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A Spectral-style Duality for Distributive Posets

Luciano J. González[1] · **Ramon Jansana**[2]

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Abstract In this paper, we present a topological duality for a category of partially ordered sets that satisfy a distributivity condition studied by David and Erné. We call these posets mo-distributive. Our duality extends a duality given by David and Erné because our category of spaces has the same objects as theirs but the class of morphisms that we consider strictly includes their morphisms. As a consequence of our duality, the duality of David and Erné easily follows. Using the dual spaces of the mo-distributive posets we prove the existence of a particular Δ_1-completion for mo-distributive posets that might be different from the canonical extension. This allows us to show that the canonical extension of a distributive meet-semilattice is a completely distributive algebraic lattice.

Keywords Posets · Distributivity · Topological duality · Completion

1 Introduction

The topological dualities for classes of algebras associated with logics arose mainly with M.H. Stone's work [21] in the mid-1930s when he developed a duality between Boolean algebras and the class of compact, Hausdorff, and zero-dimensional topological spaces, later known as Stone spaces. In the subsequent paper [22], Stone generalizes the previous duality for Boolean algebras to show that the category of bounded distributive lattices and

✉ Luciano J. González
lucianogonzalez@exactas.unlpam.edu.ar

Ramon Jansana
jansana@ub.edu

1 Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, Uruguay 151, 6300 Santa Rosa, Argentina

2 Department de Filosofia, IMUB and BGSMath, Universitat de Barcelona, Montalegre 6, 08001 Barcelona, Spain

lattice homomorphisms is dually equivalent to the category of spectral spaces and spectral maps. A spectral space is a sober topological space in which the family of all compact open subsets forms a base closed under finite intersections, and a spectral map between spectral spaces is a map such that its inverse image map preserves compact open subsets. Both topological categories, Stone spaces, and spectral spaces are subcategories of the category of all topological spaces and continuous maps. Unlike Stone spaces, spectral spaces need not be Hausdorff, and not even T_1-spaces (in fact, a spectral space is a Stone space if and only if it is T_1). This is a disadvantage to handle these spaces, but the way in which they are obtained from distributive lattices is considered by some authors the most natural way to get a duality for distributive lattices.

In [18] Grätzer introduced the notion of distributive join-semilattice that generalizes the concept of distributive lattice. And in [5] David and Erné considered a notion of distributivity for partially ordered sets (posets), already introduced in [7] and that they call ideal-distributivity, that generalizes the concept of distributive join-semilattice. A poset is ideal-distributive if the lattice of its ideals in the sense introduced by Frink in [10] is a distributive lattice. Then David and Erné developed a topological (spectral-like) representation in [5] for the ideal-distributive posets and extended it to a full duality for the category of ideal-distributive posets together with the maps between them with the property that the inverse image of a prime Frink ideal (i.e., a Frink ideal which is a prime element in the lattice of Frink ideals) is a prime Frink ideal. We can dually consider the notion of Frink filter, the associated concept of filter-distributive poset and obtain a topological representation for these posets. We decided to take this dual perspective in this paper where we extend the topological duality of David and Erné, but stated here for filter-distributive posets, to a full duality for the category with objects the filter-distributive posets and morphisms the filter-continuous maps (those for which the inverse image of a Frink filter is a Frink filter). These morphisms seem to be the most natural ones to consider for filter-distributive posets. The duals of these maps are not continuous functions meeting some other properties between the dual spaces, but relations between them. Only when the maps satisfy the additional condition of being $\vee$-stable (in the terminology of [5]) the dual relation can be turned into a continuous function. In this way, the duality of David and Erné in [5], but for filter-distributive posets, follows. We will use the topological representation to find a well-behaved completion of a filter-distributive poset different from the canonical extension (in the sense of [6]).

The paper is organized as follows. In Section 2 we introduce the basic notions and notations of order theory and topology needed throughout the paper. In Section 3 we study some algebraic notions on posets. Namely, we consider concepts of filter and ideal, a distributivity condition and a notion of morphism. These concepts on posets were introduced in the literature as generalizations of the analogous concepts in the setting of lattices. We also present in this section some new results about these algebraic notions. The distributivity condition on posets of [5] plays an important role to obtain a spectral style topological duality for this class of posets. We will call the posets satisfying this condition *meet-order distributive*. In Section 4 we introduce the definition of DP-spaces, we prove their main properties, and we develop a full topological duality for the class of meet-order distributive posets. The aim of Section 5 is to show how we can obtain the topological duality for meet-order distributive posets developed by David and Erné in [5] through the topological duality developed in the previous section. In Section 6 we consider a particular type of Δ_1-completion, in the sense of Gehrke et al. [13], for posets. We call this type of Δ_1-completion the *Frink completion*. We use the duality obtained in Section 4 to provide a topological proof of the existence of

Frink completions of meet-order distributive posets and we show that this Δ_1-completion is a completely distributive algebraic lattice. Then we show, as an immediate corollary, that the canonical extension [6] of a distributive meet-semilattice is a completely distributive algebraic lattice.

2 Preliminaries

In this section, we introduce the basic notions and notational conventions needed for what follows. Let X be a set. We denote by $\mathcal{P}(X)$ the power set of X and for every $A \subseteq X$, $A^c := X \setminus A$.

Let $\langle P, \leq \rangle$ be a poset. We often denote $\langle P, \leq \rangle$ simply by P. Let $A \subseteq P$. We say that A is an *up-set* of P if for all $a, b \in P$ such that $a \leq b$ and $a \in A$, it holds that $b \in A$. Dually we have the notion of *down-set*. Let $a \in P$. The principal up-set of P generated by a is $\uparrow a := \{x \in P : a \leq x\}$ and the principal down-set of P generated by a is $\downarrow a := \{x \in P : x \leq a\}$. A lower bound x of $A \subseteq P$ is the *meet* (*greatest lower bound* or *infimum*) of A if for every lower bound b of A, we have $b \leq x$. If the meet of A exists, then we denote it by $\bigwedge A$ and when we write $x = \bigwedge A$ we mean that the meet of A exists and it is equal to x. Similarly, an upper bound y of A is the *join* (*least upper bound* or *supremum*) of A if for every upper bound b of A, we have $y \leq b$. If the join of A exists, then we denote it by $\bigvee A$ and when we write $y = \bigvee A$ we mean that the join of A exists and it is equal to y. If A is finite and non-empty, say $A = \{a_1, \dots, a_n\}$, we write $a_1 \wedge \cdots \wedge a_n$ for $\bigwedge A$ and $a_1 \vee \cdots \vee a_n$ for $\bigvee A$. A non-empty subset U of P is *up-directed* when for every $a, b \in U$, there exists $c \in U$ such that $a \leq c$ and $b \leq c$. Dually, a non-empty subset D of P is *down-directed* when for every $a, b \in D$, there exists $c \in D$ such that $c \leq a$ and $c \leq b$.

Let P be a poset. We consider the following maps $(.)^{\ell} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ and $(.)^{u} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ defined, respectively, as follows: for every $A \subseteq P$

$$A^{\ell} := \{x \in P : (\forall a \in A)(x \leq a)\} \quad \text{and} \quad A^{u} := \{x \in P : (\forall a \in A)(a \leq x)\}.$$

The maps $(.)^{\ell}$ and $(.)^{u}$ satisfy the following conditions:

(1) $((.)^{u}, (.)^{\ell})$ is a Galois connection on $\mathcal{P}(P)$;
(2) for every $a \in P$, $\{a\}^{\ell u} = \uparrow a$ and $\{a\}^{u\ell} = \downarrow a$.

Let P and Q be posets and let $h \colon P \to Q$ be a map. We say that h is *order-preserving* if for all $a, b \in P$, $a \leq b$ implies $h(a) \leq h(b)$; and h is called an *order-embedding* if for all $a, b \in P$, $a \leq b$ if and only if $h(a) \leq h(b)$. If for all $a, b \in P$, $a \leq b$ implies $h(b) \leq h(a)$, then we say that h is *order-reversing*. A map $h \colon P \to Q$ between posets is called an *order-isomorphism* if h is an order-embedding which is onto.

Let $\langle X, \tau \rangle$ be a topological space. The collection of all closed subsets of X is denoted by $\mathsf{C}(X)$ and for $A \subseteq X$, $\mathrm{cl}(A)$ denotes the topological closure of A. For $x \in X$, we write $\mathrm{cl}(x)$ instead of $\mathrm{cl}(\{x\})$. We denote by $\mathsf{KO}(X)$ the collection of all compact open subsets of X. The *specialization quasi-order of* X is the binary relation $\preceq$ on X defined by saying, for every $x, y \in X$, that $x \preceq y$ if and only if for every $U \in \tau$, if $x \in U$ then $y \in U$. It is straightforward to check directly that the relation $\preceq$ on X is reflexive and transitive. Moreover, it should be noted that $x \preceq y$ if and only if $x \in \mathrm{cl}(y)$. Thus, $\mathrm{cl}(x) = \downarrow x = \{y \in X : y \preceq x\}$. Hence, X is a T_0-space if and only if $\preceq$ is a partial order on X. A non-empty closed subset F of X is said to be *irreducible* when for every

$F_1, F_2 \in \mathsf{C}(X)$, if $F \subseteq F_1 \cup F_2$, then $F \subseteq F_1$ or $F \subseteq F_2$. A topological space $\langle X, \tau \rangle$ is said to be *sober* when it is a T_0-space and for every irreducible closed subset F of X there exists an element x of X such that $F = \mathrm{cl}(x)$.

3 Algebraic Notions on Posets

In this section, we will study analogues for posets of some usual algebraic concepts of lattice theory like filter, ideal, homomorphism and the distributivity condition. These lattice-theoretical concepts are generalized to the setting of posets in several different ways in the literature, for instance in [5, 10, 17, 19]. Here we only consider those notions of filter, ideal, morphism and distributivity on posets that will be useful for our purposes.

3.1 Filters and Ideals on Posets

The notions of filter and ideal on posets that we consider here are due to Frink [10]. For other possible notions of filter and ideal on posets see for instance [17, 20].

Definition 3.1 Let P be a poset and $F, I \subseteq P$.

(1) F is said to be a *Frink filter* of P if for every finite $A \subseteq F$, $A^{\ell u} \subseteq F$.
(2) I is said to be a *Frink ideal* of P if for every finite $A \subseteq I$, $A^{u\ell} \subseteq I$.

We denote the collection of all Frink filters of P by $\mathsf{Fi_F}(P)$ and the collection of all Frink ideals of P by $\mathsf{Id_F}(P)$.

Notice that the empty set may be a Frink filter or a Frink ideal of a poset P. In fact, we have that for a poset P, the empty set is a Frink filter (Frink ideal) of P if and only if P has no top (bottom) element. This is a consequence of the fact that $\emptyset^{\ell u} = P^u$ ($\emptyset^{u\ell} = P^\ell$). A Frink filter F (Frink ideal I) of a poset P is called *proper* if $F \neq P$ ($I \neq P$).

Proposition 3.2 ([10]) *For every poset P, the classes $\mathsf{Fi_F}(P)$ and $\mathsf{Id_F}(P)$ are closure systems.*

Let P be a poset. The closure operators associated with $\mathsf{Fi_F}(P)$ and $\mathsf{Id_F}(P)$ are denoted by $\mathrm{C_F}(.)$ and $\mathrm{C_I}(.)$, respectively. For every $A \subseteq P$, $\mathrm{C_F}(A)$ is called the *Frink filter of P generated by A* and $\mathrm{C_I}(A)$ the *Frink ideal of P generated by A*. It is not hard to check that for every $A \subseteq P$, $\mathrm{C_F}(A) = \bigcup\{A_0^{\ell u} : A_0 \subseteq A \text{ and is finite}\}$ and $\mathrm{C_I}(A) = \bigcup\{A_0^{u\ell} : A_0 \subseteq A \text{ and is finite}\}$.

By Proposition 3.2 we have, for every poset P, that $\mathsf{Fi_F}(P)$ and $\mathsf{Id_F}(P)$ are complete lattices with respect to the inclusion order. The meet and join of a subfamily $\mathcal{F} = \{F_i : i \in I\} \subseteq \mathsf{Fi_F}(P)$ are given by

$$\bigwedge \mathcal{F} = \bigcap_{i \in I} F_i \quad \text{and} \quad \bigvee \mathcal{F} = \mathrm{C_F}\left(\bigcup_{i \in I} F_i\right),$$

respectively. Similarly for the meet and join of a subfamily of $\mathsf{Id_F}(P)$.

A proper Frink filter F of a poset P is said to be *irreducible* if it is a meet-irreducible element of the lattice $\mathsf{Fi_F}(P)$ and F is called *prime* if it is a meet-prime element of $\mathsf{Fi_F}(P)$.

We denote respectively by $\mathsf{Fi}^{\mathrm{irr}}_{\mathsf{F}}(P)$ and $\mathsf{Fi}^{\mathrm{pr}}_{\mathsf{F}}(P)$ the collections of irreducible Frink filters and prime Frink filters of P. Similarly, we have the notions of *irreducible* and *prime* Frink ideals. The following propositions are useful characterizations of the concepts of prime and irreducible Frink filter. The following proposition is known, and it is not hard to prove, so we leave the details to the reader.

Proposition 3.3 *Let P be a poset and $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ be proper. Then, $F \in \mathsf{Fi}^{\mathrm{pr}}_{\mathsf{F}}(P)$ if and only if $P \setminus F$ is an up-directed subset of P.*

In the following proposition, we present a new characterization of the notion of irreducible Frink filter, which will be useful in the next theorem.

Proposition 3.4 *Let P be a poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ be proper. Then, F is irreducible if and only if for every $a, b \notin F$ there exist $c \notin F$ and a finite $C \subseteq F$ such that $c \in (C \cup \{a\})^{\ell u}$ and $c \in (C \cup \{b\})^{\ell u}$.*

Proof Let F be a proper Frink filter of P. Suppose that F is irreducible and let $a, b \notin F$. We consider the following Frink filters of P: $F_a := \mathrm{C}_{\mathrm{F}}(F \cup \{a\})$ and $F_b := \mathrm{C}_{\mathrm{F}}(F \cup \{b\})$. It is clear that $F \neq F_a$ and $F \neq F_b$ and since F is irreducible, it follows that $F \subsetneq F_a \cap F_b$. So, let $c \in F_a \cap F_b$ be such that $c \notin F$. As $c \in F_a$, it follows that there exists a finite $A \subseteq F \cup \{a\}$ such that $c \in A^{\ell u}$ and similarly since $c \in F_b$, there exists a finite $B \subseteq F \cup \{b\}$ such that $c \in B^{\ell u}$. Taking $C = (A \cap F) \cup (B \cap F)$ we obtain that C is a finite subset of F, $c \in (C \cup \{a\})^{\ell u}$ and $c \in (C \cup \{b\})^{\ell u}$. Conversely, assume that the condition on the right hand side of the "if and only if" of the proposition is satisfied. Let F_1 and F_2 be Frink filters such that $F = F_1 \cap F_2$. Suppose $F \neq F_1$ and $F \neq F_2$. So, there are $a \in F_1 \setminus F$ and $b \in F_2 \setminus F$. Then, there exist $c \notin F$ and a finite $C \subseteq F$ such that $c \in (C \cup \{a\})^{\ell u}$ and $c \in (C \cup \{b\})^{\ell u}$. Notice that $(C \cup \{a\})^{\ell u} \subseteq F_1$ and $(C \cup \{b\})^{\ell u} \subseteq F_2$. Then, $c \in F_1 \cap F_2 = F$, which is a contradiction. Thus, $F_1 = F$ or $F_2 = F$ and therefore F is irreducible. □

The reader can obtain the dual statements of the two previous propositions for prime and irreducible Frink ideals.

Theorem 3.5 *Let P be a poset. Let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and I be an up-directed down-set of P. If $F \cap I = \emptyset$, then there exists $U \in \mathsf{Fi}^{\mathrm{irr}}_{\mathsf{F}}(P)$ such that $F \subseteq U$ and $U \cap I = \emptyset$.*

Proof Consider the following set $\mathcal{G} = \{G \in \mathsf{Fi}_{\mathsf{F}}(P) : F \subseteq G \text{ and} G \cap I = \emptyset\}$ ordered by the set-theoretic inclusion. Notice that $\mathcal{G} \neq \emptyset$ because $F \in \mathcal{G}$. And it is straightforward to show that the union of any chain of elements of $\mathcal{G}$ is in $\mathcal{G}$. Then, by Zorn's Lemma, there exists a maximal element U of $\mathcal{G}$. Now we prove that U is irreducible using Proposition 3.4. Let $a, b \notin U$. So, it is clear that $U \subsetneq F_a := \mathrm{C}_{\mathrm{F}}(U \cup \{a\})$ and $U \subsetneq F_b := \mathrm{C}_{\mathrm{F}}(U \cup \{b\})$. By the maximality of U, we have $F_a, F_b \notin \mathcal{G}$. So, $F_a \cap I \neq \emptyset$ and $F_b \cap I \neq \emptyset$. Let $x \in F_a \cap I$ and $y \in F_b \cap I$. Then, there are finite $A, B \subseteq U$ such that $x \in (A \cup \{a\})^{\ell u}$ and $y \in (B \cup \{b\})^{\ell u}$. Let $C := A \cup B$. We thus obtain that C is a finite subset of U, $x \in (C \cup \{a\})^{\ell u}$ and $y \in (C \cup \{b\})^{\ell u}$. Since $x, y \in I$ and I is up-directed, it follows that there exists $c \in I$ such that $x \leq c$ and $y \leq c$. Hence $c \notin U$, $c \in (C \cup \{a\})^{\ell u}$ and $c \in (C \cup \{b\})^{\ell u}$. Therefore, by Proposition 3.4, U is an irreducible Frink filter of P. □

3.2 A Distributivity Condition on Posets

We consider here a notion of distributivity for posets discussed by David and Erné in [5]; it is a generalization of the notion of distributivity for semilattices of Grätzer [18] and so, it is also a generalization of the usual distributivity condition for lattices.

Definition 3.6 ([5]) Let P be a poset.

(1) We will say that P is *meet-order distributive* (*mo-distributive* for short) when for every $b_1, \dots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \dots, b_n\}^{\ell u} \implies (\exists a_1, \dots, a_k \in {\uparrow} b_1 \cup \dots \cup {\uparrow} b_n)(a = a_1 \wedge \dots \wedge a_k).$$

(2) We will say that P is *join-order distributive* (*jo-distributive* for short) when for every $b_1, \dots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \dots, b_n\}^{u\ell} \implies (\exists a_1, \dots, a_k \in {\downarrow} b_1 \cup \dots \cup {\downarrow} b_n)(a = a_1 \vee \dots \vee a_k).$$

In [5] David and Erné prove that a poset is jo-distributive if and only if the lattice of its Frink ideals is distributive. Here, for the sake of completeness, we present a proof of the dual statement.

Theorem 3.7 *Let P be a poset. Then, P is mo-distributive if and only if the lattice $\mathsf{Fi}_{\mathsf{F}}(P)$ is distributive.*

Proof First we assume that $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice and we prove that P is mo-distributive. For this, let $a, b_1, \dots, b_n \in P$ be such that $a \in \{b_1, \dots, b_n\}^{\ell u}$. So, we have ${\uparrow} a \cap \{b_1, \dots, b_n\}^{\ell u} = {\uparrow} a \cap ({\uparrow} b_1 \vee \dots \vee {\uparrow} b_n) = ({\uparrow} a \cap {\uparrow} b_1) \vee \dots \vee ({\uparrow} a \cap {\uparrow} b_n)$. Given that $a \in {\uparrow} a \cap \{b_1, \dots, b_n\}^{\ell u}$, it follows that $a \in ({\uparrow} a \cap {\uparrow} b_1) \vee \dots \vee ({\uparrow} a \cap {\uparrow} b_n)$. Then, there exist $a_1, \dots, a_k \in ({\uparrow} a \cap {\uparrow} b_1) \cup \dots \cup ({\uparrow} a \cap {\uparrow} b_n)$ such that $a \in \{a_1, \dots, a_k\}^{\ell u}$. Moreover, as $a_1, \dots, a_k \in {\uparrow} a$, $a \in \{a_1, \dots, a_k\}^{\ell}$. We thus obtain $a \in \{a_1, \dots, a_k\}^{\ell u} \cap \{a_1, \dots, a_k\}^{\ell}$ and this implies that $a = a_1 \wedge \dots \wedge a_k$ and moreover it is clear that $a_1, \dots, a_k \in {\uparrow} b_1 \cup \dots \cup {\uparrow} b_n$. Therefore, P is mo-distributive.

Now we suppose that P is mo-distributive. Let $F_1, F_2, F_3 \in \mathsf{Fi}_{\mathsf{F}}(P)$. We need only to prove that $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Let $a \in F_1 \cap (F_2 \vee F_3)$. So, $a \in F_1$ and there exist $a_1, \dots, a_n \in F_2 \cup F_3$ such that $a \in \{a_1, \dots, a_n\}^{\ell u}$. Then, because P is mo-distributive, there exist $a'_1, \dots, a'_k \in {\uparrow} a_1 \cup \dots \cup {\uparrow} a_n$ such that $a = a'_1 \wedge \dots \wedge a'_k$. Clearly, $a \in \{a'_1, \dots, a'_k\}^{\ell u}$. Given that $a \in F_1$ and $a \leq a'_i$ for all $i = 1, \dots, k$, we have $a'_1, \dots, a'_k \in F_1$. It also holds $a'_1, \dots, a'_k \in {\uparrow} a_1 \cup \dots \cup {\uparrow} a_n \subseteq F_2 \cup F_3$. Thus, $a'_1, \dots, a'_k \in F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$. Hence, since $a \in \{a'_1, \dots, a'_k\}^{\ell u}$, we have $a \in (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Therefore, $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice. □

Recall that for a lattice L to be distributive is equivalent to each one of the following conditions: 1) for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and 2) for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. This does not extend to posets, that is, it is not true that a poset P is mo-distributive if and only if it is jo-distributive. The following example shows it.

Example 3.8 In Fig. 1 we display a poset P and its lattices of Frink filters and Frink ideals, respectively. The lattice $\mathsf{Fi}_{\mathsf{F}}(P)$ is distributive because it is isomorphic to the product of two distributive lattices: $\mathsf{Fi}_{\mathsf{F}}(P) \cong ((\mathbf{2} \times \mathbf{2}) \oplus \mathbb{N}) \times \mathbf{2}$, where $\mathbf{2}$ is the distributive lattice of two

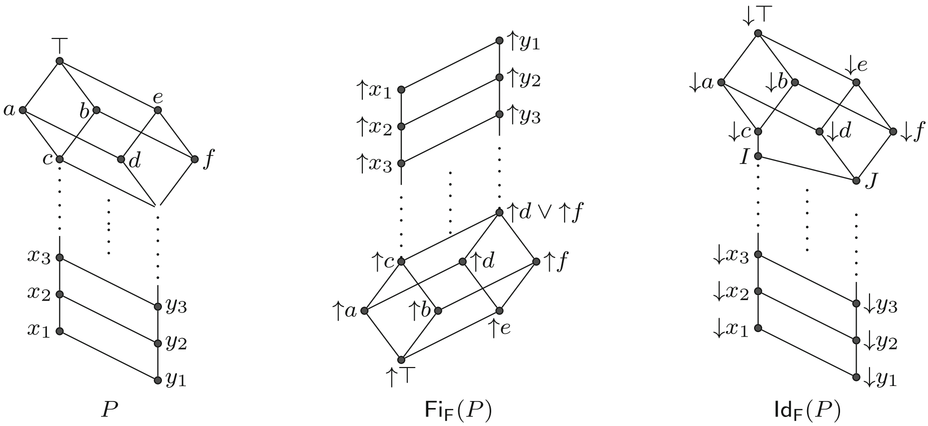

Fig. 1 A poset P and its lattices of Frink filters and Frink ideals, respectively

elements and $\mathbb{N}$ is the chain of the natural numbers with the usual order. Hence, the poset P is mo-distributive. In the lattice $\mathsf{Id}_\mathsf{F}(P)$ of Fig. 1, $I = \bigcup_{i\geq 1} \downarrow x_i$ and $J = \bigcup_{i\geq 1} \downarrow y_i$. Then we can see that the sub-lattice $\{I, \downarrow c, \downarrow b, \downarrow f, J\}$ of $\mathsf{Id}_\mathsf{F}(P)$ is not distributive. Hence the lattice $\mathsf{Id}_\mathsf{F}(P)$ is not distributive. Therefore, P is not jo-distributive.

We finish this subsection with two new characterizations of the condition of mo-distributivity.

Theorem 3.9 *Let P be a poset. The following conditions are equivalent:*

(1) *P is mo-distributive;*
(2) *Every irreducible Frink filter is prime;*
(3) *if $F \in \mathsf{Fi}_\mathsf{F}(P)$ and I is an up-directed down-set of P such that $F \cap I = \emptyset$, then there exists $U \in \mathsf{Fi}^{\mathsf{pr}}_\mathsf{F}(P)$ such that $F \subseteq U$ and $U \cap I = \emptyset$.*

Proof (1) $\Rightarrow$ (2) Assume that P is mo-distributive. Then, by Theorem 3.7, $\mathsf{Fi}_\mathsf{F}(P)$ is a distributive lattice and therefore its prime elements coincide with its irreducible elements. Thus we have (2).

(2) $\Rightarrow$ (3) It is a consequence of Theorem 3.5.

(3) $\Rightarrow$ (1) We prove that the lattice of Frink filters is distributive. Let $F_1, F_2, F_3 \in \mathsf{Fi}_\mathsf{F}(P)$. We need only to show that $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$ because the other inclusion always holds. We suppose that $F_1 \cap (F_2 \vee F_3) \nsubseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. So, let $a \in F_1 \cap (F_2 \vee F_3) \setminus (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Since $a \notin (F_1 \cap F_2) \vee (F_1 \cap F_3)$, it follows that there exists $U \in \mathsf{Fi}^{\mathsf{pr}}_\mathsf{F}(P)$ such that $a \notin U$ and $(F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq U$. Then, $F_1 \cap F_2 \subseteq U$ and $F_1 \cap F_3 \subseteq U$. As U is prime, we have

$$(F_1 \subseteq U \text{ or } F_2 \subseteq U) \text{ and } (F_1 \subseteq U \text{ or } F_3 \subseteq U).$$

Since $a \in F_1$ and $a \notin U$, we have $F_2 \subseteq U$ and $F_3 \subseteq U$. Then, $F_2 \vee F_3 \subseteq U$. We thus get $a \in U$, which is a contradiction. Hence, $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Therefore, by Theorem 3.7, P is mo-distributive. $\square$

3.3 Morphisms Between Posets

In this part of the paper, we introduce the definitions of certain morphisms between posets that intend to be a generalization of the standard notion of homomorphism for lattices. They are respectively the filter-continuous and the ideal-continuous maps considered in [5]. We also discuss the notion of $\vee$-stable and filter-continuous map between posets dual to the one considered by David and Erné in [5] in order to define the category for which they present their topological duality.

Definition 3.10 Let P and Q be two posets. We say that a map $h\colon P \to Q$

(1) is an *inf-homomorphism* if for every finite $A \subseteq P$, we have

$$a \in A^{\ell u} \text{ implies } h(a) \in h[A]^{\ell u};$$

(2) is a *sup-homomorphism* if for every finite $A \subseteq P$, we have

$$a \in A^{u\ell} \text{ implies } h(a) \in h[A]^{u\ell};$$

(3) is an *inf-sup-homomorphism* if h is inf-homomorphism and sup-homomorphism.

The map h is called an *inf-embedding* (*sup-embedding*) if h is an inf-homomorphism (a sup-homomorphism) and an order-embedding. Moreover, h is said to be an *inf-sup-embedding* if h is an inf-embedding and a sup-embedding.

Remark 3.11 Notice that if $h\colon P \to Q$ is an inf-homomorphism or sup-homomorphism, then h is order-preserving. Moreover, if h is an inf-homomorphism (sup-homomorphism), then h preserves the top (bottom) element, if it exists.

The next proposition shows that the homomorphisms just defined are indeed the filter-continuous and the ideal-continuous maps of [5].

Proposition 3.12 *Let P and Q be posets and let $h\colon P \to Q$ be a map. The following statements are true.*

(1) *h is an inf-homomorphism if and only if $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$.*
(2) *h is a sup-homomorphism if and only if $h^{-1}[J] \in \mathsf{Id}_{\mathsf{F}}(P)$ for all $J \in \mathsf{Id}_{\mathsf{F}}(Q)$.*

Proof We only prove (1), as (2) can be proved dually. Assume that h is an inf-homomorphism and let $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$. Let $A \subseteq h^{-1}[G]$ be finite and $a \in A^{\ell u}$. Since h is an inf-homomorphism, $h(a) \in h[A]^{\ell u}$. As $h[A]$ is a finite subset of G and $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$, it follows that $h[A]^{\ell u} \subseteq G$. Then, $h(a) \in G$ and hence $a \in h^{-1}[G]$. Therefore, $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$. Reciprocally, suppose that $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$. Let $A \subseteq P$ be finite and let $a \in A^{\ell u}$. By hypothesis, $h^{-1}\left[h[A]^{\ell u}\right] \in \mathsf{Fi}_{\mathsf{F}}(P)$. Moreover, notice that $A \subseteq h^{-1}[h[A]] \subseteq h^{-1}\left[h[A]^{\ell u}\right]$, consequently $A^{\ell u} \subseteq h^{-1}\left[h[A]^{\ell u}\right]$. Thus, $a \in h^{-1}\left[h[A]^{\ell u}\right]$ and hence, $h(a) \in h[A]^{\ell u}$. Therefore, h is an inf-homomorphism. □

Proposition 3.13 *Let P and Q be posets and let $h\colon P \to Q$ be a map. The following statements are equivalent:*

(1) *h is an inf-embedding;*
(2) *for every finite $A \subseteq P$ and $a \in P$, $a \in A^{\ell u}$ if and only if $h(a) \in h[A]^{\ell u}$.*

Proof (1) $\Rightarrow$ (2) We need only to prove that $h(a) \in h[A]^{\ell u}$ implies $a \in A^{\ell u}$. So, let $A \subseteq P$ be finite and let $a \in P$ be such that $h(a) \in h[A]^{\ell u}$. Let $b \in A^{\ell}$. Thus, $b \leq a'$ for all $a' \in A$. Since h is order-preserving, we have $h(b) \leq h(a')$ for all $a' \in A$. So, $h(b) \in h[A]^{\ell}$ and then $h(b) \leq h(a)$. Hence, since h is an order-embedding, it follows that $b \leq a$. Therefore $a \in A^{\ell u}$.

(2) $\Rightarrow$ (1) From (2) it is clear that h is an inf-homomorphism and so, it is also order-preserving. Let $a, b \in P$. Suppose that $h(a) \leq h(b)$. So, $h(b) \in \{h(a)\}^{\ell u}$. Then $b \in \{a\}^{\ell u}$, which implies that $a \leq b$. Hence, h is an order-embedding. □

By a dual argument we have:

Proposition 3.14 *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, the following statements are equivalent:*

(1) *h is a sup-embedding;*
(2) *for every finite $A \subseteq P$ and $a \in P$, $a \in A^{u\ell}$ if and only if $h(a) \in h[A]^{u\ell}$.*

An inf-homomorphism from a poset P to a poset Q preserves all existing finite meets and the converse is true when the poset P is mo-distributive. We proceed to prove these facts and their duals.

Proposition 3.15 *Let P be a mo-distributive (jo-distributive) poset and let Q be an arbitrary poset. Let $h\colon P \to Q$ be a map. Then, h is an inf-homomorphism (sup-homomorphism) if and only if h preserves all existing finite meets (joins).*

Proof Assume that $h\colon P \to Q$ is an inf-homomorphism. Let $A \subseteq P$ be finite and suppose that $\bigwedge A$ exists in P. If $A = \emptyset$, then $\bigwedge A$ is the top element of P and hence, by Remark 3.11, $h(\bigwedge A) = \bigwedge h[A]$ because $h[A] = \emptyset$. Now we suppose that $A \neq \emptyset$, say $A = \{a_1, \ldots, a_n\}$ and so $\bigwedge A = a_1 \wedge \cdots \wedge a_n$. Since h is order-preserving, we have $h(a_1 \wedge \cdots \wedge a_n) \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. Let $y \in Q$ be such that $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. So, $y \in \{h(a_1), \ldots, h(a_n)\}^{\ell}$. Since $a_1 \wedge \cdots \wedge a_n \in \{a_1, \ldots, a_n\}^{\ell u}$ and h is an inf-homomorphism, it follows that $h(a_1 \wedge \cdots \wedge a_n) \in \{h(a_1), \ldots, h(a_n)\}^{\ell u}$ and then $y \leq h(a_1 \wedge \cdots \wedge a_n)$. Hence, we have shown that $h(a_1 \wedge \cdots \wedge a_n)$ is the greatest lower bound of $\{h(a_1), \ldots, h(a_n)\}$, i.e., $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Therefore, h preserves all existing finite meets.

For the reverse implication, assume that h preserves all existing finite meets. Let $A \subseteq P$ be finite and $b \in A^{\ell u}$. If $A = \emptyset$, then $b = \bigwedge A$ and is the top element of P. Then, since h preserves finite meets, we have $h(b) = \bigwedge h[A]$ with $h[A] = \emptyset$. So $h(b)$ is the top element of Q and hence $h(b) \in h[A]^{\ell u}$. Now, suppose $A \neq \emptyset$ and let $A = \{a_1, \ldots, a_n\}$. So, $b \in \{a_1, \ldots, a_n\}^{\ell u}$. From the mo-distributive condition for P, there exist $b_1, \ldots, b_k \in {\uparrow}a_1 \cup \cdots \cup {\uparrow}a_n$ such that $b = b_1 \wedge \cdots \wedge b_k$. Then, by hypothesis, we have that $h(b) = h(b_1) \wedge \cdots \wedge h(b_k)$. Since h is order-preserving, we obtain $h(b_1), \ldots, h(b_k) \in {\uparrow}h(a_1) \cup \cdots \cup {\uparrow}h(a_n)$. Let $y \in \{h(a_1), \ldots, h(a_n)\}^{\ell}$. So, $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$ and thus $y \leq h(b_j)$ for all $j \in \{1, \ldots, k\}$. Then, $y \leq h(b)$. Hence, $h(b) \in \{h(a_1), \ldots, h(a_n)\}^{\ell u}$ and therefore h is an inf-homomorphism. □

The aim of the definition of the morphisms considered by David and Erné [5] between jo-distributive posets in their duality (i.e., the $\wedge$-stable and ideal-continuous maps) was that the extension map to the lattice of Frink ideals preserves not only arbitrary joins but also finite meets. More precisely, for a sup-homomorphism $h\colon P \to Q$ between jo-distributive posets they looked for necessary and sufficient conditions so that the extension map $\widehat{h}\colon \mathsf{Id_F}(P) \to \mathsf{Id_F}(Q)$ defined by $\widehat{h}(F) = \mathrm{C_I}(h[F])$ preserves finite meets. As we have been doing we will work with the dual notions.

Definition 3.16 ([5]) Let P and Q be posets. A map $h\colon P \to Q$ is called $\vee$-*stable* if for every finite $A \subseteq P$ we have

$$h[A]^u = \mathrm{C_F}\left(h[A^u]\right).$$

Proposition 3.17 ([5, Proposition 2.3]) *Let P and Q be posets. If $h\colon P \to Q$ is a $\vee$-stable map, then h is a sup-homomorphism.*

This proposition seems to show that the notion of $\vee$-stable map can be considered as a kind of morphism between posets in the sense that in the setting of join-semilattices the condition of $\vee$-stability implies that the map is a join-homomorphism, but the notion of $\vee$-stable map has two remarkable drawbacks. On the one hand, a join-homomorphism between join-semilattices is not necessarily a $\vee$-stable map and on the other hand, the composition of two $\vee$-stable maps on posets need not be a $\vee$-stable map (an example of this can be found in [8]). However, as shown (dually) in [5], the maps between posets that are $\vee$-stable and inf-homomorphisms can be considered as one generalization of the notion of lattice homomorphism.

The next proposition shows clearly what the condition of being $\vee$-stable adds to that of being an inf-homomorphism.

Proposition 3.18 ([5, Proposition 3.2]) *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, the following conditions are equivalent:*

(1) *h is a $\vee$-stable map and an inf-homomorphism;*
(2) *for every $G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q)$, we have $h^{-1}[G] \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$.*

4 A Spectral-style Duality

Let us consider the category whose objects are the mo-distributive posets and whose morphisms between mo-distributive posets are the inf-homomorphisms. We denote this category by $\mathbb{MODP}$. It should be clear that the composition of morphisms in this category is the usual set-theoretic composition of functions and for every object, the identity morphism is the identity function. In this section, we develop a topological duality for the category $\mathbb{MODP}$ using certain sober spaces that we call DP-spaces. These spaces were introduced by David and Erné in [5] to develop their topological duality for the category of jo-distributive posets and $\wedge$-stable sup-homomorphisms. In Section 4.1 we give the definition of DP-space and we show how to construct the DP-space $\mathbf{X}(P)$ from a mo-distributive poset P. We thus obtain a representation theorem for mo-distributive posets. We further show that the open subsets and compact open subsets of the space $\mathbf{X}(P)$ correspond to the Frink filters and finitely generated Frink filters of P, respectively. Then in Section 4.2 we extend the representation theorem to a full duality between the categories of mo-distributive posets and DP-spaces.

4.1 Topological Representation

We start introducing the definition of the topological spaces that will be dual to the mo-distributive posets, and we study them by showing their main properties. As we said, David and Erné [5] introduced this sort of topological spaces.

Definition 4.1 A triple $\mathbf{X} = \langle X, \tau, \mathcal{B}\rangle$ is a DP-*space* if:

(DP1) $\langle X, \tau\rangle$ is a sober topological space;
(DP2) $\mathcal{B}$ is a base for $\langle X, \tau\rangle$ of compact open subsets that is meet-dense in $\mathsf{KO}(\mathbf{X})$, that is, it satisfies the following condition:

$$\text{for every} C \in \mathsf{KO}(\mathbf{X}),\ \text{there exists } \mathcal{A} \subseteq \mathcal{B} \text{ such that } C = \bigcap \mathcal{A}. \tag{1}$$

We often denote a DP-space $\langle X, \tau, \mathcal{B}\rangle$ by $\langle \mathbf{X}, \mathcal{B}\rangle$, if the confusion is unlikely. It should be noted, from Eq. 1, that if $\langle \mathbf{X}, \mathcal{B}\rangle$ is a DP-space and there is $U \in \mathcal{B}$ such that $U \subseteq V$ for all $V \in \mathcal{B}$, then $U = \emptyset$. This fact should be kept in mind, because it will be used later on.

Proposition 4.2 *Let $\mathbf{X}$ be a topological space and let $\mathcal{B}$ be a base for $\mathbf{X}$ of compact open subsets. Then, the following conditions are equivalent:*

(1) *for every $U \in \mathcal{B}$ and every finite subfamily $\mathcal{A} \subseteq \mathcal{B}$,*

$$(\forall V \in \mathcal{B})\left(\bigcup \mathcal{A} \subseteq V \implies U \subseteq V\right) \implies U \subseteq \bigcup \mathcal{A};$$

(2) *for every $U \in \mathcal{B}$ and every $C \in \mathsf{KO}(X)$, if $U \nsubseteq C$ then there is $U_0 \in \mathcal{B}$ such that $U \nsubseteq U_0$ and $C \subseteq U_0$;*
(3) *for every $C \in \mathsf{KO}(X)$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $C = \bigcap \mathcal{A}$.*

Proof It is straightforward to show the equivalence between (1) and (2), and the equivalence between (2) and (3) is given in [5, p. 110]. □

Let $\langle \mathbf{X}, \mathcal{B}\rangle$ be a DP-space. We define the set $P_{\mathbf{X}} := \{U^c : U \in \mathcal{B}\}$ and we consider the poset $\langle P_{\mathbf{X}}, \subseteq\rangle$. The poset $\langle P_{\mathbf{X}}, \subseteq\rangle$ is called the *dual poset* of $\mathbf{X}$. For what follows we shall need the following property of the poset $\langle P_{\mathbf{X}}, \subseteq\rangle$.

Proposition 4.3 *Let $\langle X, \mathcal{B}\rangle$ be a DP-space and let $A, A_1, \ldots, A_n \in P_X$. Then, $A \in \{A_1, \ldots, A_n\}^{\ell u}$ if and only if $A_1 \cap \cdots \cap A_n \subseteq A$.*

Proof It is a consequence of condition (DP2) of Definition 4.1. □

Proposition 4.4 *Let $\langle X, \mathcal{B}\rangle$ be a DP-space. Then the poset $\langle P_X, \subseteq\rangle$ is mo-distributive.*

Proof To prove that the poset $P_{\mathbf{X}}$ is mo-distributive, let $A, B_1, \ldots, B_n \in P_{\mathbf{X}}$ be such that $A \in \{B_1, \ldots, B_n\}^{\ell u}$. By the previous proposition, we obtain $B_1 \cap \cdots \cap B_n \subseteq A$. Then, $A = A \cup (B_1 \cap \cdots \cap B_n) = (A \cup B_1) \cap \cdots \cap (A \cup B_n)$ and thus $A^c = (A^c \cap B_1^c) \cup \cdots \cup (A^c \cap B_n^c)$. Notice that for every $i \in \{1, \ldots, n\}$, we have that $A^c \cap B_i^c$ is an open of $\mathbf{X}$. Then, for every $i \in \{1, \ldots, n\}$, there exists $\{U_j^i : j \in J_i\} \subseteq \mathcal{B}$ such that $A^c \cap B_i^c = \bigcup_{j \in J_i} U_j^i$. So, for every

$i \in \{1, \ldots, n\}$ it follows that $U_j^i \subseteq \bigcup_{j \in J_i} U_j^i \subseteq B_i^c$ for all $j \in J_i$. On the other hand, we have $A^c = \left(\bigcup_{j \in J_1} U_j^1\right) \cup \cdots \cup \left(\bigcup_{j \in J_n} U_j^n\right)$. Then, since A^c is compact, it follows that there exist $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $k_1 \in J_{i_1}, \ldots, k_m \in J_{i_m}$ such that $A^c = U_{k_1}^{i_1} \cup \cdots \cup U_{k_m}^{i_m}$. Hence, $A = \left(U_{k_1}^{i_1}\right)^c \cap \cdots \cap \left(U_{k_m}^{i_m}\right)^c$ with $\left(U_{k_1}^{i_1}\right)^c, \ldots, \left(U_{k_m}^{i_m}\right)^c \in {\uparrow} B_1 \cup \cdots \cup {\uparrow} B_n$. Therefore, $P_{\mathbf{X}}$ is a mo-distributive poset. □

Let P be a fixed but arbitrary mo-distributive poset. We define the *dual space* of P as the topological space $\mathbf{X}(P) := \left\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P), \tau_P \right\rangle$ with τ_P the topology on $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ generated by the family $\{\varphi(a)^c : a \in P\}$ where

$$\varphi(a) := \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) : a \in F\} \qquad \text{and} \qquad \varphi(a)^c := \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) : \ a \notin F\}$$

for every $a \in P$.

Lemma 4.5 *Let P be a mo-distributive poset.*

(1) *For every $a, b \in P$ and every $F \in \varphi(a)^c \cap \varphi(b)^c$, there exists $c \in P$ such that $F \in \varphi(c)^c \subseteq \varphi(a)^c \cap \varphi(b)^c$.*
(2) $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) = \bigcup_{a \in P} \varphi(a)^c$.

Proof (1) is an immediate consequence of Proposition 3.3. (2) follows from the fact that every prime Frink filter of P is proper. □

The following corollary follows directly from the previous lemma.

Corollary 4.6 *Let P be a mo-distributive poset. Then, the family $\mathcal{B}_P := \{\varphi(a)^c \ : \ a \in P\}$ is a base for the topological space $\boldsymbol{X}(P)$.*

Proposition 4.7 *For every $a \in P$, $\varphi(a)^c$ is a compact subset of $\boldsymbol{X}(P)$.*

Proof Let $a \in P$. Suppose that $\{a_i \ : \ i \in I\} \subseteq P$ is such that $\varphi(a)^c \subseteq \bigcup_{i \in I} \varphi(a_i)^c$, having thus that $\bigcap_{i \in I} \varphi(a_i) \subseteq \varphi(a)$. Let us consider the Frink filter F generated by the set $\{a_i \ : \ i \in I\}$, i.e., $F := \mathrm{C_F}\left(\{a_i \ : \ i \in I\}\right)$. Then, by Theorem 3.9, we have that $a \in F$. Consequently, there exist $i_1, \ldots, i_n \in I$ such that $a \in \{a_{i_1}, \ldots, a_{i_n}\}^{\ell u}$. Now it is easy to check that $\varphi(a_{i_1}) \cap \ldots \cap \varphi(a_{i_n}) \subseteq \varphi(a)$ and this implies that $\varphi(a)^c \subseteq \varphi(a_{i_1})^c \cup \ldots \cup \varphi(a_{i_n})^c$. Therefore, $\varphi(a)^c$ is compact. □

We have proved that $\mathcal{B}_P$ is a base of compact open subsets of the space $\mathbf{X}(P)$. We want to show that the dual space $\mathbf{X}(P)$ of a mo-distributive poset P is, in fact, a DP-space. In order to attain this, the next step is to prove that the space $\mathbf{X}(P)$ is sober. And for this, we need to obtain a characterization of all open subsets of the space $\mathbf{X}(P)$.

Let us introduce the following notation. For $A \subseteq P$, we define

$$\widehat{\varphi}(A) := \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) : A \subseteq F\} \qquad \text{and} \qquad \widehat{\varphi}(A)^c := \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) : A \nsubseteq F\}.$$

Notice that for every $A \subseteq P$, we have $\widehat{\varphi}(A) = \widehat{\varphi}(\mathrm{C_F}(A))$.

Proposition 4.8 *A set $U \subseteq \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ is an open subset of $\mathbf{X}(P)$ if and only if there exists a Frink filter F of P such that $\widehat{\varphi}(F)^c = U$.*

Proof Let $U \subseteq \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ be an open subset of $\mathbf{X}(P)$. Since $\mathcal{B}_P$ is a base, it follows that there exists $A \subseteq P$ such that $U = \bigcup_{a \in A} \varphi(a)^c$. It is not hard to see that $U = \widehat{\varphi}(A)^c$. Hence, $U = \widehat{\varphi}(F)^c$ where $F := \mathrm{C}_{\mathrm{F}}(A)$. Conversely, let F be a Frink filter of P. Notice that $\widehat{\varphi}(F) = \bigcap_{a \in F} \varphi(a)$. Then, $\widehat{\varphi}(F)^c = \bigcup_{a \in F} \varphi(a)^c$ and therefore $\widehat{\varphi}(F)^c$ is an open subset of $\mathbf{X}(P)$. □

Proposition 4.9 *A set $U \subseteq \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ is a compact open subset of $\mathbf{X}(P)$ if and only if there exists a finite $A \subseteq P$ such that $U = \widehat{\varphi}\left(A^{\ell u}\right)^c$.*

Proof Let U be a compact open subset of the space $\mathbf{X}(P)$. By the previous proposition, there is $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ such that $U = \widehat{\varphi}(F)^c = \bigcup_{a \in F} \varphi(a)^c$. Since U is compact, it follows that there is a finite $A \subseteq F$ such that $U = \bigcup_{a \in A} \varphi(a)^c = \widehat{\varphi}(A)^c = \widehat{\varphi}\left(A^{\ell u}\right)^c$. Conversely, let $A \subseteq P$ be finite. We want to prove that $\widehat{\varphi}(A^{\ell u})^c$ is a compact open subset of the space $\mathbf{X}(P)$. Since $\widehat{\varphi}\left(A^{\ell u}\right)^c = \widehat{\varphi}(A)^c = \bigcup_{a \in A} \varphi(a)^c$ and each $\varphi(a)^c$ is a compact open subset, we have $\widehat{\varphi}\left(A^{\ell u}\right)^c$ is a finite union of compact open subsets. Hence, $\widehat{\varphi}\left(A^{\ell u}\right)^c$ is a compact open subset of the space $\mathbf{X}(P)$. □

Let us denote by $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P)$ the collection of all finitely generated Frink filters of P. The following corollary is an immediate consequence of the two previous propositions and using Theorem 3.9.

Corollary 4.10 *The map $\widehat{\varphi}(.)^c\colon \mathsf{Fi}_{\mathsf{F}}(P) \to \tau_P$ is an order-isomorphism. Moreover, the restriction $\widehat{\varphi}(.)^c\colon \mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P) \to \mathsf{KO}^*(\boldsymbol{X}(P))$ is an order-isomorphism, where we consider $\mathsf{KO}^*(\boldsymbol{X}(P)) = \mathsf{KO}(\boldsymbol{X}(P))$, in case P has top, and $\mathsf{KO}^*(\boldsymbol{X}(P)) = \mathsf{KO}(\boldsymbol{X}(P)) \setminus \{\emptyset\}$, otherwise.*

It is straightforward to check directly that the specialization quasi-order $\preceq$ of $\mathbf{X}(P)$ is the dual of the inclusion order, i.e., for every $F, G \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$, $F \preceq G$ if and only if $G \subseteq F$. Therefore, $\mathbf{X}(P)$ is a T_0-space. Moreover, it follows that in the poset $\langle \mathbf{X}(P), \preceq \rangle$ we have $\downarrow F = \widehat{\varphi}(F)$ for every $F \in \mathbf{X}(P)$.

Proposition 4.11 *The space $\boldsymbol{X}(P) = \left\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P), \tau_P \right\rangle$ is sober.*

Proof We already know that $\mathbf{X}(P)$ is a T_0-space. Let Z be an irreducible closed subset of $\mathbf{X}(P)$. Since Z^c is an open subset, it follows by Proposition 4.8 that there exists a Frink filter F of P such that $Z^c = \widehat{\varphi}(F)^c$. So, $Z = \widehat{\varphi}(F)$. Now we show that F is a prime Frink filter of P. Since $Z \neq \emptyset$, we have $F \neq P$. Let $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$ be such that $F_1 \cap F_2 \subseteq F$. By Corollary 4.10, we have that $\widehat{\varphi}(F_1)$ and $\widehat{\varphi}(F_2)$ are closed subsets of $\mathbf{X}(P)$ and $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_1) \cup \widehat{\varphi}(F_2)$. Since $\widehat{\varphi}(F) = Z$ is an irreducible closed set, we obtain $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_1)$ or $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_2)$; which implies, by Corollary 4.10, $F_1 \subseteq F$ or $F_2 \subseteq F$. Hence, F is a prime Frink filter of P and $Z = \widehat{\varphi}(F) = \downarrow F$. Therefore, $\mathbf{X}(P)$ is sober. □

Now we are ready to show that the dual space $\mathbf{X}(P)$ of a mo-distributive poset P is a DP-space.

Proposition 4.12 *Let P be a mo-distributive poset. Then, $\langle \mathbf{X}(P), \mathcal{B}_P\rangle$ is a* DP*-space.*

Proof By Proposition 4.11, we have $\mathbf{X}(P)$ is sober. By Corollary 4.6 and by Proposition 4.7, $\mathcal{B}_P$ is a base of compact open subsets for $\mathbf{X}(P)$. It only remains to prove that $\mathcal{B}_P$ satisfies (1) of condition (DP2). To attain this, we show that $\mathcal{B}_P$ satisfies condition (1) in Proposition 4.2. Let $A \subseteq P$ be finite and let $b \in P$ be such that

$$(\forall x \in P)\left(\bigcup_{a\in A} \varphi(a)^c \subseteq \varphi(x)^c \implies \varphi(b)^c \subseteq \varphi(x)^c\right). \tag{2}$$

We need to prove $\varphi(b)^c \subseteq \bigcup_{a\in A} \varphi(a)^c$. To this end, let us show that Eq. 2 implies $b \in A^{\ell u}$. Let $x \in P$ be such that $x \leq a$ for all $a \in A$. So $\varphi(x) \subseteq \bigcap_{a\in A} \varphi(a)$. Then, by Eq. 2, we have $\varphi(x) \subseteq \varphi(b)$. Hence, by Theorem 3.9, we obtain that $x \leq b$. Thus $b \in A^{\ell u}$. Now, let $F \in \varphi(b)^c$. So $b \notin F$. Since $b \in A^{\ell u}$, it follows that $A \cap F = \emptyset$ and thus $F \in \varphi(a)^c$ for all $a \in A$. Hence $\varphi(b)^c \subseteq \bigcup_{a\in A} \varphi(a)^c$. Therefore, $\langle \mathbf{X}(P), \mathcal{B}_P\rangle$ is a DP-space. □

Let P be a mo-distributive poset. Since $\mathbf{X}(P)$ is a DP-space, we have its dual poset $P_{\mathbf{X}(P)}$ and moreover we have that $P_{\mathbf{X}(P)} = \{\varphi(a) : a \in P\}$.

Proposition 4.13 (Representation theorem) *The map $\varphi\colon P \to P_{\mathbf{X}(P)}$ is an order-isomorphism.*

Proof It is clear that φ is an onto map. That it is order-preserving follows from the fact that filters are up-sets. Reciprocally, assume $\varphi(a) \subseteq \varphi(b)$. If $a \nleq b$, then $\uparrow a \cap \downarrow b = \emptyset$. So, by Theorem 3.9, there exists $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $a \in F$ and $b \notin F$. We thus obtain $F \in \varphi(a)$ and $F \notin \varphi(b)$, which is a contradiction. Hence, $a \leq b$. We have proved that φ is an order-embedding. Therefore, φ is an order-isomorphism. □

Let $\langle \mathbf{X}, \mathcal{B}\rangle$ be a DP-space and let $\langle P_{\mathbf{X}}, \subseteq\rangle$ its dual mo-distributive poset. By Proposition 4.12, we can consider the dual DP-space $\mathbf{X}(P_{\mathbf{X}})$ of the poset $P_{\mathbf{X}}$. We want to prove that the DP-spaces $\mathbf{X}$ and $\mathbf{X}(P_{\mathbf{X}})$ are homeomorphic. To this end, we define the map $\theta_{\mathbf{X}}\colon \mathbf{X} \to \mathcal{P}(P_{\mathbf{X}})$ by setting

$$\theta_{\mathbf{X}}(x) := \{A \in P_{\mathbf{X}} :\ x \in A\}$$

for every $x \in X$. As usual, we omit the subscript on θ whenever confusion is unlikely. The next proposition shows that the range of this map is included in $\mathbf{X}(P_{\mathbf{X}})$.

Proposition 4.14 *Let $\langle X, \mathcal{B}\rangle$ be a* DP*-space. For every $x \in X$, $\theta(x)$ is a prime Frink filter of $P_{\mathbf{X}}$.*

Proof Let $\mathbf{X}$ be a DP-space and $x \in X$. Let $\mathcal{A} \subseteq \theta(x)$ be finite. First suppose that $\mathcal{A} = \emptyset$. If $P_{\mathbf{X}}$ has not a top element, then $\mathcal{A}^{\ell u} = \emptyset$ and if $P_{\mathbf{X}}$ has a top element, then $\mathcal{A}^{\ell u} = \{X\}$. In both cases we have $\mathcal{A}^{\ell u} \subseteq \theta(x)$. Now suppose $\mathcal{A}$ is non-empty and let $B \in \mathcal{A}^{\ell u}$. Then, by Proposition 4.3, we obtain $\bigcap \mathcal{A} \subseteq B$. Since $x \in \bigcap \mathcal{A}$, it follows that $x \in B$ and so $B \in \theta(x)$. Thus $\theta(x)$ is a Frink filter of $P_{\mathbf{X}}$ and, since $\mathcal{B}$ is a base for $\mathbf{X}$, we have that $\theta(x) \neq P_{\mathbf{X}}$. We show now that the Frink filter $\theta(x)$ is prime. To prove this, we show that $\theta(x)^c$ is an up-directed subset of $P_{\mathbf{X}}$. Let $A, B \in \theta(x)^c$. So, $x \in A^c \cap B^c$. Since $A^c \cap B^c$ is an open subset, it follows that there is $U \in \mathcal{B}$ such that $x \in U \subseteq A^c \cap B^c$. Then, $U^c \in \theta(x)^c$

and $A, B \subseteq U^c$. Hence, $\theta(x)^c$ is an up-directed subset and therefore $\theta(x)$ is a prime Frink filter of $P_{\mathbf{X}}$. □

Theorem 4.15 *Let $\langle X, \mathcal{B}\rangle$ be a* DP*-space. Then, $\theta\colon \boldsymbol{X} \to \boldsymbol{X}(P_{\boldsymbol{X}})$ is a homeomorphism such that $\{\theta[U] : U \in \mathcal{B}\}$ is the corresponding basis of the* DP*-space $\langle \boldsymbol{X}(P_{\boldsymbol{X}}), \mathcal{B}_{P_{\boldsymbol{X}}}\rangle$.*

Proof From the previous proposition we know that θ is well-defined. Notice that the basic open subsets of the DP-space $\mathbf{X}(P_{\mathbf{X}})$ are of the form $\varphi(A)^c = \{F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}}) : A \notin F\}$ for each $A \in P_{\mathbf{X}}$. So, to prove that θ is continuous, let $A \in P_{\mathbf{X}}$ and $x \in X$. Then, $x \in \theta^{-1}[\varphi(A)^c]$ if and only if $x \in A^c$, and since $A^c \in \mathcal{B}$, it follows that $\theta^{-1}[\varphi(A)^c]$ is an open subset of $\mathbf{X}$. Hence, θ is continuous. Next, we show θ is an onto map. Let $F \in \mathbf{X}(P_{\mathbf{X}}) = \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. It is straightforward to show directly that $C_F := \bigcap F$ is an irreducible closed subset of $\mathbf{X}$. Now, since $\mathbf{X}$ is a sober space, we have that there exists a unique $x \in X$ such that $\mathrm{cl}(x) = C_F$. Let $A \in P_{\mathbf{X}}$, then we have $A \in \theta(x) \iff x \in A \iff \mathrm{cl}(x) \subseteq A \iff C_F \subseteq A \iff A \in F$. We thus obtain that $\theta(x) = F$ and therefore θ is onto. Now, let us show that θ is an open map. Let $U \in \mathcal{B}$. So, we have

$$F \in \theta[U] \iff (\exists x \in U)(F = \theta(x)) \iff (\exists x \in X)(U^c \notin \theta(x) = F) \iff F \in \varphi(U^c)^c.$$

Hence, θ is an open map. It should be noted that to justify the last equivalence it is necessary to use the fact that θ is an onto map. Finally, since $\mathbf{X}$ is a T_0-space, it is clear that θ is an injective map. Therefore, $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is a homeomorphism. Moreover from the proof above showing that θ is an open map follows that $\{\theta[U] : U \in \mathcal{B}\} = \mathcal{B}_{P_{\mathbf{X}}}$. □

Corollary 4.16 *Let P be a mo-distributive poset and let $\boldsymbol{X}$ be a* DP*-space. Then, $\boldsymbol{X}(P)$ is a* DP*-space, $P_{\boldsymbol{X}}$ is a mo-distributive poset and $P \cong P_{\boldsymbol{X}(P)}$ and $\boldsymbol{X} \cong \boldsymbol{X}(P_{\boldsymbol{X}})$.*

4.2 Functorial Duality Between the Categories $\mathbb{MODP}$ and $\mathbb{DPS}$

The primary purpose of this subsection is to extend the results obtained in the previous subsection to a full categorical duality between the category $\mathbb{MODP}$ and a certain category of DP-spaces. The first step to achieving this goal is to give an adequate definition of morphism between DP-spaces. The work of Celani in [2] (see also [3]) motivates the kind of morphism that we define between DP-spaces.

Let $\langle \mathbf{X}, \mathcal{B}_{\mathbf{X}}\rangle$ and $\langle \mathbf{Y}, \mathcal{B}_{\mathbf{Y}}\rangle$ be DP-spaces and let $R \subseteq X \times Y$ be a binary relation. We define the map $h_R\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by setting $h_R(Z) := \{x \in X : R[x] \subseteq Z\}$ for every $Z \subseteq Y$.

Definition 4.17 Let $\langle \mathbf{X}, \mathcal{B}_{\mathbf{X}}\rangle$ and $\langle \mathbf{Y}, \mathcal{B}_{\mathbf{Y}}\rangle$ be DP-spaces. A binary relation $R \subseteq X \times Y$ is said to be a DP-*morphism* if

(M1) for every $B \in P_{\mathbf{Y}}$, $h_R(B) \in P_{\mathbf{X}}$;
(M2) for every $x \in X$, $R[x]$ is a closed subset of $\mathbf{Y}$.

In this case, we write $R \subseteq \mathbf{X} \times \mathbf{Y}$.

Notice that condition (M1) tells us that the restriction of h_R to $P_{\mathbf{Y}}$ is a map from the poset $P_{\mathbf{Y}}$ to the poset $P_{\mathbf{X}}$. Moreover, it is not hard to check that for every $Z_1, Z_2 \subseteq Y$, $h_R(Z_1 \cap Z_2) = h_R(Z_1) \cap h_R(Z_2)$ and $h_R(Y) = X$.

Proposition 4.18 *Let $\langle \boldsymbol{X}, \mathcal{B}_{\boldsymbol{X}}\rangle$ and $\langle \boldsymbol{Y}, \mathcal{B}_{\boldsymbol{Y}}\rangle$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a* DP*-morphism. Then, the map $h_R\colon P_{\boldsymbol{Y}} \to P_{\boldsymbol{X}}$ is an inf-homomorphism.*

Proof Notice that if $B_1, B_2 \in P_{\mathbf{Y}}$ and $B_1 \wedge B_2$ exists in $P_{\mathbf{Y}}$, then $B_1 \wedge B_2 = B_1 \cap B_2$. Then, by the previous observation, we have that h_R preserves all existing finite meets. Hence, since $P_{\mathbf{Y}}$ is a mo-distributive poset, by Proposition 3.15 it follows that h_R is an inf-homomorphism. □

Unfortunately, the usual set-theoretic relational composition of two DP-morphisms may not be a DP-morphism. So, we need to introduce an adequate notion of composition between DP-morphisms. To this end, we follow the same approach as Celani and Calomino followed in [3].

Definition 4.19 Let $\langle \mathbf{X}, \mathcal{B}_{\mathbf{X}}\rangle$, $\langle \mathbf{Y}, \mathcal{B}_{\mathbf{Y}}\rangle$ and $\langle \mathbf{Z}, \mathcal{B}_{\mathbf{Z}}\rangle$ be DP-spaces and let $R \subseteq \mathbf{X} \times \mathbf{Y}$ and $S \subseteq \mathbf{Y} \times \mathbf{Z}$ be DP-morphisms. We define the binary relation $S * R \subseteq X \times Z$ as follows: for every $x \in X$,

$$(S * R)[x] := \mathrm{cl}((R \circ S)[x]). \tag{3}$$

Remark 4.20 Since $\mathcal{B}_{\mathbf{Z}}$ is a base for $\mathbf{Z}$ and $P_{\mathbf{Z}} = \{U^c : U \in \mathcal{B}_{\mathbf{Z}}\}$, it follows that for every $(x, z) \in X \times Z$, $(x, z) \in S * R \iff (\forall C \in P_{\mathbf{Z}})((R \circ S)[x] \subseteq C \implies z \in C)$.

The following facts gathered in the next proposition are not difficult to prove. We leave the details to the reader.

Proposition 4.21 *Let $\langle \mathbf{X}, \mathcal{B}_{\mathbf{X}}\rangle$, $\langle \mathbf{Y}, \mathcal{B}_{\mathbf{Y}}\rangle$, $\langle \mathbf{Z}, \mathcal{B}_{\mathbf{Z}}\rangle$ and $\langle \mathbf{W}, \mathcal{B}_{\mathbf{W}}\rangle$ be* DP*-spaces and let $R \subseteq \mathbf{X} \times \mathbf{Y}$, $S \subseteq \mathbf{Y} \times \mathbf{Z}$ and $T \subseteq \mathbf{Z} \times \mathbf{W}$ be* DP*-morphisms. Then,*

(1) *for every $C \in P_{\mathbf{Z}}$, $h_{S*R}(C) = (h_R \circ h_S)(C)$;*
(2) *$S * R$ is a* DP*-morphism;*
(3) *$T * (S * R) = (T * S) * R$;*
(4) *the dual specialization order of $\mathbf{X}$, $\succeq_{\mathbf{X}}$, is a* DP*-morphism;*
(5) *$R * \succeq_{\mathbf{X}} = R$ and $\succeq_{\mathbf{Y}} * R = R$;*
(6) *$h_{\succeq_{\mathbf{X}}} = \mathrm{id}_{P_{\mathbf{X}}}$.*

Now we can define the category of all DP-spaces and all DP-morphisms, where the composition between DP-morphisms is $*$ and for every DP-space $\mathbf{X}$ the identity DP-morphism is $\succeq_{\mathbf{X}}$. We denote this category by $\mathbb{DPS}$.

We define $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ as follows:

- for every DP-space $\mathbf{X}$, $\Delta(\mathbf{X}) := \langle P_{\mathbf{X}}, \subseteq\rangle$;
- for every morphism $R \subseteq \mathbf{X} \times \mathbf{Y}$ of $\mathbb{DPS}$, $\Delta(R) := h_R\colon P_{\mathbf{Y}} \to P_{\mathbf{X}}$.

By Propositions 4.4 and 4.18, we have that Δ sends objects and morphisms from $\mathbb{DPS}$ to objects and morphisms of $\mathbb{MODP}$, respectively, and by Proposition 4.21 we obtain that Δ is a contravariant functor.

Now we want to find a contravariant functor from $\mathbb{MODP}$ to $\mathbb{DPS}$. To this end, by Proposition 4.12, we need only define the image of morphisms of the category $\mathbb{MODP}$. So, let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. We define a binary relation $R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P)$ as follows:

$$G R_h F \iff h^{-1}[G] \subseteq F$$

for every pair $(G, F) \in \mathbf{X}(Q) \times \mathbf{X}(P)$.

Proposition 4.22 *Let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. Then,*

(1) *for every $a \in P$, $h_{R_h}(\varphi(a)) = \varphi(h(a))$;*
(2) *R_h is a DP-morphism;*
(3) *$R_{\mathrm{id}_P} = \succeq_{X(P)}$.*

Proof (1) Let $a \in P$ and let $G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q)$. Then, by Proposition 3.12 and Theorem 3.9, we have

$$G \in \varphi(h(a)) \iff a \in h^{-1}[G] \iff (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P))(h^{-1}[G] \subseteq F \implies a \in F)$$
$$\iff (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P))(G R_h F \implies F \in \varphi(a)) \iff R_h[G] \subseteq \varphi(a) \iff G \in h_{R_h}(\varphi(a)).$$

Hence, $h_{R_h}(\varphi(a)) = \varphi(h(a))$ for all $a \in P$.

(2) By (1), we have that condition (M1) of Definition 4.17 holds. To prove condition (M2), let $G \in \mathbf{X}(Q)$. It is clear that $R_h[G] \subseteq \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : R_h[G] \subseteq \varphi(a)\}$. In order to show the other inclusion let $F \in \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : R_h[G] \subseteq \varphi(a)\}$ and let $b \in h^{-1}[G]$. So, by (1), we have $R_h[G] \subseteq \varphi(b)$. Then $F \in \varphi(b)$, which implies $b \in F$. We thus obtain $h^{-1}[G] \subseteq F$. Then, $F \in R_h[G]$. Hence, $R_h[G] = \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : R_h[G] \subseteq \varphi(a)\}$. Therefore, R_h is a DP-morphism.

(3) It follows directly from the definition of R_{id_P}. □

Proposition 4.23 *Let P_1, P_2 and P_3 be mo-distributive posets and let $h_1\colon P_1 \to P_2$ and $h_2\colon P_2 \to P_3$ be inf-homomorphisms. Then, $R_{(h_2 \circ h_1)} = R_{h_1} * R_{h_2}$.*

Proof It is not difficult to see that for every DP-morphisms $R_1, R_2 \subseteq \mathbf{X} \times \mathbf{Y}$, if the restrictions of h_{R_1} and h_{R_2} to $P_{\mathbf{Y}}$ are the same, then $R_1 = R_2$. Thus, it is enough to show that $h_{R_{(h_2 \circ h_1)}} = h_{R_{h_1} * R_{h_2}}$. Let $a \in P_1$. Then, using Proposition 4.22 we have

$$h_{R_{(h_2 \circ h_1)}}(\varphi(a)) = \varphi\left((h_2 \circ h_1)(a)\right) = \varphi\left(h_2(h_1(a))\right)$$
$$= h_{R_{h_2}}\left(\varphi(h_1(a))\right) = h_{R_{h_2}}(h_{R_{h_1}}(\varphi(a))) = (h_{R_{h_2}} \circ h_{R_{h_1}})(\varphi(a)).$$

We thus obtain $h_{R_{(h_2 \circ h_1)}} = h_{R_{h_2}} \circ h_{R_{h_1}}$. Now, by condition (1) of Proposition 4.21, we know that $h_{R_{h_2}} \circ h_{R_{h_1}} = h_{(R_{h_1} * R_{h_2})}$. We thus obtain $h_{R_{(h_2 \circ h_1)}} = h_{(R_{h_1} * R_{h_2})}$ and this implies that $R_{(h_2 \circ h_1)} = R_{h_1} * R_{h_2}$. This completes the proof. □

Now we define $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$ as follows:

- for every mo-distributive poset P, $\Gamma(P) := \langle \mathbf{X}(P), \mathcal{B}_P \rangle$;
- for every morphism $h\colon P \to Q$ of the category $\mathbb{MODP}$, $\Gamma(h) := R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P)$.

Therefore, by Propositions 4.12 and 4.22, we have that Γ sends objects and morphisms from $\mathbb{MODP}$ to objects and morphisms of $\mathbb{DPS}$ and by Propositions 4.22 and 4.23, we obtain that Γ is a contravariant functor.

Our purpose is to prove that the categories $\mathbb{MODP}$ and $\mathbb{DPS}$ are dually equivalent via the functors $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ and $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$. So, we need to define natural equivalences $\eta\colon \mathrm{Id}_{\mathbb{DPS}} \cong \Gamma \circ \Delta$ and $\mu\colon \mathrm{Id}_{\mathbb{MODP}} \cong \Delta \circ \Gamma$, where $\mathrm{Id}_{\mathbb{MODP}}$ and $\mathrm{Id}_{\mathbb{DPS}}$ are the identity functors on the categories $\mathbb{MODP}$ and $\mathbb{DPS}$, respectively.

Let $\langle \mathbf{X}, \mathcal{B}\rangle$ be a DP-space. Recall that we have defined the homeomorphism $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ by $\theta(x) = \{A \in P_{\mathbf{X}} : x \in A\}$. We define the binary relation $R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}})$ as follows: for every $x \in X$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$,

$$x R_\theta F \iff \theta(x) \subseteq F.$$

Notice that for every $x \in X$, $x R_\theta \theta(x)$ because $\theta(x) \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. We need also to consider the inverse homeomorphism $\theta^{-1}\colon \mathbf{X}(P_{\mathbf{X}}) \to \mathbf{X}$. We can define the binary relation $R_{\theta^{-1}} \subseteq \mathbf{X}(P_{\mathbf{X}}) \times \mathbf{X}$ as follows: for every $x \in X$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$,

$$F R_{\theta^{-1}} x \iff \theta^{-1}(F) \succeq_{\mathbf{X}} x.$$

Proposition 4.24 *Let $\langle X, \mathcal{B}\rangle$ be a DP-space. Then,*

(1) *R_θ is a DP-morphism;*
(2) *$R_{\theta^{-1}}$ is a DP-morphism;*
(3) *$R_{\theta^{-1}} * R_\theta = \succeq_X$ and $R_\theta * R_{\theta^{-1}} = \succeq_{X(P_X)}$.*

Proof (1) To prove condition (M1), let $A \in P_{\mathbf{X}}$. We show that $h_{R_\theta}(\varphi(A)) = A$. Let $x \in h_{R_\theta}(\varphi(A))$. So, $R_\theta[x] \subseteq \varphi(A)$ and since $\theta(x) \in R_\theta[x]$, we have $A \in \theta(x)$. That is, $x \in A$. Conversely, let $x \in A$. So, $A \in \theta(x)$. Let $F \in R_\theta[x]$. Thus, $\theta(x) \subseteq F$; this implies that $A \in F$. Then, $F \in \varphi(A)$. Hence, we have proved that $R_\theta[x] \subseteq \varphi(A)$ and thus, $x \in h_{R_\theta}(\varphi(A))$. Therefore, condition (M1) holds. To prove (M2), let $x \in X$. It should be noted that

$$R_\theta[x] = \bigcap\{\varphi(A) : \ A \in P_{\mathbf{X}} \text{ and} x \in A\}.$$

Thus $R_\theta[x]$ is a closed subset of $\mathbf{X}(P_{\mathbf{X}})$ and hence, condition (M2) holds. Therefore, R_θ is a DP-morphism.

(2) Notice that for every $A \in P_{\mathbf{X}}$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$, we have that

$$R_{\theta^{-1}}[F] \subseteq A \text{ if and only if} A \in F. \tag{4}$$

Let $A \in P_{\mathbf{X}}$. We prove that $h_{R_{\theta^{-1}}}(A) = \varphi(A)$. Let $F \in h_{R_{\theta^{-1}}}(A)$. So, $R_{\theta^{-1}}[F] \subseteq A$ and from Eq. 4, it follows that $A \in F$. Hence, $F \in \varphi(A)$. We now assume that $F \in \varphi(A)$. So, by Eq. 4, we have that $R_{\theta^{-1}}[F] \subseteq A$; this implies that $F \in h_{R_{\theta^{-1}}}(A)$. Hence, we have proved that $h_{R_{\theta^{-1}}}(A) = \varphi(A) \in P_{\mathbf{X}(P_{\mathbf{X}})}$ for all $A \in P_{\mathbf{X}}$. Therefore, condition (M1) holds. To prove condition (M2), let $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. We prove that $R_{\theta^{-1}}[F] = \bigcap\{A \in P_{\mathbf{X}} : \ A \in F\}$. It is immediate that $R_{\theta^{-1}}[F] \subseteq \bigcap\{A \in P_{\mathbf{X}} : \ A \in F\}$. Let $x \in \bigcap\{A \in P_{\mathbf{X}} : \ A \in F\}$ and let $A \in P_{\mathbf{X}}$ be such that $\theta^{-1}(F) \in A$. So, $A \in \theta(\theta^{-1}(F)) = F$ and then $x \in A$. Thus, $\theta^{-1}(F) \succeq_{\mathbf{X}} x$. That is, $x \in R_{\theta^{-1}}[F]$. Then $R_{\theta^{-1}}[F]$ is a closed subset of $\mathbf{X}$ and hence condition (M2) holds. Therefore, $R_{\theta^{-1}}$ is a DP-morphism.

(3) First we show that $R_{\theta^{-1}} * R_\theta = \succeq_{\mathbf{X}}$. So, let $x, x' \in X$. We assume $x(R_{\theta^{-1}} * R_\theta)x'$. Thus, for every $A \in P_{\mathbf{X}}$, if $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$ then $x' \in A$. To prove that $x \succeq_{\mathbf{X}} x'$, let $A \in P_{\mathbf{X}}$ be such that $x \in A$. Let $x'' \in (R_\theta \circ R_{\theta^{-1}})[x]$. So, there exists $F \in \mathbf{X}(P_{\mathbf{X}})$ such that $x R_\theta F$ and $F R_{\theta^{-1}} x''$. That is, $\theta(x) \subseteq F$ and $\theta^{-1}(F) \succeq_{\mathbf{X}} x''$. As $x \in A$, we have $A \in \theta(x)$, which implies that $A \in F = \theta(\theta^{-1}(F))$. Then $\theta^{-1}(F) \in A$ and, since $\theta^{-1}(F) \succeq_{\mathbf{X}} x''$, it follows that $x'' \in A$. Thus, $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$ and then, by hypothesis, $x' \in A$. Hence, we have proved that for every $A \in P_{\mathbf{X}}$, if $x \in A$ then $x' \in A$; which implies that $x \succeq_{\mathbf{X}} x'$. Conversely, we suppose that $x \succeq_{\mathbf{X}} x'$. We need to prove that $x(R_{\theta^{-1}} * R_\theta)x'$. Let $A \in P_{\mathbf{X}}$ be such that $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$. Notice that $x(R_\theta \circ R_{\theta^{-1}})x$, because $x R_\theta \theta(x)$ and $\theta(x) R_{\theta^{-1}} x$. We thus obtain that $x \in A$ and, since $x \succeq_{\mathbf{X}} x'$, we have $x' \in A$. Hence, $x(R_{\theta^{-1}} * R_\theta)x'$. Therefore, $R_{\theta^{-1}} * R_\theta = \succeq_{\mathbf{X}}$.

Now we prove that $R_\theta * R_{\theta^{-1}} = \succeq_{\mathbf{X}(P_{\mathbf{X}})}$. Let $F_1, F_2 \in \mathbf{X}(P_{\mathbf{X}})$. Let us first assume that $F_1(R_\theta * R_{\theta^{-1}})F_2$. Then, we have

$$(\forall A \in P_{\mathbf{X}})\left((R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A) \implies F_2 \in \varphi(A)\right).$$

Given that $\mathcal{B}_{P_{\mathbf{X}}} = \{\varphi(A)^c : A \in P_{\mathbf{X}}\}$ is a base for the DP-space $\mathbf{X}(P_{\mathbf{X}})$, it follows that the dual specialization order $\succeq_{\mathbf{X}(P_{\mathbf{X}})}$ of $\mathbf{X}(P_{\mathbf{X}})$ can be defined as

$$F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2 \iff (\forall A \in P_{\mathbf{X}})\,(F_1 \in \varphi(A) \implies F_2 \in \varphi(A)).$$

Let $A \in P_{\mathbf{X}}$ be such that $F_1 \in \varphi(A)$. We show that $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. Let $F \in (R_{\theta^{-1}} \circ R_\theta)[F_1]$. Then, there exists $x \in X$ such that $F_1 R_{\theta^{-1}} x$ and $x R_\theta F$. That is, $\theta^{-1}(F_1) \succeq_{\mathbf{X}} x$ and $\theta(x) \subseteq F$. Notice that $\theta(x) \subseteq F$ is equivalent to $\theta(x) \succeq_{\mathbf{X}(P_{\mathbf{X}})} F$. Since $\theta^{-1} : \mathbf{X}(P_{\mathbf{X}}) \to \mathbf{X}$ is a homeomorphism, it follows that θ^{-1} is order-preserving with respect to the specialization order. We thus obtain $\theta^{-1}(\theta(x)) \succeq_{\mathbf{X}} \theta^{-1}(F)$ and then $x \succeq_{\mathbf{X}} \theta^{-1}(F)$. By the transitivity of $\succeq_{\mathbf{X}}$, we obtain $\theta^{-1}(F_1) \succeq_{\mathbf{X}} \theta^{-1}(F)$. Using the fact that θ is order-preserving, because it is a homeomorphism, we have that $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F$ and, since $F_1 \in \varphi(A)$, it follows that $F \in \varphi(A)$. Thus, $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. Hence, by hypothesis, $F_2 \in \varphi(A)$. Therefore, $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2$. Now, conversely, we assume that $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2$. It follows that $(\forall A \in P_{\mathbf{X}})\,(F_1 \in \varphi(A) \implies F_2 \in \varphi(A))$. Let $A \in P_{\mathbf{X}}$ be such that $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. It should be noted that $F_1 \in (R_{\theta^{-1}} \circ R_\theta)[F_1]$ because $F_1 R_{\theta^{-1}} \theta^{-1}(F_1)$ and $\theta^{-1}(F_1) R_\theta F_1$. Then, $F_1 \in \varphi(A)$ and so, $F_2 \in \varphi(A)$. Hence, $F_1(R_\theta * R_{\theta^{-1}})F_2$. This finishes the proof. $\square$

We are now ready to establish the main result of this section.

Theorem 4.25 *The categories $\mathbb{MODP}$ and $\mathbb{DPS}$ are dually equivalent via the above functors $\Delta : \mathbb{DPS} \to \mathbb{MODP}$ and $\Gamma : \mathbb{MODP} \to \mathbb{DPS}$.*

Proof As outlined above, it only remains to define the natural equivalences

$$\mu : \mathrm{Id}_{\mathbb{MODP}} \cong \Delta \circ \Gamma \quad \text{and} \quad \eta : \mathrm{Id}_{\mathbb{DPS}} \cong \Gamma \circ \Delta.$$

We consider the following definitions:

- for every mo-distributive poset P, $\mu(P) = \varphi : P \to P_{\mathbf{X}(P)}$;
- for every DP-space $\mathbf{X}$, $\eta(\mathbf{X}) = R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}})$.

By Propositions 4.13 and 4.24 we have that for every mo-distributive poset P and every DP-space $\mathbf{X}$, $\mu(P) = \varphi$ and $\eta(\mathbf{X}) = R_\theta$ are isomorphisms of the categories $\mathbb{MODP}$ and $\mathbb{DPS}$, respectively. Lastly, we show that for every morphism $h : P_1 \to P_2$ of the category $\mathbb{MODP}$ and every morphism $R \subseteq \mathbf{X}_1 \times \mathbf{X}_2$ of the category $\mathbb{DPS}$ the diagrams in Fig. 2 commute.

$$\begin{tikzcd} P_1 \arrow[r, "h"] \arrow[d, "\varphi_1"'] & P_2 \arrow[d, "\varphi_2"] \\ P_{\mathbf{X}(P_1)} \arrow[r, "h_{R_h}"'] & P_{\mathbf{X}(P_2)} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X}_1 \arrow[r, "R"] \arrow[d, "R_{\theta_1}"'] & \mathbf{X}_2 \arrow[d, "R_{\theta_2}"] \\ \mathbf{X}(P_{\mathbf{X}_1}) \arrow[r, "R_{h_R}"'] & \mathbf{X}(P_{\mathbf{X}_2}) \end{tikzcd}$$

Fig. 2 Commutative diagrams of morphisms in the categories $\mathbb{MODP}$ and $\mathbb{DPS}$

That the diagram on the left hand side of Fig. 2 commutes is a consequence of (1) of Proposition 4.22. For the diagram on the right hand side of Fig. 2, we must show that $R_{h_R} * R_{\theta_1} = R_{\theta_2} * R$. To this end, we first show that for every $x \in X_1$

$$(R_{\theta_1} \circ R_{h_R})[x] = (R \circ R_{\theta_2})[x]. \tag{5}$$

Let $G \in (R_{\theta_1} \circ R_{h_R})[x]$. So, there exists $F \in \mathbf{X}(P_{\mathbf{X}_1})$ such that $xR_{\theta_1}F$ and $FR_{h_R}G$. Then $\theta_1(x) \subseteq F$ and $h_R^{-1}[F] \subseteq G$. Since $G \in \mathbf{X}(P_{\mathbf{X}_2})$, it follows that there is $x_2 \in \mathbf{X}_2$ such that $G = \theta_2(x_2)$ and thus it is clear that $x_2 R_{\theta_2} G$. Now, we want to show that xRx_2. Let $B \in P_{\mathbf{X}_2}$ be such that $R[x] \subseteq B$. Then, we have the following implications:

$$\begin{aligned} R[x] \subseteq B \implies x \in h_R(B) \implies h_R(B) \in \theta_1(x) \implies h_R(B) \in F \\ \implies B \in h_R^{-1}[F] \implies B \in \theta_2(x_2) \implies x_2 \in B. \end{aligned}$$

Hence, by (M2) of Definition 4.17, $x_2 \in R[x]$. We thus obtain xRx_2 and $x_2 R_{\theta_2} G$. Hence, $G \in (R \circ R_{\theta_2})[x]$. Conversely, let $G \in (R \circ R_{\theta_2})[x]$. So, there is $x_2 \in \mathbf{X}_2$ such that xRx_2 and $x_2 R_{\theta_2} G$. Then, $x_2 \in R[x]$ and $\theta_2(x_2) \subseteq G$. Given that $xR_{\theta_1}\theta_1(x)$, we want to show that $\theta_1(x) R_{h_R} G$. Let $B \in h_R^{-1}[\theta_1(x)]$. So, $h_R(B) \in \theta_1(x)$ and this implies that $x \in h_R(B)$. Then, $R[x] \subseteq B$ and thus $x_2 \in B$. That is, $B \in \theta_2(x_2)$ and whereupon $B \in G$. Thus, $h_R^{-1}[\theta_1(x)] \subseteq G$ and hence $\theta_1(x) R_{h_R} G$. Therefore, we have $xR_{\theta_1}\theta_1(x)$ and $\theta_1(x) R_{h_R} G$, i.e., $G \in (R_{\theta_1} \circ R_{h_R})[x]$. Hence, Eq. 5 holds. Then, taking the topological closure to both sides of equality in Eq. 5 and by the definition of $*$, we obtain that $R_{h_R} * R_{\theta_1} = R_{\theta_2} * R$. This completes the proof. □

5 Connection with the Work of David and Erné

In this section, we will see how to obtain, from our spectral-style duality, the topological duality due to David and Erné [5]. We do it for the category of mo-distributive posets and maps that are $\vee$-stable and inf-homomorphisms.

Let us recall that for every mo-distributive poset P, the specialization order of the DP-space $\mathbf{X}(P)$ is the dual of the inclusion order. In this section, whenever we refer to an order or a notion related to an order of a DP-space, we refer to the specialization order.

Definition 5.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces. A DP-morphism $R \subseteq \mathbf{X} \times \mathbf{Y}$ is called *functional* if for every $x \in X$ there exists $y \in Y$ such that $R[x] = \downarrow y$, where $\downarrow y = \{y' \in Y : y' \preceq y\}$.

Proposition 5.2 *Let P and Q be mo-distributive posets and let $h \colon P \to Q$ be an inf-homomorphism. Then, h is $\vee$-stable if and only if the* DP*-morphism R_h is functional.*

Proof We prove first that for each $G \in \mathbf{X}(Q)$, $R_h[G]$ has a top element if and only if $h^{-1}[G] \in \mathbf{X}(P)$. Let $G \in \mathbf{X}(Q)$. Suppose that $R_h[G]$ has a top element $F \in R_h[G]$. So, $h^{-1}[G] \subseteq F$. If $F \neq h^{-1}[G]$, then there exists $a \in F$ such that $a \notin h^{-1}[G]$. Since h is an inf-homomorphism, we have $h^{-1}[G]$ is a Frink filter of P. Then, by Theorem 3.9, there exists $F' \in \mathbf{X}(P)$ such that $h^{-1}[G] \subseteq F'$ and $a \notin F'$. We thus get $F' \in R_h[G]$ and $F' \not\preceq F$, which is a contradiction because F is the top element of $R_h[G]$. Hence, $h^{-1}[G] = F \in \mathbf{X}(P)$. Reciprocally, if $h^{-1}[G] \in \mathbf{X}(P)$ then $h^{-1}[G]$ is the top element of $R_h[G]$ in $\mathbf{X}(P)$. Hence, by Lemma 3.18, it follows that h is a $\vee$-stable map if and only if $R_h[G]$ has a top element for every $G \in \mathbf{X}(Q)$. □

The following lemma shows that the composition $*$ between functional DP-morphisms is the usual set-theoretical composition between relations.

Lemma 5.3 *Let* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be* DP*-spaces and let* $R \subseteq \mathbf{X} \times \mathbf{Y}$ *and* $S \subseteq \mathbf{Y} \times \mathbf{Z}$ *be functional* DP*-morphisms. Then* $S * R = R \circ S$*.*

Proof Let $x \in X$. So, there is $y \in Y$ such that $R[x] = \downarrow y$. Then, there exists $z \in Z$ such that $S[y] = \downarrow z$. We prove that $(R \circ S)[x] = \downarrow z$. Let $z' \in (R \circ S)[x]$. Thus, there is $y' \in Y$ such that $y' \in R[x]$ and $z' \in S[y']$. We thus obtain $y' \preceq y$ and this implies that $S[y'] \subseteq S[y]$. Then, $z' \in S[y] = \downarrow z$. Hence, $(R \circ S)[x] \subseteq \downarrow z$. To show the other inclusion, let $z' \in \downarrow z = S[y]$. Since xRy and ySz', it follows that $z' \in (R \circ S)[x]$. Hence, $\downarrow z \subseteq (R \circ S)[x]$. Then, $(R \circ S)[x] = \downarrow z$ and so it is a closed subset of $\mathbf{Z}$. Hence $(S * R)[x] = \mathrm{cl}\,((R \circ S)[x]) = (R \circ S)[x]$. Thus we obtain $(S * R)[x] = (R \circ S)[x]$ for all $x \in X$ and therefore $S * R = R \circ S$. □

From the previous lemma, it is not hard to check the following proposition.

Proposition 5.4 *Let* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be* DP*-spaces. If* $R \subseteq \mathbf{X} \times \mathbf{Y}$ *and* $S \subseteq \mathbf{Y} \times \mathbf{Z}$ *are functional* DP*-morphisms, then* $S * R$ *is functional.*

It should be noted that for every DP-space $\mathbf{X}$, the identity DP-morphism $\succeq_{\mathbf{X}}$ is functional. Hence, by the previous proposition, we can consider the category of DP-spaces and functional DP-morphisms. We denote this category by $\mathbb{DPS}^{\mathrm{F}}$, which is a subcategory of $\mathbb{DPS}$. Let us also consider the category formed by all mo-distributive posets and all maps between mo-distributive posets that are inf-homomorphism and $\vee$-stable. This category is denoted by $\mathbb{MODP}^{\mathrm{sta}}$. It is clear that $\mathbb{MODP}^{\mathrm{sta}}$ is a subcategory of $\mathbb{MODP}$.

Proposition 5.5 *The categories* $\mathbb{MODP}^{\mathrm{sta}}$ *and* $\mathbb{DPS}^{\mathrm{F}}$ *are dually equivalent via the functors* $\Delta\colon \mathbb{DPS}^{\mathrm{F}} \to \mathbb{MODP}^{\mathrm{sta}}$ *and* $\Gamma\colon \mathbb{MODP}^{\mathrm{sta}} \to \mathbb{DPS}^{\mathrm{F}}$*, which are the restrictions of the functors defined on pages 16 and 17, respectively.*

Now we are going to introduce the topological category considered by David and Erné [5] to establish their dual categorical equivalence. Let $\langle \mathbf{X}, \mathcal{B}_{\mathbf{X}} \rangle$ and $\langle \mathbf{Y}, \mathcal{B}_{\mathbf{Y}} \rangle$ be DP-spaces. A map $f\colon X \to Y$ is called a DP-*function* if for every $V \in \mathcal{B}_{\mathbf{Y}}$, $f^{-1}[V] \in \mathcal{B}_{\mathbf{X}}$. Let us denote by $\mathbb{DPS}^{\mathrm{sta}}$ the category of DP-spaces and DP-functions (this category is denoted in [5, pp. 110] by **SB**).

Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces. For each functional DP-morphism $R \subseteq \mathbf{X} \times \mathbf{Y}$ we define the map $f^R\colon X \to Y$ by setting $f^R(x) :=$ the greatest element of $R[x]$, for every $x \in X$. And for each DP-function $f\colon \mathbf{X} \to \mathbf{Y}$ we define the binary relation $R^f \subseteq X \times Y$ as follows: $xR^f y \iff y \preceq f(x)$, for every pair $(x, y) \in X \times Y$. Using the definitions, the following two lemmas are not hard to prove.

Lemma 5.6 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* DP*-spaces and let* $R \subseteq \mathbf{X} \times \mathbf{Y}$ *be a functional* DP*-morphism. Then, the map* $f^R\colon X \to Y$ *is a* DP*-function. Moreover, if* $\mathbf{Z}$ *is a* DP*-space and* $S \subseteq \mathbf{Y} \times \mathbf{Z}$ *is a functional* DP*-morphism, then we have* $f^{S \circ R} = f^S \circ f^R$*.*

Proof To prove that f^R is a DP-function, let $V \in \mathcal{B}(\mathbf{Y})$ and let $x \in X$. Then, we have

$$x \in f^{R^{-1}}[V^c] \Longleftrightarrow f^R(x) \in V^c \Longleftrightarrow \downarrow f^R(x) \subseteq V^c \Longleftrightarrow R[x] \subseteq V^c \Longleftrightarrow x \in h_R(V^c).$$

So, $f^{R^{-1}}[V^c] = h_R(V^c) \in P_{\mathbf{X}}$ and hence $f^{R^{-1}}[V] \in \mathcal{B}(\mathbf{X})$. Therefore, f^R is a DP-function. Now, let $\mathbf{Z}$ be a DP-space and let $S \subseteq \mathbf{Y} \times \mathbf{Z}$ be a functional DP-morphism. Let $x \in X$. Then,

$$\begin{aligned} f^{R\circ S}(x) &= \text{the greatest element of}(R \circ S)[x] \\ &= \text{the greatest element of} S[y],\ \text{where} y \text{ is the greatest element of} R[x] \\ &= f^S(y),\ \text{where} y \text{ is the greatest element of} R[x] \\ &= f^S(f^R(x)) \\ &= (f^S \circ f^R)(x). \end{aligned}$$

Therefore, $f^{R\circ S} = f^S \circ f^R$. □

Lemma 5.7 *Let X and Y be* DP*-spaces and let $f: X \to Y$ be a* DP*-function. Then, the relation $R^f \subseteq X \times Y$ is a functional* DP*-morphism. Moreover, if $\mathbf{Z}$ is a* DP*-space and $g: Y \to \mathbf{Z}$ is a* DP*-function, then $R^{g\circ f} = R^g \circ R^f$.*

Proof We need to prove conditions (M1) and (M2) of Definition 4.17. Let $B \in P_{\mathbf{Y}}$. Since $R^f[x] = \downarrow f(x)$ for all $x \in X$, we have

$$\begin{aligned} h_{R^f}(B) = \{x \in X :\ R^f[x] \subseteq B\} &= \{x \in X :\ \downarrow f(x) \subseteq B\} \\ &= \{x \in X :\ f(x) \in B\} = f^{-1}[B] \in P_{\mathbf{X}}. \end{aligned}$$

Then, condition (M1) holds. Let now $x \in X$. Since $R^f[x] = \downarrow f(x) = \mathrm{cl}(f(x))$, we obtain that $R^f[x]$ is a closed subset of $\mathbf{Y}$. Then, R^f satisfies condition (M2). Hence, $R^f \subseteq \mathbf{X} \times \mathbf{Y}$ is a DP-morphism. Since $R^f[x] = \downarrow f(x)$ for all $x \in X$, it follows that R^f is functional. This completes the proof of the first part of the lemma.

Let $\mathbf{Z}$ be a DP-space and let $g: \mathbf{Y} \to \mathbf{Z}$ be a DP-function. Let $x \in X$ and $z \in Z$. Then,

$$\begin{aligned} xR^{g\circ f}z \iff z \preceq (g \circ f)(x) \iff z \preceq g(f(x)) &\iff f(x)R^g z \\ &\iff xR^f f(x) \text{ and} f(x)R^g z \iff x(R^f \circ R^g)z. \end{aligned}$$

Hence, $R^{g\circ f} = R^f \circ R^g$. □

The following lemma follows immediately from definitions of f^R and R^f and so we leave the details to the reader.

Lemma 5.8 *Let X and Y be* DP*-spaces. Let $R \subseteq X \times Y$ be a functional* DP*-morphism and let $f: X \to Y$ be a* DP*-function. Then, $R^{f^R} = R$ and $f^{R^f} = f$.*

Putting the last three lemmas together we obtain the following proposition, whose proof we omit.

Proposition 5.9 *The categories $\mathbb{DPS}^{\mathrm{F}}$ and $\mathbb{DPS}^{\mathrm{sta}}$ are isomorphic.*

Then, by the previous proposition and Proposition 5.5, we can directly derive the duality established by David and Erné in [5, Theorem 4.2] (but stated now for mo-distributive posets).

Theorem 5.10 *The categories $\mathbb{MODP}^{\mathrm{sta}}$ and $\mathbb{DPS}^{\mathrm{sta}}$ are dually equivalent.*

Finally, we want to set the explicit construction of the functors that give the dual equivalence between the categories $\mathbb{MODP}^{\text{sta}}$ and $\mathbb{DPS}^{\text{sta}}$. These functors correspond to those defined in [5] by David and Erné to establish their dual equivalence. To this end, the following lemma is central.

Lemma 5.11

(1) *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $f: \boldsymbol{X} \to \boldsymbol{Y}$ be a* DP*-function. Then, for every $B \in P_{\mathbf{Y}}$, $h_{R^f}(B) = f^{-1}[B]$.*

(2) *Let P and Q be mo-distributive posets and let $h: P \to Q$ be a $\vee$-stable inf-homomorphism. Then, $f^{R_h}(G) = h^{-1}[G]$ for all $G \in \boldsymbol{X}(Q)$.*

Proof (1) Let $B \in P_{\mathbf{Y}}$ and let $x \in X$. Then,

$$x \in h_{R^f}(B) \iff R^f[x] \subseteq B \iff {\downarrow} f(x) \subseteq B \iff f(x) \in B \iff x \in f^{-1}[B].$$

Hence, $h_{R^f}(B) = f^{-1}[B]$ for all $B \in P_{\mathbf{Y}}$.

(2) Given that $h: P \to Q$ is a $\vee$-stable inf-homomorphism, we have that $R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P)$ is a functional DP-morphism where $h^{-1}[G]$ is the greatest element of $R_h[G]$ in $\mathbf{X}(P)$ for every $G \in \mathbf{X}(Q)$. By definition of $f^{R_h}: \mathbf{X}(Q) \to \mathbf{X}(P)$, we have $f^{R_h}(G) = h^{-1}[G]$ for every $G \in \mathbf{X}(Q)$. □

Now, by Lemma 5.11, we can make explicit the corresponding functors that establish the dual equivalence in Theorem 5.10:

► $\Gamma^*: \mathbb{MODP}^{\text{sta}} \to \mathbb{DPS}^{\text{sta}}$ is defined as follows:

- for every mo-distributive poset P, $\Gamma^*(P) := \langle \mathbf{X}(P), \mathcal{B}_P \rangle = \Gamma(P)$;
- for every morphism $h: P \to Q$ in $\mathbb{MODP}^{\text{sta}}$, $\Gamma^*(h) := h^{-1}: \mathbf{X}(Q) \to \mathbf{X}(P)$.

► $\Delta^*: \mathbb{DPS}^{\text{sta}} \to \mathbb{MODP}^{\text{sta}}$ is defined as follows:

- for every DP-space $\mathbf{X}$, $\Delta^*(\mathbf{X}) := P_{\mathbf{X}} = \Delta(\mathbf{X})$;
- for every morphism $f: \mathbf{X} \to \mathbf{Y}$ in $\mathbb{DPS}^{\text{sta}}$, $\Delta^*(f) := f^{-1}: P_{\mathbf{Y}} \to P_{\mathbf{X}}$.

6 A Completion for Mo-distributive Posets

The main aim of this section is to provide a proof of the existence of a particular completion for the mo-distributive posets using the dual DP-spaces. This completion will be a Δ_1-completion in the sense of [13].

Let P be a poset. A *completion* of P is a complete lattice L together with an order-embedding $e: P \hookrightarrow L$. A Δ_1*-completion* of P is a completion for which each element can be obtained both as a join of meets of elements of $e[P]$ and as a meet of joins of elements of $e[P]$. A nice and important way to obtain Δ_1-completions is by means of Δ_1-polarities as defined in [13] (see also [12, Remark 2.8]). Given a poset P, we consider here for our purposes only some of these Δ_1-polarities, the triples $(\mathcal{F}, \mathcal{I}, R)$ where $\mathcal{F}$ is a collection of up-sets of P such that all principal up-sets of P belong to $\mathcal{F}$, $\mathcal{I}$ is a collection of down-sets of P such that all principal down-sets of P belong to $\mathcal{I}$ and $R \subseteq \mathcal{F} \times \mathcal{I}$ is the binary relation of non-empty intersection, that is, FRI if and only if $F \cap I \neq \emptyset$, for every $F \in \mathcal{F}$

and $I \in \mathcal{I}$. Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ one of these Δ_1-polarities. It provides a Δ_1-completion of P by considering the Galois connection associated with it, which is given by the maps $\Phi_R \colon \mathcal{P}(\mathcal{F}) \to \mathcal{P}(\mathcal{I})$ and $\Psi_R \colon \mathcal{P}(\mathcal{I}) \to \mathcal{P}(\mathcal{F})$ defined by

- $\Phi_R(A) := \{I \in \mathcal{I} : \forall F(F \in A \Rightarrow FRI)\}$ and
- $\Psi_R(B) := \{F \in \mathcal{F} : \forall I(I \in B \Rightarrow FRI)\}$,

respectively. The Δ_1-completion of P provided by $(\mathcal{F}, \mathcal{I}, R)$ is the complete lattice $L = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ of Galois closed subsets of $\mathcal{F}$, and the embedding e is given by $a \mapsto \{F \in \mathcal{F} : a \in F\}$, for every $a \in P$. In the terminology of [13] the completion $L = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ is an $(\mathcal{F}, \mathcal{I})$*-completion of* P, which means that it has the following two important properties:

1. $(\mathcal{F}, \mathcal{I})$-compactness: for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$ if $\bigwedge e[F] \leq \bigvee e[I]$, then $F \cap I \neq \emptyset$;
2. $(\mathcal{F}, \mathcal{I})$-density: for every $a \in L$,
   - $a = \bigvee\{\bigwedge e[F] : F \in \mathcal{F}$ and $\bigwedge e[F] \leq a\}$
   - $a = \bigwedge\{\bigvee e[I] : I \in \mathcal{I}$ and$a \leq \bigvee e[I]\}$.

(see [13, Definitions 5.1, 5.7 and Theorem 5.10]). For further details and background on polarities, a notion that goes back to G. Birkhoff [1], see [11–13] and [4, Chapters 3 and 7].

Given a poset P, recall that $\mathsf{Fi}_\mathsf{F}(P)$ denotes the collection of all Frink filters of P and let us denote by $\mathsf{Id}(P)$ the collection of all non-empty up-directed down-sets of P. Theorem 5.10 in [13] applied to $\mathsf{Fi}_\mathsf{F}(P)$ and $\mathsf{Id}(P)$ gives the next theorem.

Theorem 6.1 *Let P be a poset. Then, there exists a unique, up to isomorphism, completion* $\langle L, e\rangle$ *of P which is* $(\mathsf{Fi}_\mathsf{F}(P), \mathsf{Id}(P))$*-compact as well as* $(\mathsf{Fi}_\mathsf{F}(P), \mathsf{Id}(P))$*-dense, that is, that satisfies the following conditions :*

(1) *for every* $F \in \mathsf{Fi}_\mathsf{F}(P)$ *and* $I \in \mathsf{Id}(P)$ *if* $\bigwedge e[F] \leq \bigvee e[I]$*, then* $F \cap I \neq \emptyset$*;*
(2) *for every element* $a \in L$*,* $a = \bigvee\{\bigwedge e[F] : F \in \mathsf{Fi}_\mathsf{F}(P)$ *and* $\bigwedge e[F] \leq a\}$ *and* $a = \bigwedge\{\bigvee e[I] : I \in \mathsf{Id}(P)$ *anda* $\leq \bigvee e[I]\}$.

The unique up to isomorphism completion of P satisfying the previous conditions is the $(\mathsf{Fi}_\mathsf{F}(P), \mathsf{Id}(P))$-completion of P, in the terminology of [13]. We give a name to this completion of a poset.

Definition 6.2 Let P be a poset. The *Frink completion* of P is the unique up to isomorphism completion of P such that conditions (1) and (2) in Theorem 6.1 hold. We refer to it by P^{Fr}.

Another important completion of a poset considered in the literature is the canonical extension as defined in [6]. The concept of canonical extension for posets is a generalization of the canonical extensions for lattices [12] and for distributive lattices [14–16]. The canonical extension of a poset P is the $(\mathsf{F}(P), \mathsf{Id}(P))$-completion of P, where $\mathsf{F}(P)$ is here the collection of all non-empty down-directed up-sets of P, and it is denoted by P^σ (see [13] and [20]). In the following example, we show that the Frink completion and the canonical extension of a poset may be different, even if the poset is mo-distributive. In contrast they coincide for meet-semilattices.

Example 6.3 We consider the poset P given on the right hand side in Fig. 3. The canonical extension P^σ and the Frink completion P^{Fr} of P are also shown in Fig. 3. Thus we observe that P^σ and P^{Fr} are not isomorphic. Moreover, it is clear that the poset P is mo-distributive.

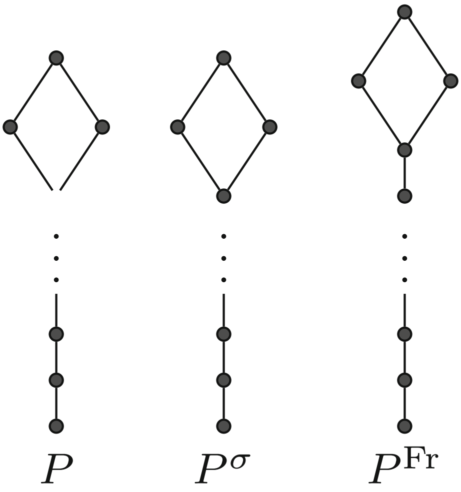

Fig. 3 A mo-distributive poset P and its canonical extension P^σ and Frink completion P^{Fr}

Proposition 6.4 *Let M be a meet-semilattice. Then the canonical extension of M coincides with the Frink completion of M. That is, $M^\sigma \cong M^{\mathrm{Fr}}$.*

Proof It is an immediate consequence since $\mathsf{Fi}_{\mathsf{F}}(M)$ coincides with the collection of all filters (in the sense of lattices) of M because M is a meet-semilattice. □

Now we will use the topological representation for mo-distributive posets presented in Section 4 to provide a topological proof of the existence of the Frink completion of a mo-distributive poset. This completion will be obtained via the topological duality proved in Theorem 4.25 in an analogous fashion as the canonical extension for bounded distributive lattices was obtained via the Priestley duality [14]. This allows us to show that the Frink completion of a mo-distributive poset has very nice properties, and we also get an interesting result about the canonical extensions of distributive meet-semilattices.

Let P be a fixed but arbitrary mo-distributive poset and $\mathbf{X}(P) = \langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \tau_P, \mathcal{B}_P\rangle$ its dual DP-space. To simplify the notation we let $\mathbf{X} := \mathbf{X}(P)$ and $P_{\mathbf{X}} := P_{\mathbf{X}(P)}$. Recall that the specialization order of $\mathbf{X}$ is the dual of the inclusion order of $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. Let $\mathsf{Down}(\mathbf{X})$ be the collection of all down-sets of the poset $\langle \mathbf{X}, \preceq\rangle$. It is well known that $\mathsf{Down}(\mathbf{X})$ is a completely distributive algebraic lattice where the meet is the intersection and the join is the union. We also know that the collections of all completely join-irreducible elements and of all completely meet-irreducible elements of $\mathsf{Down}(\mathbf{X})$ are $\mathcal{J}^\infty(\mathsf{Down}(\mathbf{X})) := \{{\downarrow} F : F \in \mathbf{X}\}$ and $\mathcal{M}^\infty(\mathsf{Down}(\mathbf{X})) := \{({\uparrow} F)^c : F \in \mathbf{X}\}$, respectively. It is clear that $P_{\mathbf{X}} \subseteq \mathsf{C}(\mathbf{X}) \subseteq \mathsf{Down}(\mathbf{X})$ and $\mathsf{C}(\mathbf{X})$ is a sub-lattice of $\mathsf{Down}(\mathbf{X})$. Hence $\mathsf{Down}(\mathbf{X})$ is a completion of $P_{\mathbf{X}}$ and therefore it is a completion of P.

Theorem 6.5 *Let P be a mo-distributive poset and $X = X(P)$ its dual DP-space. Then, $\mathsf{Down}(X)$ is the Frink completion of P.*

Proof Since P and $P_{\mathbf{X}}$ are isomorphic, it is enough to prove that $\mathsf{Down}(\mathbf{X})$ is the Frink completion of $P_{\mathbf{X}}$. Thus, we need to show that the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ is such that conditions (1) and (2) in Theorem 6.1 hold. To prove condition (1), let $\mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$ and $\mathcal{I} \in \mathsf{Id}(P_{\mathbf{X}})$ and assume that $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{I}$. Suppose towards a contradiction that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Let $F := \varphi^{-1}[\mathcal{F}]$ and $I := \varphi^{-1}[\mathcal{I}]$. It is clear that $F \cap I = \emptyset$ and since $\varphi\colon P \to P_{\mathbf{X}}$ is an order-isomorphism, we have $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}(P)$. Since P is mo-distributive, it follows that there is $H \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ such that $F \subseteq H$ and $H \cap I = \emptyset$. We thus get $H \in \bigcap \mathcal{F}$ and $H \notin \bigcup \mathcal{I}$, which is a contradiction. Then, $\mathcal{F} \cap \mathcal{I} \neq \emptyset$. Hence, the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ satisfies condition (1).

Now to prove condition (2), let $D \in \mathsf{Down}(\mathbf{X})$. Notice that

$$\bigcup \left\{ \bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \text{ and} \bigcap \mathcal{F} \subseteq D \right\} \subseteq D.$$

Let now $F \in D$. Since $F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ and $\varphi\colon P \to P_{\mathbf{X}}$ is an order-isomorphism, it follows that $\varphi[F] \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathbf{X}})$. Let $G \in \bigcap \varphi[F]$. So, $G \in \varphi(a)$ for all $a \in F$ and then $F \subseteq G$. Thus $G \preceq F$ and since $D \in \mathsf{Down}(\mathbf{X})$, we have that $G \in D$. Hence $\bigcap \varphi[F] \subseteq D$ and it is clear that $F \in \bigcap \varphi[F]$. Then, $F \in \bigcup \{\bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \text{ and} \bigcap \mathcal{F} \subseteq D\}$ and therefore

$$D = \bigcup \left\{ \bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \text{ and} \bigcap \mathcal{F} \subseteq D \right\}.$$

To prove the second part of condition (2), we note that

$$D \subseteq \bigcap \left\{ \bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}(P_{\mathbf{X}}) \text{ and} D \subseteq \bigcup \mathcal{I} \right\}.$$

To prove the other inclusion, let $F \in \bigcup \mathcal{I}$ for all $\mathcal{I} \in \mathsf{Id}(P_{\mathbf{X}})$ such that $D \subseteq \bigcup \mathcal{I}$. As $F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$, we have $F^c \in \mathsf{Id}(P)$ and since φ is an order-isomorphism, it follows that $\varphi[F^c] \in \mathsf{Id}(P_{\mathbf{X}})$. We suppose that $F \notin D$. So, $F \npreceq G$ for all $G \in D$ and this implies that for every $G \in D$ there exists $a_G \in G \setminus F$. Then, we have that $D \subseteq \bigcup \varphi[F^c]$ and $F \notin \bigcup \varphi[F^c]$; which is a contradiction. Thus, $F \in D$ and hence

$$D = \bigcap \left\{ \bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}(P_{\mathbf{X}(P)}) \text{ and} D \subseteq \bigcup \mathcal{I} \right\}.$$

Then, the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ satisfies condition (2). Therefore, by Theorem 6.1 we have proved that $\mathsf{Down}(\mathbf{X})$ is the Frink completion of $P_{\mathbf{X}}$ and thus it is the Frink completion of P. □

Therefore the following properties of the Frink completion of a mo-distributive poset follow:

Corollary 6.6 *Let P be a mo-distributive poset and P^{Fr} its Frink completion. Then,*

(1) *P^{Fr} is a completely distributive algebraic lattice;*
(2) *$\mathcal{J}^{\infty}(P^{\mathrm{Fr}})$ and $\mathcal{M}^{\infty}(P^{\mathrm{Fr}})$ are isomorphic posets.*

These properties are the same that hold in the canonical extension of a distributive lattice, see [14]. We finish this section by showing that the canonical extension (in the sense of Dunn et al. [6]) of a distributive meet-semilattice may be considered a good completion for it. And this provides an answer to a comment introduced in [9, pp. 69] related to a notion of "canonical extension" in the setting of abstract algebraic logic.

Corollary 6.7 *The canonical extension of a distributive meet-semilattice is a completely distributive algebraic lattice.*

Proof It is a consequence of Proposition 6.4 and Corollary 6.6. □

Remark 6.8 To define the Frink completion of a mo-distributive poset P, we choose the polarity $(\mathcal{F}, \mathcal{I})$ with $\mathcal{F}$ being the Frink filters and $\mathcal{I}$ being the up-directed downs-sets of P. We have chosen the up-directed down-sets in place of the Frink ideals (which may seem at first sight more natural) because we have the prime Frink filter theorem (Theorem 3.9) for the up-directed down-sets. Thus we were able to prove the existence of the Frink completion through the topological duality. The analogous of the prime Frink filter theorem but for Frink ideals instead of up-directed down-sets is easily seen to fail.

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