider distributive nearlattices are useful for different purposes.

Definition 1.1. Let $\mathbf{A}$ be a join-semilattice with greatest element. We say that $\mathbf{A}$ is a *distributive nearlattice* if each principal filter is a bounded distributive lattice.

Distributive nearlattices are in a one-to-one correspondence with certain ternary algebras satisfying some identities. This fact was proved in [18, 10], and in [2] the authors found a smaller equational base for the ternary algebras.

Theorem 1.2. ([2, 10]) *Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a distributive nearlattice. Let $m \colon A^3 \to A$ be the ternary operation given by $m(x, y, z) = (x \vee z) \wedge_z (y \vee z)$, where $\wedge_z$ denotes the meet in $[z)$. Then the structure $\mathbf{A}_* = \langle A, m, 1\rangle$ satisfies the following identities:*

(1) $m(x, x, 1) = 1$,

(2) $m(x, y, x) = x$,

(3) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,

(4) $m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w))$.

Conversely, let $\mathbf{A} = \langle A, m, 1\rangle$ *be an algebra of type* $(3, 0)$ *satisfying the identities* $(1) - (4)$. *If we define the binary operation* $x \vee y = m(x, x, y)$, *then* $\mathbf{A}^* = \langle A, \vee, 1\rangle$ *is a distributive nearlattice. Moreover,* $(\mathbf{A}_*)^* = \mathbf{A}$ *and* $(\mathbf{A}^*)_* = \mathbf{A}$.

EXAMPLE 1.3. Each bounded distributive lattice is a distributive nearlattice. Also, every implication algebra, in the sense of [1], is a distributive nearlattice.

REMARK 1.4. If $\mathbf{A}$ is a distributive nearlattice, then $\mathrm{X}_m(A) \subseteq \mathrm{X}(A)$. Let $a, b \in A$ and $U \in \mathrm{X}_m(A)$ be such that $a \wedge b$ exists and $a \wedge b \in U$. Suppose that $a \notin U$ and $b \notin U$. Then $U = \mathrm{Idg}\,(U \cup \{a\}) \cap \mathrm{Idg}\,(U \cup \{b\})$. Indeed, if $x \in \mathrm{Idg}\,(U \cup \{a\}) \cap \mathrm{Idg}\,(U \cup \{b\})$, then there are $u_1, u_2 \in U$ such that $x \leq u_1 \vee a$ and $x \leq u_2 \vee b$. Thus, $u = u_1 \vee u_2 \in U$ and since $[x)$ is a bounded distributive lattice, we have $x \leq (u \vee a) \wedge (u \vee b) = u \vee (a \wedge b)$. As $a \wedge b \in U$, it follows that $x \in U$. The other inclusion is immediate. On the other hand, since $U \subset \mathrm{Idg}\,(U \cup \{a\})$, $U \subset \mathrm{Idg}\,(U \cup \{b\})$ and U is maximal, we have $\mathrm{Idg}\,(U \cup \{a\}) = \mathrm{Idg}\,(U \cup \{b\}) = A$ and $U = A$, which is a contradiction because U is proper. Therefore, $\mathrm{X}_m(A) \subseteq \mathrm{X}(A)$.

Let $\mathbf{A}$ be a distributive nearlattice. We consider $\mathrm{Fi}(\mathbf{A}) = \langle \mathrm{Fi}(A), \veebar, \cap, \{1\}, A\rangle$, where the least element is $\{1\}$, the greatest element is A, and for each $G, H \in \mathrm{Fi}(A)$, we have $G \veebar H = \mathrm{Fig}(G \cup H)$.

THEOREM 1.5. ([13]) *Let* $\mathbf{A}$ *be a distributive nearlattice. Then* $\mathrm{Fi}(\mathbf{A})$ *is a bounded distributive lattice.*

THEOREM 1.6. ([17]) *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $I \in \mathrm{Id}(A)$ *and* $F \in \mathrm{Fi}(A)$ *be such that* $I \cap F = \emptyset$. *Then there exists* $P \in \mathrm{X}(A)$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$.

The following definition was given in [4] as an alternative to [9].

DEFINITION 1.7. Let $\mathbf{A}$ be a join-semilattice with greatest element and $a, b \in A$. The *annihilator of* a *relative to* b is the set

$$a \circ b = \{x \in A \colon b \leq x \vee a\}.$$

In particular, $a^\top = a \circ 1 = \{x \in A \colon x \vee a = 1\}$ is called the *annihilator of* a.

If $\mathbf{A}$ is a distributive nearlattice, then $a \circ b \in \mathrm{Fi}(A)$, for all $a, b \in A$ ([4]). Let $a \in A$ and we consider

$$a^{\top\top} = \{y \in A \colon \forall x \in a^\top\ (y \vee x = 1)\} = \bigcap \{x^\top : x \in a^\top\}.$$

Notice that $a^\top$ and $a^{\top\top}$ are filters of A, for all $a \in A$.

LEMMA 1.8. ([3, 4]) *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $a, b \in A$ *and* $I \in \mathrm{Id}(A)$. *The following properties are satisfied:*

(1) $a \leq b$ *implies* $a^{\top} \subseteq b^{\top}$.

(2) $a^{\top} \subseteq b^{\top}$ *if and only if* $b^{\top\top} \subseteq a^{\top\top}$.

(3) $(a \wedge b)^{\top} = a^{\top} \cap b^{\top}$, *whenever* $a \wedge b$ *exists.*

(4) $I \cap a^{\top} = \emptyset$ *if and only if there exists* $U \in \mathrm{X}_m(A)$ *such that* $I \subseteq U$ *and* $a \in U$.

(5) *If* $U \in \mathrm{Id}(A)$, *then* $U \in \mathrm{X}_m(A)$ *if and only if* $\forall a \in A \ \left(a \notin U \Leftrightarrow U \cap a^{\top} \neq \emptyset\right)$.

(6) *If* $U \in \mathrm{X}_m(A)$, *then* $\forall a \in A \ \left(a \notin U \Leftrightarrow U \cap a^{\top\top} = \emptyset\right)$.

We are interested in studying the class of normal distributive nearlattices introduced in [4], which are a generalization of the normal lattices given in [11].

DEFINITION 1.9. Let $\mathbf{A}$ be a distributive nearlattice. We say that $\mathbf{A}$ is *normal* if each prime ideal is contained in a unique maximal ideal.

THEOREM 1.10. ([4]) *Let* $\mathbf{A}$ *be a distributive nearlattice. Then* $\mathbf{A}$ *is normal if and only if* $(a \vee b)^{\top} = a^{\top} \veebar b^{\top}$, *for all* $a, b \in A$.

Let $\mathbf{A}$ be a normal distributive nearlattice, and consider the set $\mathrm{R}(A) = \{a^{\top} : a \in A\}$. Note that $\mathrm{R}(A) \subseteq \mathrm{Fi}(A)$. If we define $\overline{m}\colon \mathrm{R}(A)^3 \to \mathrm{R}(A)$ by $\overline{m}\left(a^{\top}, b^{\top}, c^{\top}\right) = \left(a^{\top} \veebar c^{\top}\right) \cap \left(b^{\top} \veebar c^{\top}\right)$, then by Lemma 1.8 and Theorem 1.10, the structure

$$\mathrm{R}(\mathbf{A}) = \langle \mathrm{R}(A), \overline{m}, A \rangle$$

is a distributive nearlattice, and it is called the *distributive nearlattice of the annihilators of* $\mathbf{A}$ (for more details see [3]). Let us denote by $\mathrm{Fi}(\mathrm{R}(\mathbf{A})) = \langle \mathrm{Fi}(\mathrm{R}(A)), \sqcup, \cap \rangle$ the distributive lattice of filters of $\mathrm{R}(\mathbf{A})$.

2. α-ideals.

The main aim of this section is to introduce the class of α-ideals in distributive nearlattices and prove that there is a one-to-one correspondence between α-ideals of a normal distributive nearlattice and ideals of the distributive nearlattice of the annihilators.

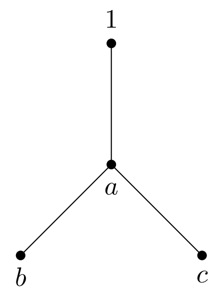
DEFINITION 2.1. Let $\mathbf{A}$ be a distributive nearlattice and $I \in \mathrm{Id}(A)$. We say that I is an *α-ideal* if for each $a \in A$, $I \cap a^{\top\top} \neq \emptyset$ implies $a \in I$.

Denote by $\mathrm{Id}_{\alpha}(A)$ and $\mathrm{X}_{\alpha}(A)$ the set of all α-ideals and prime α-ideals of A, respectively. We consider the structure $\mathrm{Id}_{\alpha}(\mathbf{A}) = \langle \mathrm{Id}_{\alpha}(A), \subseteq \rangle$.

EXAMPLE 2.2. Every maximal ideal of a distributive nearlattice is an α-ideal.

EXAMPLE 2.3. Let $\mathbf{A}$ be a distributive nearlattice. An element $a \in A$ is *dense* if $a^{\top} = \{1\}$. Denote by $\mathrm{D}(A)$ the set of all dense elements of A. By Lemma 1.8, we have $\mathrm{D}(A) \in \mathrm{Id}(A)$. Also, $\mathrm{D}(A)$ is an α-ideal. Indeed, if $a \in A$ such that $\mathrm{D}(A) \cap a^{\top\top} \neq \emptyset$, then there is $x \in \mathrm{D}(A)$ such that $x \in a^{\top\top}$, i.e., $x^{\top} = \{1\}$ and $a^{\top} \subseteq x^{\top}$. Therefore, $a^{\top} = \{1\}$ and $a \in \mathrm{D}(A)$. Note that $\mathrm{D}(A)$ is the smallest α-ideal of A. If $I \in \mathrm{Id}_{\alpha}(A)$ and $a \in \mathrm{D}(A)$, then $a^{\top} = \{1\}$ and $a^{\top\top} = A$. Thus, $I \cap a^{\top\top} \neq \emptyset$ and since I is an α-ideal, $a \in I$. So, $\mathrm{D}(A) \subseteq I$, for all $I \in \mathrm{Id}_{\alpha}(A)$.

REMARK 2.4. Not every ideal is an α-ideal. We consider the following distributive nearlattice $\mathbf{A}$ given by



and the ideal $I = \{c\}$. Then, since $b^{\top\top} = A$, we have $I \cap b^{\top\top} \neq \emptyset$ but $b \notin I$.

Let $\mathbf{A}$ be a distributive nearlattice and $a \in A$. We define the set $(a]_{\alpha} = \{x \in A \colon x^{\top} \subseteq a^{\top}\}$. Notice that $(a]_{\alpha} = (b]_{\alpha}$ if and only if $a^{\top} = b^{\top}$. We have the following result, which characterizes the α-ideals.

THEOREM 2.5. *Let $\mathbf{A}$ be a normal distributive nearlattice and $I \in \mathrm{Id}(A)$. The following conditions are equivalent:*

(1) *I is an α-ideal.*

(2) *If $a \in I$, then $(a]_{\alpha} \subseteq I$.*

(3) *If $a^{\top} = b^{\top}$ and $a \in I$, then $b \in I$.*

(4) *$I = \bigcup\{(a]_{\alpha} \colon a \in I\}$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in I$ and $x \in (a]_{\alpha}$. So, $x^{\top} \subseteq a^{\top}$ and $a^{\top\top} \subseteq x^{\top\top}$. As $a \in a^{\top\top}$, we have $a \in x^{\top\top}$. Thus, $I \cap x^{\top\top} \neq \emptyset$, and since I is an α-ideal, $x \in I$. Then $(a]_{\alpha} \subseteq I$.

(2) $\Rightarrow$ (3) Let $a, b \in A$ be such that $a^{\top} = b^{\top}$ and $a \in I$. Thus, $(a]_{\alpha} = (b]_{\alpha}$ and by hypothesis, $(a]_{\alpha} \subseteq I$. Then $(b]_{\alpha} \subseteq I$, and as $b \in (b]_{\alpha}$, it follows that $b \in I$.

(3) $\Rightarrow$ (4) Since $a \in (a]_{\alpha}$, for all $a \in A$, we have $I \subseteq \bigcup\{(a]_{\alpha} \colon a \in I\}$. Conversely, if $x \in \bigcup\{(a]_{\alpha} \colon a \in I\}$, then there is $a \in I$ such that $x \in (a]_{\alpha}$. Then $x^{\top} \subseteq a^{\top}$. By Theorem 1.10, we have $a^{\top} = a^{\top} \veebar x^{\top} = (a \vee x)^{\top}$. By hypothesis, $a \vee x \in I$. Hence, $x \in I$ and $I = \bigcup\{(a]_{\alpha} \colon a \in I\}$.

(4) $\Rightarrow$ (1) Let $b \in A$ be such that $I \cap b^{\top\top} \neq \emptyset$. Then there is $a \in I$ such that $a \in b^{\top\top}$. It is easy to see that $b^{\top} \subseteq a^{\top}$. Then $b \in (a]_{\alpha}$ and $b \in \bigcup\{(a]_{\alpha} \colon a \in I\} = I$.

So, I is an α-ideal. □

PROPOSITION 2.6. *Let* **A** *be a normal distributive nearlattice. Then* $\mathrm{Id}_\alpha(A)$ *is an algebraic closure system. Moreover, for each subset* $X \subseteq A$*, the set*

$$\mathrm{Idg}_\alpha(X) = \{a \in A \colon \exists x_1, \ldots, x_n \in X \ \left(a^\top \subseteq x_1^\top \veebar \ldots \veebar x_n^\top\right)\}$$

is the least α*-ideal of* A *containing the subset* X*.*

Proof. It is straightforward from Theorem 2.5 that $\mathrm{Id}_\alpha(A)$ is closed under arbitrary intersections and unions of chains. Hence, $\mathrm{Id}_\alpha(A)$ is an algebraic closure system.

Let $X \subseteq A$. By Lemma 1.8 and Theorem 1.10, it is easy to show that $\mathrm{Idg}_\alpha(X)$ is an ideal of A. Let $a \in A$ be such that $\mathrm{Idg}_\alpha(X) \cap a^{\top\top} \neq \emptyset$. So, there is $y \in \mathrm{Idg}_\alpha(X)$ and $y \in a^{\top\top}$. Then, there are $x_1, \ldots, x_n \in X$ such that $y^\top \subseteq x_1^\top \veebar \ldots \veebar x_n^\top$ and $a^\top \subseteq y^\top$. Thus, $a^\top \subseteq x_1^\top \veebar \ldots \veebar x_n^\top$ and $a \in \mathrm{Idg}_\alpha(X)$. Therefore, $\mathrm{Idg}_\alpha(X)$ is an α-ideal. It is clear that $X \subseteq \mathrm{Idg}_\alpha(X)$. Let now J be an α-ideal such that $X \subseteq J$. Let $a \in \mathrm{Idg}_\alpha(X)$. So, there are $x_1, \ldots, x_n \in X$ such that $a^\top \subseteq x_1^\top \veebar \ldots \veebar x_n^\top$. Then $x = x_1 \vee \ldots \vee x_n \in J$. Since **A** is normal, it follows that $x_1^\top \veebar \ldots \veebar x_n^\top = (x_1 \vee \ldots \vee x_n)^\top = x^\top$. Then $a^\top \subseteq x^\top$ and $x \in a^{\top\top} \cap J$. Since J is an α-ideal, we have $a \in J$. Hence, $\mathrm{Idg}_\alpha(X) \subseteq J$. □

REMARK 2.7. Let **A** be a normal distributive nearlattice and $I \in \mathrm{Id}(A)$. Then

$$\mathrm{Idg}_\alpha(I) = \{x \in A \colon \exists i \in I \ \left(x^\top \subseteq i^\top\right)\}.$$

In particular, for each $a \in A$, $\mathrm{Idg}_\alpha((a]) = (a]_\alpha = \{x \in A \colon x^\top \subseteq a^\top\}$.

Now, we consider the concept of α-filter introduced in [3]. In Theorem 2.12 we will see that the α-filters are closely related to the α-ideals.

DEFINITION 2.8. Let **A** be a distributive nearlattice and $F \in \mathrm{Fi}(A)$. We say that F is an *α-filter* if $a^{\top\top} \subseteq F$, for all $a \in F$.

Denote by $\mathrm{Fi}_\alpha(A)$ the set of all α-filters of A.

EXAMPLE 2.9. Let **A** be a distributive nearlattice. Then $a^\top$ and $a^{\top\top}$ are α-filters, for all $a \in A$.

EXAMPLE 2.10. Let I be a non-empty ideal of a distributive nearlattice **A**. Then

$$F_I = \{x \in A \colon \exists i \in I \ \left(i \in x^\top\right)\}$$

is an α-filter. As I is a non-empty set, it follows that $1 \in F_I$ and F_I is increasing. Let $x, y \in F_I$ and suppose that $x \wedge y$ exists. Then there exist $i, j \in I$ such that $i \in x^\top$ and $j \in y^\top$. Since I is an ideal, $k = i \vee j \in I$. By Lemma 1.8, $k \in x^\top \cap y^\top = (x \wedge y)^\top$ and $x \wedge y \in F_I$. So, F_I is a filter. Let $a \in F_I$. If $x \in a^{\top\top}$, then $a^\top \subseteq x^\top$. As $a \in F_I$, there is $i \in I$ such that $i \in a^\top \subseteq x^\top$. Thus, $i \in x^\top$ and $x \in F_I$. Therefore, $a^{\top\top} \subseteq F_I$ and F_I is an α-filter. Moreover, if I is proper, then $I \cap F_I = \emptyset$.

PROPOSITION 2.11. *Let* $\mathbf{A}$ *be a distributive nearlattice. Then* $\mathrm{Fi}_{\alpha}(A)$ *is an algebraic closure system.*

Theorem 1.6 allows us to separate ideals and filters through prime ideals. We have the following separation theorem between ideals and α-filters via α-ideals.

THEOREM 2.12. *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $I \in \mathrm{Id}(A)$ *and* $F \in \mathrm{Fi}_{\alpha}(A)$ *be such that* $I \cap F = \emptyset$*. Then there exists* $P \in \mathrm{X}_{\alpha}(A)$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$*.*

Proof. Let us consider the set

$$\mathcal{F} = \{H \in \mathrm{Id}(A) \colon I \subseteq H \text{ and } H \cap F = \emptyset\}.$$

Since $I \in \mathcal{F}$, we have $\mathcal{F} \neq \emptyset$. The union of a chain of elements of $\mathcal{F}$ is also in $\mathcal{F}$. Then, by Zorn's Lemma, there exists an ideal P maximal in $\mathcal{F}$. It is easy to see that P is prime. We prove that P is an α-ideal. Let $a \in A$ be such that $P \cap a^{\top\top} \neq \emptyset$ and suppose that $a \notin P$. We consider the ideal $\mathrm{Idg}(P \cup \{a\})$. Since P is maximal in $\mathcal{F}$, then $\mathrm{Idg}(P \cup \{a\}) \cap F \neq \emptyset$, i.e., there is $p \in P$ such that $p \vee a \in F$. As F is an α-filter, $(p \vee a)^{\top\top} \subseteq F$. On the other hand, since $P \cap a^{\top\top} \neq \emptyset$, there is $b \in P$ such that $b \in a^{\top\top}$. So, $p \vee b \in p^{\top\top}$ and $p \vee b \in a^{\top\top}$. By Lemma 1.8, it follows that

$$p \vee b \in p^{\top\top} \cap a^{\top\top} = (p \vee a)^{\top\top} \subseteq F$$

and $P \cap F \neq \emptyset$, which is a contradiction. Therefore, P is a prime α-ideal. □

Recall that $\mathrm{D}(A) = \{a \in A : a^{\top} = \{1\}\}$.

LEMMA 2.13. *Let* $\mathbf{A}$ *be a distributive nearlattice. Then* $\mathrm{D}(A) = \bigcap\{P \colon P \in \mathrm{X}_{\alpha}(A)\}$*.*

Proof. By Example 2.3, $\mathrm{D}(A) \in \mathrm{Id}_{\alpha}(A)$ and $\mathrm{D}(A) \subseteq \bigcap\{P \colon P \in \mathrm{X}_{\alpha}(A)\}$. Reciprocally, suppose there is $a \in \bigcap\{P \colon P \in \mathrm{X}_{\alpha}(A)\}$ such that $a \notin \mathrm{D}(A)$. So, since $\mathrm{D}(A)$ is an α-ideal, we have $\mathrm{D}(A) \cap a^{\top\top} = \emptyset$. By Theorem 2.12, there exists $Q \in \mathrm{X}_{\alpha}(A)$ such that $\mathrm{D}(A) \subseteq Q$ and $Q \cap a^{\top\top} = \emptyset$. As $a \in a^{\top\top}$, it follows that $a \notin Q$. On the other hand, $a \in \bigcap\{P \colon P \in \mathrm{X}_{\alpha}(A)\} \subseteq Q$, which is a contradiction. We conclude that $\mathrm{D}(A) = \bigcap\{P \colon P \in \mathrm{X}_{\alpha}(A)\}$. □

We present the main result of this section.

THEOREM 2.14. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\mathrm{Id}_{\alpha}(\mathbf{A})$ *is isomorphic to* $\mathrm{Id}(\mathrm{R}(\mathbf{A}))$*.*

Proof. Let $\phi \colon \mathrm{Id}_{\alpha}(A) \to \mathrm{Id}(\mathrm{R}(\mathrm{A}))$ be the mapping given by

$$\phi(I) = \{a^{\top} : a \in I\}.$$

Firstly, we see that ϕ is well-defined. Let $I \in \mathrm{Id}_\alpha(A)$ and $a^\top, b^\top \in \mathrm{R}(A)$ be such that $b^\top \subseteq a^\top$ and $a^\top \in \phi(I)$. By Lemma 1.8, $a^{\top\top} \subseteq b^{\top\top}$ and $a \in I$. As $a \in a^{\top\top}$, we have $a \in b^{\top\top}$. Thus, $I \cap b^{\top\top} \neq \emptyset$ and since I is an α-ideal, we have $b \in I$. Hence, $b^\top \in \phi(I)$ and $\phi(I)$ is decreasing. Let $a^\top, b^\top \in \phi(I)$. Then $a, b \in I$, and so $a \vee b \in I$. By Theorem 1.10, it follows that $a^\top \veebar b^\top = (a \vee b)^\top \in \phi(I)$. Thus, $\phi(I) \in \mathrm{Id}(\mathrm{R}(A))$.

Let $I, J \in \mathrm{Id}_\alpha(A)$. It is clear that $I \subseteq J$ implies $\phi(I) \subseteq \phi(J)$. Assume now that $\phi(I) \subseteq \phi(J)$. Let $a \in I$. So, $a^\top \in \phi(I)$. Thus, $a^\top \in \phi(J)$. Then there is $b \in J$ such that $a^\top = b^\top$. Since J is an α-ideal, it follows by Theorem 2.5 that $a \in J$. Hence, $I \subseteq J$.

It only remains to show that ϕ is onto. Let $G \in \mathrm{Id}(\mathrm{R(A)})$. If $I_G = \{a \in A \colon a^\top \in G\}$, then by Lemma 1.8 and Theorem 1.10, it is easy to see that $I_G \in \mathrm{Id}_\alpha(A)$ and $\phi(I_G) = G$. So, ϕ is onto. □

3. α-congruences. Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a distributive nearlattice. An equivalence relation $\theta \subseteq A \times A$ is said to be a *congruence* of $\mathbf{A}$ if: (i) whenever $(a,b),(c,d) \in \theta$, then $(a \vee c, b \vee d) \in \theta$, and (ii) if $(a,b),(c,d) \in \theta$ and $a \wedge c$, $b \wedge d$ exist, then $(a \wedge c, b \wedge d) \in \theta$ (see [18]). Let us denote by $\mathrm{Con}(A)$ the set of all congruences of A. The structure $\mathrm{Con}(\mathbf{A}) = \langle \mathrm{Con}(A), \vee, \cap, \Delta, \nabla\rangle$ is a complete distributive lattice, where the least element is $\Delta = \{(a,a) \colon a \in A\}$, the greatest element is $\nabla = A \times A$, and for $\{\Theta_i \colon i \in I\} \subseteq \mathrm{Con}(A)$, $\bigwedge_{i\in I} \Theta_i = \bigcap_{i \in I} \Theta_i$ and $(a,b) \in \bigvee_{i \in I} \Theta_i$ if and only if there exist $z_0 = a, z_1, \ldots, z_n = b \in A$ such that $(z_j, z_{j+1}) \in \bigcup_{i\in I} \Theta_i$, for all $j = 0, \ldots, n-1$.

EXAMPLE 3.1. If $\mathbf{A}$ is a distributive nearlattice and $Y \subseteq \mathrm{X}(A)$, then

$$\Theta(Y) = \{(a,b) \in A \times A \colon \varphi_{\mathbf{A}}(a)^c \cap Y = \varphi_{\mathbf{A}}(b)^c \cap Y\}$$

is a congruence of A, where $\varphi_{\mathbf{A}} : A \to \mathcal{P}_d(\mathrm{X}(A))$ is the mapping defined by $\varphi_{\mathbf{A}}(a) = \{P \in \mathrm{X}(A) \colon a \notin P\}$.

Now, we introduce the class of α-congruences.

DEFINITION 3.2. Let $\mathbf{A}$ be a distributive nearlattice and $\Theta \in \mathrm{Con}(A)$. We say that Θ is an *α-congruence* if for each $a, b, c, d \in A$ such that $(a,b) \in \Theta$, $a^\top = c^\top$ and $b^\top = d^\top$ implies $(c,d) \in \Theta$.

Denote by $\mathrm{Con}_\alpha(A)$ the set of all α-congruences of A.

EXAMPLE 3.3. Let $\mathbf{A}$ be a normal distributive nearlattice. The relation $\Theta^\top \subseteq A \times A$ given by

$$(a,b) \in \Theta^\top \iff a^\top = b^\top$$

is a congruence of A such that $\mathbf{A}/\Theta^\top$ is isomorphic to $\mathrm{R}(\mathbf{A})$ (see [3]). It is easy to see that $\Theta^\top$ is an α-congruence. Moreover, $\Theta^\top$ is the smallest α-congruence of $\mathrm{Con}_\alpha(A)$. Indeed, let $\Psi \in \mathrm{Con}_\alpha(A)$ and $(a,b) \in \Theta^\top$. Since $(a,a) \in \Psi$, $a^\top = b^\top$ and Ψ is an α-congruence, we have $(a,b) \in \Psi$. Hence, $\Theta^\top \subseteq \Psi$.

REMARK 3.4. Not every congruence is an α-congruence. Following Remark 2.4, it is easy to see that the congruence Δ is not an α-congruence.

Following the notation of Examples 3.1 and 3.3, we have the next result.

THEOREM 3.5. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\Theta^{\top} = \Theta(\mathrm{X}_m(A))$.

Proof. Let $(a, b) \in \Theta^{\top}$ and suppose that $(a, b) \notin \Theta(\mathrm{X}_m(A))$. Then, $a^{\top} = b^{\top}$ and

$$\varphi_{\mathbf{A}}(a)^c \cap \mathrm{X}_m(A) \neq \varphi_{\mathbf{A}}(b)^c \cap \mathrm{X}_m(A).$$

If $\varphi_{\mathbf{A}}(a)^c \cap \mathrm{X}_m(A) \nsubseteq \varphi_{\mathbf{A}}(b)^c \cap \mathrm{X}_m(A)$, then there exists $P \in \mathrm{X}_m(A)$ such that $P \in \varphi_{\mathbf{A}}(a)^c$ and $P \notin \varphi_{\mathbf{A}}(b)^c$. So, $a \in P$ and $b \notin P$. By Lemma 1.8, we have $P \cap b^{\top\top} = \emptyset$. Since $a^{\top} = b^{\top}$, it follows that $a \in P \cap a^{\top\top} = P \cap b^{\top\top}$, which is a contradiction. Hence, $(a, b) \in \Theta(\mathrm{X}_m(A))$.

Conversely, let $(a, b) \in \Theta(\mathrm{X}_m(A))$ and suppose that $(a, b) \notin \Theta^{\top}$. Then $a^{\top} \neq b^{\top}$. Suppose that $a^{\top} \nsubseteq b^{\top}$. Thus, there is $x \in a^{\top}$ such that $x \notin b^{\top}$. Since $b^{\top} \in \mathrm{Fi}(A)$, by Theorem 1.6 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $P \cap b^{\top} = \emptyset$. By Lemma 1.8, there exists $U \in \mathrm{X}_m(A)$ such that $P \subseteq U$ and $b \in U$. Then, $U \in \varphi_{\mathbf{A}}(b)^c \cap \mathrm{X}_m(A) = \varphi_{\mathbf{A}}(a)^c \cap \mathrm{X}_m(A)$. Thus, $a \in U$. As $x \in P$ and $P \subseteq U$, $x \in U$. Hence, $1 = x \vee a \in U$, which is a contradiction because U is maximal. So, $(a, b) \in \Theta^{\top}$. Therefore, $\Theta^{\top} = \Theta(\mathrm{X}_m(A))$. □

PROPOSITION 3.6. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\mathrm{Con}_{\alpha}(\mathbf{A})$ *is a complete sublattice of* $\mathrm{Con}(\mathbf{A})$.

Proof. Let $\{\Theta_i : i \in I\} \subseteq \mathrm{Con}_{\alpha}(A)$. It is immediate that $\bigcap_{i \in I} \Theta_i \in \mathrm{Con}_{\alpha}(A)$. Let $(a, b) \in \bigvee_{i \in I} \Theta_i$. Then, there exist $z_0 = a, z_1, \ldots, z_n = b \in A$ such that $(z_j, z_{j+1}) \in \bigcup_{i \in I} \Theta_i$, for all $j = 0, \ldots, n-1$. Let $c, d \in A$ be such that $a^{\top} = c^{\top}$ and $b^{\top} = d^{\top}$. So, $(a, z_1) \in \bigcup_{i \in I} \Theta_i$, $a^{\top} = c^{\top}$ and $z_1^{\top} = z_1^{\top}$. Since $\{\Theta_i : i \in I\}$ are α-congruences, we have $(c, z_1) \in \bigcup_{i \in I} \Theta_i$. Analogously, $(z_{n-1}, b) \in \bigcup_{i \in I} \Theta_i$, $z_{n-1}^{\top} = z_{n-1}^{\top}$ and $b^{\top} = d^{\top}$. Then $(z_{n-1}, d) \in \bigcup_{i \in I} \Theta_i$. It follows that $(c, z_1), (z_1, z_2), \ldots, (z_{n-1}, d) \in \bigcup_{i \in I} \Theta_i$, i.e., $(c, d) \in \bigvee_{i \in I} \Theta_i$. Therefore, $\bigvee_{i \in I} \Theta_i$ is an α-congruence. □

Let $\mathbf{A}$ be a distributive nearlattice and $a, b \in A$. Let $\Theta(a, b)$ be the principal congruence generated by (a, b), i.e., $\Theta(a, b)$ is the smallest congruence of A containing (a, b). The following result will be useful.

LEMMA 3.7. ([8]) *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $a, b \in A$ *be such that* $b \leq a$. *Then*

$$\Theta(a, b) = \{(x, y) \in A \times A : x \vee a = y \vee a \ \textit{and} \ [x) \veebar [b) = [y) \veebar [b)\}.$$

We denote by $\Theta_{\alpha}(a, b)$ the smallest α-congruence of A containing (a, b), called the *principal* α*-congruence generated by* (a, b). Our next goal is to characterize the principal α-congruences.

PROPOSITION 3.8. *Let* **A** *be a normal distributive nearlattice. Let* $a, b \in A$ *be such that* $b \leq a$. *Then* $(x, y) \in \Theta_\alpha(a, b)$ *if and only if*

$$(*) \qquad x^\top \veebar a^\top = y^\top \veebar a^\top \text{ and } \left[x^\top\right) \sqcup \left[b^\top\right) = \left[y^\top\right) \sqcup \left[b^\top\right).$$

Proof. Since **A** is normal, it follows straightforward that the relation $\Phi_{(a,b)}$ defined by $(*)$ is an α-congruence and $(a, b) \in \Phi_{(a,b)}$. Let Ψ be an α-congruence such that $(a, b) \in \Psi$. Let $(x, y) \in \Phi_{(a,b)}$. Then $x^\top \veebar a^\top = y^\top \veebar a^\top$ and $\left[x^\top\right) \sqcup \left[b^\top\right) = \left[y^\top\right) \sqcup \left[b^\top\right)$. Thus, $(m(a, b, x), y \vee a) \in \Psi$. Indeed, by Lemma 1.8, $b \leq a$ implies $b^\top \subseteq a^\top$ and by Theorem 1.10, thus

$$\begin{aligned}(x \vee b)^\top &= x^\top \veebar b^\top = x^\top \veebar \left(b^\top \cap a^\top\right) = \left(x^\top \veebar b^\top\right) \cap \left(x^\top \veebar a^\top\right) \\ &= \overline{m}\left(a^\top, b^\top, x^\top\right) = m(a, b, x)^\top.\end{aligned}$$

Then $(x \vee a, x \vee b) \in \Psi$, $(x \vee b)^\top = m(a, b, x)^\top$ and $(x \vee a)^\top = (y \vee a)^\top$, and since Ψ is an α-congruence, we have $(m(a, b, x), y \vee a) \in \Psi$. Analogously, it is easy to prove that $(m(a, b, y), x \vee a), (m(a, b, x), x \vee b), (m(a, b, y), y \vee b) \in \Psi$. So, it follows that $(x \vee a, y \vee b), (x \vee b, y \vee a) \in \Psi$. Hence,

$$((x \vee y) \wedge (x \vee a), (x \vee y) \wedge (y \vee b)) = (m(a, y, x), m(b, x, y)) \in \Psi$$

and

$$((x \vee y) \wedge (x \vee b), (x \vee y) \wedge (y \vee a)) = (m(b, y, x), m(a, x, y)) \in \Psi.$$

Also, $m(a, x, y)^\top = \left(a^\top \veebar y^\top\right) \cap \left(x^\top \veebar y^\top\right) = \left(a^\top \veebar x^\top\right) \cap \left(x^\top \veebar y^\top\right) = m(a, y, x)^\top$ and as Ψ is an α-congruence, $(m(b, y, x), m(a, y, x)) \in \Psi$. Then, by transitivity, $(m(b, y, x), m(b, x, y)) \in \Psi$.

On the other hand, since $\left[x^\top\right) \sqcup \left[b^\top\right) = \left[y^\top\right) \sqcup \left[b^\top\right)$, we have

$$\begin{aligned}\left[x^\top\right) &= \left[x^\top\right) \cap \left(\left[y^\top\right) \sqcup \left[b^\top\right)\right) = \left(\left[x^\top\right) \cap \left[y^\top\right)\right) \sqcup \left(\left[x^\top\right) \cap \left[b^\top\right)\right) \\ &= \left[x^\top \veebar y^\top\right) \sqcup \left[x^\top \veebar b^\top\right) = \left[\left(x^\top \veebar y^\top\right) \cap \left(x^\top \veebar b^\top\right)\right) \\ &= \left[\overline{m}\left(b^\top, y^\top, x^\top\right)\right) = \left[m(b, y, x)^\top\right).\end{aligned}$$

Thus, $x^\top = m(b, y, x)^\top$. Similarly, we can prove that $y^\top = m(b, x, y)^\top$. Then, since Ψ is an α-congruence and $(m(b, y, x), m(b, x, y)) \in \Psi$, it follows that $(x, y) \in \Psi$. Hence, $\Phi_{(a,b)} \subseteq \Psi$. Therefore, $\Phi_{(a,b)} = \Theta_\alpha(a, b)$. □

Let **A** be a normal distributive nearlattice. Let $f \colon \mathrm{Con}(\mathrm{R}(A)) \to \mathrm{Con}_\alpha(A)$ be the mapping defined by

$$(\star) \qquad (a, b) \in f(\Theta) \Longleftrightarrow \left(a^\top, b^\top\right) \in \Theta.$$

LEMMA 3.9. *Let* **A** *be a normal distributive nearlattice. Let* f *be the mapping given by* $(\star)$. *The following properties are satisfied:*

(1) *For every* $\{\Theta_i \colon i \in I\} \subseteq \mathrm{Con}(\mathrm{R}(A))$,

$$f\left(\bigcap_{i\in I} \Theta_i\right) = \bigcap_{i\in I} f(\Theta_i)$$

and

$$f\left(\bigvee_{i\in I} \Theta_i\right) = \bigvee_{i\in I} f(\Theta_i).$$

(2) *If* $b \leq a$, *then* $f\left(\Theta\left(a^\top, b^\top\right)\right) = \Theta_\alpha(a,b)$.

Proof. (1) By Lemma 1.8 and Theorem 1.10, it follows that f is well-defined. It is straightforward to show directly that $f\left(\bigcap\{\Theta_i \colon i \in I\}\right) = \bigcap\{f(\Theta_i) \colon i \in I\}$. Let $(a,b) \in f\left(\bigvee\{\Theta_i \colon i \in I\}\right)$. Thus, $(a^\top, b^\top) \in \bigvee\{\Theta_i \colon i \in I\}$, i.e., there exist $z_0^\top = a^\top, z_1^\top, \dots, z_n^\top = b^\top \in \mathrm{R}(A)$ such that $(z_j^\top, z_{j+1}^\top) \in \bigcup\{\Theta_i \colon i \in I\}$, for all $j = 0, \dots, n-1$. So, we have that $(a, z_1), (z_1, z_2), \dots, (z_{n-1}, b) \in \bigcup\{f(\Theta_i) \colon i \in I\}$ and $(a,b) \in \bigvee\{f(\Theta_i) \colon i \in I\}$. Then $f\left(\bigvee\{\Theta_i \colon i \in I\}\right) \subseteq \bigvee\{f(\Theta_i) \colon i \in I\}$. The other inclusion is similar.

(2) Since $\mathrm{R}(\mathbf{A})$ is a distributive nearlattice, by Lemma 3.7 and Proposition 3.8, we have

$$\begin{aligned}(x,y) \in f\left(\Theta\left(a^\top, b^\top\right)\right) &\iff \left(x^\top, y^\top\right) \in \Theta\left(a^\top, b^\top\right)\\ &\iff x^\top \veebar a^\top = y^\top \veebar a^\top \text{ and } \left[x^\top\right) \sqcup \left[b^\top\right) = \left[y^\top\right) \sqcup \left[b^\top\right)\\ &\iff (x,y) \in \Theta_\alpha(a,b).\end{aligned}$$

Thus, $f\left(\Theta\left(a^\top, b^\top\right)\right) = \Theta_\alpha(a,b)$. □

THEOREM 3.10. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\mathrm{Con}_\alpha(\mathbf{A})$ *is isomorphic to* $\mathrm{Con}(\mathrm{R}(\mathbf{A}))$.

Proof. By Lemma 3.9, $f \colon \mathrm{Con}(\mathrm{R}(A)) \to \mathrm{Con}_\alpha(A)$ defined by $(\star)$ is a lattice homomorphism. It is easy to prove that f is 1-1. We prove that f is onto. We know by Lemma 3.9 that if $b \leq a$, then $f\left(\Theta\left(a^\top, b^\top\right)\right) = \Theta_\alpha(a,b)$. On the other hand, $\Psi = \bigvee\{\Theta_\alpha(a,b) \colon (a,b) \in \Psi \text{ and } b \leq a\}$, for all $\Psi \in \mathrm{Con}_\alpha(A)$. Indeed, if $(x,y) \in \Psi$, then $(x, x \vee y), (y, x \vee y) \in \Psi$ and $(x,y) \in \Theta_\alpha(x, x\vee y) \vee \Theta_\alpha(y, x \vee y)$. The other inclusion is immediate. Then

$$\begin{aligned}\Psi &= \bigvee \{\Theta_\alpha(a,b) \colon (a,b) \in \Psi \text{ and } b \leq a\}\\ &= \bigvee \left\{f\left(\Theta\left(a^\top, b^\top\right)\right) : (a,b) \in \Psi \text{ and } b \leq a\right\}\\ &= f\left(\bigvee \left\{\Theta\left(a^\top, b^\top\right) : (a,b) \in \Psi \text{ and } b \leq a\right\}\right)\end{aligned}$$

and f is onto. Therefore, f is an isomorphism. □

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