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The Free Distributive Semilattice Extension of a Distributive Poset

Luciano J. González[1]

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Abstract
The main aim of this paper is to obtain a free distributive semilattice extension for some ordered sets satisfying a distributivity condition. That is, for an ordered set P satisfying a distributivity condition, we prove the existence of a distributive semilattice M and a monomorphism $e\colon P \hookrightarrow M$ such that for every distributive semilattice L and every monomorphism $f\colon P \hookrightarrow L$ there exists a unique semilattice embedding $\widehat{f}\colon M \hookrightarrow L$ such that $f = \widehat{f} \circ e$. To attain this, we will need to consider and study some concepts on ordered sets like filters and a distributivity condition. We consider three notions of filters on posets known in the literature, and we show some new relationships between them. We also introduce and investigate three definitions of morphisms between posets.

Keywords Ordered set · Filters · Morphisms · Distributivity condition · Free extension

1 Introduction

In this paper, we present the definition of the free distributive meet-semilattice extension of a poset, and we prove its existence for those posets satisfying a distributivity condition (Section 6); we call these posets *meet-order distributive* (see Definition 4.1). To attain this aim, we consider the concept of Frink filter introduced by Frink [6]. In Section 3, we establish some connections between Frink filters and other two notions of filters well known in the literature. In Section 4, we present a distributivity condition on posets, and we study their main properties. Section 5 is devoted to introducing three different definitions of morphisms between posets; we study the relations between these three kinds of morphisms and how are they related to the concepts of filters presented in the above section. In Section 7, we establish a connection between the Frink filters of a meet-order distributive poset and the filters of its distributive meet-semilattice extension.

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✉ Luciano J. González
lucianogonzalez@exactas.unlpam.edu.ar

[1] Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, Santa Rosa, Argentina

A consequence of having a free distributive meet-semilattice extension is that the category of distributive meet-semilattices and meet-homomorphisms is a reflective subcategory of the category of meet-order distributive posets and certain morphisms. Moreover, the fact that every meet-order distributive poset has a free distributive meet-semilattice extension is related to a Priestley-type duality, see [7].

In Section 8, we present some conclusions concerning the results previously obtaining and some possible directions for future works.

2 Preliminaries

In this section, we introduce a few notations and terminologies. For more details about order theoretical concepts, we refer the reader to [4, 9].

Let X be an arbitrary set. We write $Y \subseteq_\omega X$ to concisely say that Y is a (possibly empty) finite subset of X.

Let P be a poset. A subset $A \subseteq P$ is called an *up-set* of P when for all $a \in A$ and $b \in P$, if $a \leq b$, then $b \in A$. Dually, we have the notion of *down-set*. For every $a \in P$, $\uparrow a$ denotes the up-set $\{x \in P : a \leq x\}$ and dually, $\downarrow a := \{x \in P : x \leq a\}$. For a subset $A \subseteq P$, we define the sets $\uparrow A = \{x \in P : \text{for some} a \in A, a \leq x\}$ and dually $\downarrow A$. A subset $A \subseteq P$ is said to be *up-directed* if for all $a, b \in A$, there exists $c \in A$ such that $a, b \leq c$. Dually, a subset B is said to be *down-directed* if for all $a, b \in B$, there exists $c \in B$ such that $c \leq a, b$. For every subset $X \subseteq P$, let X^u denote the set of upper bounds of X and X^ℓ the set of lower bounds of X. The two induced maps $(.)^u$ and $(.)^\ell$ on the power set $\mathcal{P}(P)$ form a Galois connection with respect to the relation $\subseteq$. If the greatest lower bound of a (possible empty) finite subset $A = \{a_0, \ldots, a_{n-1}\}$ exists in P, we denote it by $\bigwedge A$ or $a_0 \wedge \cdots \wedge a_{n-1}$; and dually, if the least upper bound of A exists, we denote it by $\bigvee A$ or $a_0 \vee \cdots \vee a_{n-1}$. We consider that $\bigwedge \emptyset$ exists in P if and only if P has a top element 1_P; and in such a case, $\bigwedge \emptyset = 1_P$.

A *semilattice* is an algebra $\langle S, * \rangle$ of type (2) such that the operation $*$ is idempotent, associative and commutative. Every semilattice $\langle S, * \rangle$ has naturally associated two partial orders: $x \leq_\wedge y \iff x * y = x$, and $x \leq_\vee y \iff x * y = y$. We will say that $\langle S, * \rangle$ is a *meet-semilattice* if it is a semilattice with the partial order $\leq_\wedge$ associated to S. Dually, a *join-semilattice* is a semilattice $\langle S, * \rangle$ with the partial order $\leq_\vee$ associated to S.

If $\langle S, * \rangle$ is a meet-semilattice, then $\langle S, \leq_\wedge \rangle$ is a poset such that for all $a, b \in S$, $a * b$ is the greatest lower bound of a and b in S. Conversely, if $\langle P, \leq \rangle$ is a poset such that the greatest lower bound exists for every pair of elements in P, then $\langle P, * \rangle$ with the operation $*$ defined by $a * b = \inf\{a, b\}$ for all $a, b \in P$ is a meet-semilattice and $\leq = \leq_\wedge$. Thus, it is usual for every meet-semilattice $\langle S, * \rangle$ to denote the operation $*$ by $\wedge$. Dually, for a join-semilattice $\langle S, * \rangle$, $a * b$ is the least upper bound of a and b. We denote the binary operation of a join-semilattice by $\vee$.

Let $\langle M, \wedge \rangle$ be a meet-semilattice. A non-empty subset $F \subseteq M$ is said to be a *filter* of M if is an up-set and closed under $\wedge$. Let $\langle J, \vee \rangle$ be a join-semilattice. A non-empty subset $I \subseteq J$ is said to be an *ideal* if is a down-set and closed under $\vee$.

The following definition can be found in [9, pp. 167]:

Definition 2.1 A join-semilattice $\langle J, \vee \rangle$ is called *distributive* when for all $a, b_0, b_1 \in J$, if $a \leq b_0 \vee b_1$, then there exist $a_0, a_1 \in J$ such that $a = a_0 \vee a_1$, $a_0 \leq b_0$ and $a_1 \leq b_1$. Dually, a meet-semilattice $\langle M, \wedge \rangle$ is *distributive* when for all $a, b_0, b_1 \in M$, if $b_0 \wedge b_1 \leq a$, then there exist $a_0, a_1 \in M$ such that $a = a_0 \wedge a_1$, $b_0 \leq a_0$ and $b_1 \leq a_1$.

3 Filters and Ideals on Posets

We will introduce three notions of filter and ideal for posets that are known in the literature. We study the relations between these three concepts, and we present some new results about them. The three different definitions of filter and ideal for posets that we consider are natural generalisations of the notions of filter and ideal for lattices. For more details about the concepts and results in this section, we refer the reader to [7, 14].

Definition 3.1 Let P be a poset. A non-empty subset $F \subseteq P$ is said to be an *order filter* of P if is a down-directed up-set. Dually, a non-empty subset $I \subseteq P$ is said to be an *order ideal* of P if is an up-directed down-set.

We denote by $\mathsf{Fi}_{\mathsf{or}}(P)$ the family of all order filters of P and by $\mathsf{Id}_{\mathsf{or}}(P)$ the family of all order ideals of P. It is clear that for every element $a \in P$, $\uparrow a$ is an order filter of P and $\downarrow a$ is an order ideal of P. Notice that the families $\mathsf{Fi}_{\mathsf{or}}(P)$ and $\mathsf{Id}_{\mathsf{or}}(P)$ are not necessarily closure systems because they are not necessarily closed under arbitrary intersections.

It is straightforward to check directly that if $\langle M, \wedge\rangle$ is a meet-semilattice, then the collection of all filters of M, $\mathsf{Fi}(M)$, coincide with the collection of all order filters $\mathsf{Fi}_{\mathsf{or}}(M)$ of the poset induced by M. That is, $\mathsf{Fi}(M) = \mathsf{Fi}_{\mathsf{or}}(M)$. Dually, if J is a join-semilattice, then $\mathsf{Id}(J) = \mathsf{Id}_{\mathsf{or}}(J)$. In particular, if M is a meet-semilattice with a top element, then $\mathsf{Fi}_{\mathsf{or}}(M) = \mathsf{Fi}(M)$ is a closure system. The following proposition expresses the converse of the previous statement, providing a new characterisation of when a poset P is a meet-semilattice with a top element.

Proposition 3.2 *Let P be a poset. Then, $\mathsf{Fi}_{\mathsf{or}}(P)$ is a closure system if and only if P is a meet-semilattice with top element.*

Proof Let P be a poset and assume that $\mathsf{Fi}_{\mathsf{or}}(P)$ is a closure system on P. We denote by $\mathrm{Fi}_{\mathrm{or}}(.)$ the closure operator associated with the closure system $\mathsf{Fi}_{\mathsf{or}}(P)$. Let $a, b \in P$. Since $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ is an order filter of P and $a, b \in \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$, there exists $c \in \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ such that $c \leq a$ and $c \leq b$. So, we have $\uparrow c \subseteq \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ and $\uparrow a \cup \uparrow b \subseteq \uparrow c$. Then, $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b) = \uparrow c$. Now, we show $c = a \wedge b$. We know that c is a lower bound of a and b. Let $d \in P$ be such that $d \leq a$ and $d \leq b$. So, $\uparrow a \cup \uparrow b \subseteq \uparrow d$ and this implies that $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b) \subseteq \uparrow d$. Then, $\uparrow c \subseteq \uparrow d$ and thus $d \leq c$. Therefore $c = a \wedge b$. In order to prove that P has a top element, consider the set $F = \bigcap\{G : G \in \mathsf{Fi}_{\mathsf{or}}(P)\}$. Since $\mathsf{Fi}_{\mathsf{or}}(P)$ is closure system, it is closed under arbitrary intersection. Then, $F \in \mathsf{Fi}_{\mathsf{or}}(P)$. So, $F \neq \emptyset$. Let $a \in F$. We want to show that a is the top element of P. Let $b \in P$. Since $\uparrow b$ is an order filter of P, $a \in \uparrow b$. Consequently, $b \leq a$. Hence a is the top element of P. Therefore, we have proved that P is a meet-semilattice with a top element. The converse implication was shown in the paragraph previous to this proposition. □

Definition 3.3 ([6]) Let P be a poset. A subset F of P is said to be a *Frink filter* of P if for every $A \subseteq_\omega F$, we have $A^{\ell u} \subseteq F$. A subset I of P is said to be a *Frink ideal* of P if for every $A \subseteq_\omega I$, we have $A^{u\ell} \subseteq I$.

Let us denote by $\mathsf{Fi}_{\mathsf{F}}(P)$ the collection of all Frink filters of P and by $\mathsf{Id}_{\mathsf{F}}(P)$ the collection of all Frink ideals of P.

Notice that the empty set may be a Frink filter or a Frink ideal of a poset P. In fact, for a poset P, we have that the empty set is a Frink filter (Frink ideal) of P if and only if P has no top (bottom) element. This is a consequence of the fact that $\emptyset^{\ell u} = P^u$ ($\emptyset^{u\ell} = P^\ell$). It is easy to check that each Frink filter is an up-set and each Frink ideal is a down-set. Moreover, given a poset P, we have that for every $a \in P$, $\uparrow a$ is a Frink filter of P and $\downarrow a$ is a Frink ideal of P. If M is a meet-semilattice, then $\mathsf{Fi}_\mathsf{F}(M) \setminus \{\emptyset\} = \mathsf{Fi}(M)$; and dually, if J is a join-semilattice, then $\mathsf{Id}_\mathsf{F}(J) \setminus \{\emptyset\} = \mathsf{Id}(J)$. Therefore, if L is a lattice, then $\mathsf{Fi}_\mathsf{F}(L) \setminus \{\emptyset\} = \mathsf{Fi}(L)$ and $\mathsf{Id}_\mathsf{F}(L) \setminus \{\emptyset\} = \mathsf{Id}(L)$.

Proposition 3.5 ([6]) *Given an arbitrary poset P, $\mathsf{Fi}_\mathsf{F}(P)$ and $\mathsf{Id}_\mathsf{F}(P)$ are closure systems.*

Let P be a poset. We denote by $\mathrm{Fig}_\mathrm{F}(.)$ the closure operator associated with $\mathsf{Fi}_\mathsf{F}(P)$. Thus, for every $X \subseteq P$, $\mathrm{Fig}_\mathrm{F}(X)$ is the least Frink filter of P containing X and it is called the *Frink filter generated by X*.

Proposition 3.6 *Let P be a poset and let $X \subseteq P$. Then,*

$$\mathrm{Fig}_\mathrm{F}(X) = \bigcup \left\{ X_0^{\ell u} : X_0 \subseteq_\omega X \right\}.$$

It should be noted that $\mathrm{Fig}_\mathrm{F}(A) = A^{\ell u}$ whenever $A \subseteq_\omega P$. For every poset P, we have by Proposition 3.4 that $\mathsf{Fi}_\mathsf{F}(P)$ is a complete lattice, where for every family $\mathcal{F} \subseteq \mathsf{Fi}_\mathsf{F}(P)$ the meet and join are given by

$$\bigwedge \mathcal{F} = \bigcap \mathcal{F} \quad \text{and} \quad \bigvee \mathcal{F} = \mathrm{Fig}_\mathrm{F}\left(\bigcup \mathcal{F}\right).$$

A Frink filter is said to be *finitely generated* if is a Frink filter generated by a (possible empty) finite subset of P. Let us denote by $\mathsf{Fi}^\mathsf{f}_\mathsf{F}(P)$ the collection of all finitely generated Frink filters of P. Notice that $\langle \mathsf{Fi}^\mathsf{f}_\mathsf{F}(P), \vee, \emptyset^{\ell u}\rangle$ is a sub-join-semilattice with bottom element $\emptyset^{\ell u}$ of the lattice $\mathsf{Fi}_\mathsf{F}(P)$. Indeed, for every $X, Y \subseteq_\omega P$, $\mathrm{Fig}_\mathrm{F}(X) \vee \mathrm{Fig}_\mathrm{F}(Y) = \mathrm{Fig}_\mathrm{F}(X \cup Y)$ with $X \cup Y \subseteq_\omega P$. Moreover, it should be noted that $\emptyset^{\ell u} = \{1_P\}$, if P has top element 1_P and $\emptyset^{\ell u} = \emptyset$, if P has no top element.

Recall that a closure operator C on a set X is said to be *finitary* if for every $Y \subseteq X$, $C(Y) = \bigcup\{C(Y_0) : Y_0 \text{ is a finite subset of } Y\}$. A closure system $\mathcal{C}$ on a set X is said to be *algebraic* if it is closed under unions of chains. We also recall that if C is a closure operator and $\mathcal{C}$ is its associated closure system, then C is finitary if and only if $\mathcal{C}$ is algebraic.

From Proposition 3.5, we have that the closure operator $\mathrm{Fig}_\mathrm{F}(.)$ is finitary, and thus the closure system $\mathsf{Fi}_\mathsf{F}(P)$ is algebraic.

The next propositions shows the connection between order filters and Frink filters (order ideals and Frink ideals). The proof can be found in [7, Lemma 2.1.8 and 2.1.9] and [14].

Proposition 3.7 *Let P be a poset. Then, $\mathsf{Fi}_\mathsf{or}(P) \subseteq \mathsf{Fi}_\mathsf{F}(P)$ and $\mathsf{Id}_\mathsf{or}(P) \subseteq \mathsf{Id}_\mathsf{F}(P)$. In general, these inclusions are strict.*

Proposition 3.8 *Let P be a poset with a top (bottom) element. Then, P is a meet-semilattice (join-semilattice) if and only if $\mathsf{Fi}_\mathsf{or}(P) = \mathsf{Fi}_\mathsf{F}(P)$ ($\mathsf{Id}_\mathsf{or}(P) = \mathsf{Id}_\mathsf{F}(P)$).*

We have shown that for every poset P, $\mathsf{Fi}_\mathsf{or}(P) \subseteq \mathsf{Fi}_\mathsf{F}(P)$ and they are not necessarily equal. Now, let us see how the Frink filters can be reached from the order filters. Let P be a poset. Let $\mathsf{C}_\mathsf{or}(P)$ be the closure system on P generated by the collection $\mathsf{Fi}_\mathsf{or}(P)$ and we denote by $C_{\mathsf{or}} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ the closure operator associated with $\mathsf{C}_{\mathsf{or}}(P)$. Thus, for every $A \subseteq P$, $C_{\mathsf{or}}(A) = \bigcap\{F \in \mathsf{Fi}_{\mathsf{or}}(P) : A \subseteq F\}$. Now we consider the operator $C^{\mathsf{f}}_{\mathsf{or}} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ defined as

$$C^{\mathsf{f}}_{\mathsf{or}}(A) = \bigcup\{C_{\mathsf{or}}(A_0) : A_0 \subseteq_\omega A\}$$

for each $A \subseteq P$. Then, the operator $C^{\mathsf{f}}_{\mathsf{or}}$ has the following properties:

(1) $C^{\mathsf{f}}_{\mathsf{or}}(A) = C_{\mathsf{or}}(A)$ whenever $A \subseteq_\omega P$;
(2) $C^{\mathsf{f}}_{\mathsf{or}}$ is a finitary closure operator and $C^{\mathsf{f}}_{\mathsf{or}} \leq C_{\mathsf{or}}$ (that is, $C^{\mathsf{f}}_{\mathsf{or}}(A) \subseteq C_{\mathsf{or}}(A)$ for $A \subseteq P$);
(3) $C^{\mathsf{f}}_{\mathsf{or}}$ is the strongest of all finitary closure operators C on P such that $C \leq C_{\mathsf{or}}$.

The closure operator $C^{\mathsf{f}}_{\mathsf{or}}$ is sometime called *the finitary companion of* C_{or}. Let us denote by $\mathsf{C}^{\mathsf{f}}_{\mathsf{or}}(P)$ the closure system associated with $C^{\mathsf{f}}_{\mathsf{or}}$. As $\mathsf{Fi}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$ and since $C^{\mathsf{f}}_{\mathsf{or}}(A_0) = C_{\mathsf{or}}(A_0)$ is an intersection of order filters for every $A_0 \subseteq_\omega P$, it follows that $C^{\mathsf{f}}_{\mathsf{or}}(A_0) \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $A_0 \subseteq_\omega P$. Then, given that $\mathsf{Fi}_{\mathsf{F}}(P)$ is an algebraic closure system and since for every $A \subseteq P$ the set $\{C_{\mathsf{or}}(A_0) : A_0 \subseteq_\omega A\}$ is up-directed, it follows that

$$C^{\mathsf{f}}_{\mathsf{or}}(A) = \bigcup\{C_{\mathsf{or}}(A_0) : A_0 \subseteq_\omega A\} \in \mathsf{Fi}_{\mathsf{F}}(P)$$

for all $A \subseteq P$.

Proposition 3.9 *Let P be a poset and $A_0 \subseteq_\omega P$. Then $C^{\mathsf{f}}_{\mathsf{or}}(A_0) = A_0^{\ell u}$.*

Proof Let $A_0 \subseteq_\omega P$. Since $C^{\mathsf{f}}_{\mathsf{or}}(A_0) \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $A_0 \subseteq C^{\mathsf{f}}_{\mathsf{or}}(A_0)$, it follows that $A_0^{\ell u} \subseteq C^{\mathsf{f}}_{\mathsf{or}}(A_0)$. Let $a \in C^{\mathsf{f}}_{\mathsf{or}}(A_0)$ and let $b \in A_0^{\ell}$. So $A_0 \subseteq \uparrow b$. Because $a \in C^{\mathsf{f}}_{\mathsf{or}}(A_0)$, a belongs to each order filter that contains A_0. Then, $a \in \uparrow b$ and thus $b \leq a$. So $a \in A_0^{\ell u}$ and hence $C^{\mathsf{f}}_{\mathsf{or}}(A_0) \subseteq A_0^{\ell u}$. Therefore $C^{\mathsf{f}}_{\mathsf{or}}(A_0) = A_0^{\ell u}$. □

Proposition 3.10 *Let P be a poset. Then $\mathsf{C}^{\mathsf{f}}_{\mathsf{or}}(P) = \mathsf{Fi}_{\mathsf{F}}(P)$.*

Proof We already know that $\mathsf{C}^{\mathsf{f}}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$. Now let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. Let us show that $C^{\mathsf{f}}_{\mathsf{or}}(F) = F$. Let $a \in C^{\mathsf{f}}_{\mathsf{or}}(F)$. So, there is $A_0 \subseteq_\omega F$ such that $a \in C_{\mathsf{or}}(A_0)$. So $A_0^{\ell u} \subseteq F$ and then, by Proposition 3.8, we have $C_{\mathsf{or}}(A_0) \subseteq F$. Thus $a \in F$. Hence $C^{\mathsf{f}}_{\mathsf{or}}(F) = F$. □

Now we consider the third notion of filter and ideal on posets.

Definition 3.4 Let P be a poset. A subset $F \subseteq P$ is said to be a $\wedge$-*filter* of P when F is an up-set and if $a_1, \ldots, a_n \in F$ and $a_1 \wedge \cdots \wedge a_n$ exists in P, then $a_1 \wedge \cdots \wedge a_n \in F$. Dually, a subset $I \subseteq P$ is said to be a $\vee$-*ideal* of P when I is a down-set and if $a_1, \ldots, a_n \in I$ and $a_1 \vee \cdots \vee a_n$ exists in P, then $a_1 \vee \cdots \vee a_n \in I$.

Notice that for any poset P, the empty set is always a $\wedge$-filter ($\vee$-ideal), even if P has a top (bottom) element. For this reason, we consider the following notation depending if P has or not a top (bottom) element. Let P be a poset. If P has no a top element, then $\mathsf{Fi}_\wedge(P)$ denotes the collection of all $\wedge$-filters of P including the empty set, and if P has a top element, then $\mathsf{Fi}_\wedge(P)$ denotes the collection of all non-empty $\wedge$-filters of P. Dually, $\mathsf{Id}_\vee(P)$ denotes the collection of all $\vee$-ideals of P if P has no a bottom element, and $\mathsf{Id}_\vee(P)$ denotes the collection of all non-empty $\vee$-ideals of P, if P has a bottom element.

Proposition 3.11 *Let P be a poset. Then,*

$$\mathsf{Fi}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P) \subseteq \mathsf{Fi}_{\wedge}(P) \quad \textit{and} \quad \mathsf{Id}_{\mathsf{or}}(P) \subseteq \mathsf{Id}_{\mathsf{F}}(P) \subseteq \mathsf{Id}_{\vee}(P).$$

Proposition 3.12 *Let P be an arbitrary poset. Then, $\mathsf{Fi}_{\wedge}(P)$ and $\mathsf{Id}_{\vee}(P)$ are algebraic closure systems on P.*

4 Distributive posets

The condition of distributivity for posets that we discuss in this paper is due to David and Erné [5]. This concept of distributivity on posets is a generalisation of the usual notion of distributivity in Lattice Theory. We also have to say that there are other possible generalisations of the concept of distributivity on posets, see for instance [3, 11–13].

Definition 4.1 ([5]) Let P be a poset.

(1.) We say that P is *meet-order distributive* if for every $b_1, \dots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \dots, b_n\}^{\ell u} \implies \text{there are } a_1, \dots, a_k \in {\uparrow} b_1 \cup \dots \cup {\uparrow} b_n \text{ such that } a = a_1 \wedge \dots \wedge a_k. \tag{4.1}$$

(2.) We say that P is *join-order distributive* if for every $b_1, \dots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \dots, b_n\}^{\ell u} \implies \text{there are } a_1, \dots, a_k \in {\downarrow} b_1 \cup \dots \cup {\downarrow} b_n \text{ such that } a = a_1 \vee \dots \vee a_k. \tag{4.2}$$

Remark 4.2 In [5], a join-order distributive poset is called an ideal-distributive poset. In the above definition, when we write $a = a_1 \wedge \dots \wedge a_k$ ($a = a_1 \vee \dots \vee a_k$), we mean that the meet (join) of $a_1, \dots, a_k$ exists and is equal to a. Moreover, it is straightforward to show directly that in each poset P the converse implications of Eqs. 4.1 and 4.2 always hold.

Proposition 4.3 *Let P be a poset.*

(1) *If P is meet-order distributive, then $\mathsf{Fi}_{\mathsf{F}}(P) = \mathsf{Fi}_{\wedge}(P)$.*
(2) *If P is join-order distributive, then $\mathsf{Id}_{\mathsf{F}}(P) = \mathsf{Id}_{\vee}(P)$.*

Proof (1) Assume that P is a meet-order distributive poset. By Proposition 3.10, we have $\mathsf{Fi}_{\mathsf{F}}(P) \subseteq \mathsf{Fi}_{\wedge}(P)$. Now let $F \in \mathsf{Fi}_{\wedge}(P)$. Let $a_1, \dots, a_n \in F$ and let $a \in \{a_1, \dots, a_n\}^{\ell u}$. Since P is meet-order distributive, there exist $b_1, \dots, b_k \in {\uparrow} a_1 \cup \dots \cup {\uparrow} a_n$ such that $a = b_1 \wedge \dots \wedge b_k$. Thus, since F is an up-set, it follows that $b_1, \dots, b_k \in F$. So, $a = b_1 \wedge \dots \wedge b_k \in F$. Then, F is a Frink filter and therefore $\mathsf{Fi}_{\wedge}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$.

(2) It can be shown dually. □

The following result is a nice characterisation of the meet-order distributivity and join-order distributivity.

Theorem 4.4 ([5]) *Let P be a poset. Then, P is meet-order distributive if and only if the lattice $\mathsf{Fi}_{\mathsf{F}}(P)$ is distributive. Dually, P is join-order distributive if and only if the lattice $\mathsf{Id}_{\mathsf{F}}(P)$ is distributive.*

Examples of posets that are meet-order distributive and join-order distributive, and posets that are meet-order distributive but are not join-order distributive can be found in [7, Example 2.2.14].

Let P be a poset. Recall that $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is the join-semilattice of all finitely generated Frink filters of P. The following proposition is a new characterisation of the meet-order distributivity condition through $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$.

Proposition 4.5 *Let P be a poset. P is meet-order distributive if and only if the join-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is distributive.*

Proof Let P be a poset. We assume first that P is meet-order distributive. In order to prove that $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is a distributive join-semilattice, let $F_1 := A_1^{\ell u}$, $F_2 := A_2^{\ell u}$ and $G := B^{\ell u}$ for some $A_1, A_2, B \subseteq_\omega P$ and assume that $G \subseteq F_1 \vee F_2$. We observe that $B \subseteq G \subseteq F_1 \vee F_2 = (A_1 \cup A_2)^{\ell u}$. So, since P is meet-order distributive, for every $b \in B$ there exists $A_b \subseteq_\omega {\uparrow}(A_1 \cup A_2)$ such that $b = \bigwedge A_b$. Since $G = B^{\ell u}$, we have $A_b \subseteq G$ for all $b \in B$. For every $b \in B$, let $A'_b := \{x \in A_b : (\exists y \in A_1)(y \leq x)\}$ and $A''_b := \{x \in A_b : (\exists y \in A_2)(y \leq x)\}$. Let now $A' := \bigcup_{b\in B} A'_b$ and $A'' := \bigcup_{b \in B} A''_b$. Then, since $A' \subseteq F_1$ and $A'' \subseteq F_2$, we obtain that $G_1 := A'^{\ell u} \subseteq F_1$ and $G_2 := A''^{\ell u} \subseteq F_2$. It only remain to show that $G = G_1 \vee G_2$. Notice that for every $b \in B$, we have $A_b = A'_b \cup A''_b$. Then,

$$G_1 \vee G_2 = \left(A' \cup A''\right)^{\ell u} = \left(\bigcup_{b\in B} A'_b \cup \bigcup_{b\in B} A''_b\right)^{\ell u} = \left(\bigcup_{b\in B} A_b\right)^{\ell u} \subseteq G.$$

Hence, $G_1 \vee G_2 \subseteq G$. On the other hand, observe that for every $b \in B$, $A_b \subseteq_\omega \bigcup_{b\in B} A_b$. So, $b = \bigwedge A_b \in (\bigcup_{b\in B} A_b)^{\ell u}$ for all $b \in B$. Then, $B \subseteq G_1 \vee G_2$ and this implies $G \subseteq G_1 \vee G_2$. Hence, $G = G_1 \vee G_2$ for $G_1, G_2 \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Therefore, $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is a distributive join-semilattice.

Conversely, assume that the join-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is distributive. Let $a, a_1, \ldots, a_n \in P$ be such that $a \in \{a_1, \ldots, a_n\}^{\ell u}$. So, ${\uparrow}a \subseteq {\uparrow}a_1 \vee \cdots \vee {\uparrow}a_n$. Then, there exist $F_1, \ldots, F_n \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ such that ${\uparrow}a = F_1 \vee \cdots \vee F_n$ and $F_i \subseteq {\uparrow}a_i$ for all $i = 1, \ldots, n$. For every $i = 1, \ldots, n$, let $F_i = A_i^{\ell u}$ for some $A_i \subseteq_\omega P$ and let $B := \bigcup_{i=1}^n A_i$. Then, ${\uparrow}a = F_1 \vee \cdots \vee F_n = A_1^{\ell u} \vee \cdots \vee A_n^{\ell u} = B^{\ell u}$. Hence, $a = \bigwedge B$ and $B \subseteq F_1 \cup \cdots \cup F_n \subseteq {\uparrow}a_1 \cup \cdots \cup {\uparrow}a_n$. Therefore, P is meet-order distributive. □

Corollary 4.6 *If P is a finite meet-order distributive poset, then $\mathsf{Fi}_{\mathsf{F}}(P) = \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$. Therefore, $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ is a bounded distributive lattice.*

We close this section noting that the condition of meet-order distributivity (join-order distributivity) behaves well with the formation of finite direct products (see [4, 1.25]), and with finite linear sums (see [4, 1.24]) of bounded meet-order distributive (join-order distributive) posets. That is, the finite direct product (linear sum) of bounded meet-order distributive posets is a meet-order distributive poset.

5 Morphism Between Posets

In this part, we introduce the definitions of certain morphisms between posets that intend to be a generalisation of the notion of homomorphism in Lattice Theory.

Definition 5.1 Let P and Q be posets and let $h\colon P \to Q$ be an order preserving map. We say that h is a *down-directed morphism* (*dd-morphism* for short) when (1) for all $a, b \in P$ and $u \in Q$, if $u \leq h(a), h(b)$, then there exists $c \in P$ such that $u \leq h(c)$ and $c \leq a, b$; and (2) if P has a top element 1_P, then $h(1_P)$ is the top element of Q. Dually, we have the notion of *up-directed morphism* (*ud-morphism* for short). We will say that h is an *order-morphism* if h is a dd-morphism and ud-morphism.

It is easy to check that for all meet-semilattices M and L, a map $h\colon M \to L$ is a meet-homomorphism (preserving top element, if it exists) if and only if h is a down-directed morphism. Dually for join-semilattices and up-directed morphism. Hence, a map $h\colon M \to L$ from a lattice M to a lattice L is a lattice homomorphism (preserving bounds, if they exist) if and only if h is an order-morphism.

Proposition 5.2 *Let P and Q be posets with top elements. A map $h\colon P \to Q$ is a down-directed morphism if and only if $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{or}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{or}}(Q)$.*

Proof Assume that h is a down-directed morphism and let $G \in \mathsf{Fi}_{\mathsf{or}}(Q)$. Since $h(1_P)$ is the top element of Q, it follows that $h^{-1}[G] \neq \emptyset$. Let $a \in h^{-1}[G]$ and $a \leq b$. Thus, $h(a) \in G$ and $h(a) \leq h(b)$. Then $b \in h^{-1}[G]$. Let now $a, b \in h^{-1}[G]$. Since G is an order filter, it follows that there is $u \in G$ such that $u \leq h(a), h(b)$. Then, there exists $c \in P$ such that $u \leq h(c)$ and $c \leq a, b$. Hence $c \in h^{-1}[G]$ and $c \leq a, b$. Thus $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{or}}(P)$. Conversely, assume that $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{or}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{or}}(Q)$. First, let us show that h is order preserving. Let $a, b \in P$ be such that $a \leq b$. Since $a \in h^{-1}[\uparrow h(a)] \in \mathsf{Fi}_{\mathsf{or}}(P)$, we have that $b \in h^{-1}[\uparrow h(a)]$ and thus $h(a) \leq h(b)$. Let now $a, b \in P$ and $u \in Q$ be such that $u \leq h(a), h(b)$. Since $a, b \in h^{-1}[\uparrow u] \in \mathsf{Fi}_{\mathsf{or}}(P)$, there exists $c \in h^{-1}[\uparrow u]$ such that $c \leq a, b$. Finally, since $1_P \in h^{-1}[\{1_Q\}]$, it follows that $h(1_P) = 1_Q$. Hence h is a down-directed morphism. □

Definition 5.3 Let P and Q be posets. A map $h\colon P \to Q$ is said to be an *ℓu-morphism* if for every $A \subseteq_\omega P$, we have

$$a \in A^{\ell u} \text{ implies } h(a) \in h[A]^{\ell u}.$$

Dually, the dual notion is that one of an *ℓu-morphism*. We will say that h is an *ℓu-ℓu-morphism* if h is an ℓu-morphism and ℓu-morphism.

The notion of ℓu-morphism was defined in [8] under the name inf-homomorphism and the concept of ℓu-morphism was considered in [1, 2] in the setting of semilattices under the name of sup-homomorphism.

The proof of the following proposition is not hard, and thus we leave the details to the reader.

Proposition 5.4 *Let P and Q be posets and let $h\colon P \to Q$ be a map. If h is an ℓu-morphism and P has a top element 1_P, then $h(1_P)$ is the top element of Q.*

It is straightforward to show that ℓu-morphisms and meet-homomorphisms (preserving top element, if it exists) coincide in the setting of meet-semilattices.

Proposition 5.5 ([8]) *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, h is an ℓu-morphism if and only if $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$.*

Proposition 5.6 *Let P and Q be posets. If $h\colon P \to Q$ is a down-directed morphism, then h is an ℓu-morphism.*

Proof Let $A \subseteq_\omega P$ and $a \in A^{\ell u}$. If $A = \emptyset$, then a is the top element of P and thus $h(a)$ is the top element of Q. Then $h(a) \in h[A]^{\ell u}$. Suppose that $A \neq \emptyset$ and $A = \{a_1, \ldots, a_n\}$. Let $u \in h[A]^{\ell}$. So $u \leq h(a_i)$ for all $i = 1, \ldots, n$. Since h is a down-directed morphism, there exists $c \in P$ such that $u \leq h(c)$ and $c \leq a_i$ for all $i = 1, \ldots, n$. That is, $c \in A^{\ell}$ and then $c \leq a$. Thus $u \leq h(c) \leq h(a)$. Hence $h(a) \in h[A]^{\ell u}$. Therefore, h is an ℓu-morphism. □

The map in Fig. 1 is an example of an ℓu-morphism that is not a down-directed morphism between two posets. Moreover, notice that the posets in Fig. 1 are meet-order and join-order distributive.

Definition 5.7 Let P and Q be posets. A map $h\colon P \to Q$ is called an *ℓu-embedding* (*ℓu-embedding*) if is an ℓu-morphism (a ℓu-morphism) and an order-embedding. Moreover, h is said to be an *ℓu-ℓu-embedding* if h is an ℓu-embedding and a ℓu-embedding.

Proposition 5.8 ([8]) *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, h is an ℓu-embedding if and only if for every $A \subseteq_\omega P$ and $a \in P$,*

$$a \in A^{\ell u} \iff h(a) \in h[A]^{\ell u}. \tag{5.1}$$

Now we present the third notion of morphism between posets.

Definition 5.9 Let P and Q be posets. We say that a map $h\colon P \to Q$ is a *$\wedge$-morphism* if h preserves all existing finite meets and $h(1_P)$ is the top element of Q, if P has a top element 1_P. That is, h is a $\wedge$-morphism if and only if for each $a_1, \ldots, a_n \in P$ such that $a_1 \wedge \cdots \wedge a_n$ exists in P, then $h(a_1) \wedge \cdots \wedge h(a_n)$ exists in Q and $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$; and $h(1_P)$ is the top element of Q, if 1_P is the top element of P. Th concept of *$\vee$-morphism* can be defined dually.

Notice that every $\wedge$-morphism or $\vee$-morphism is order preserving.

Proposition 5.10 *Let P and Q be posets and $h\colon P \to Q$ a map. Then, h is a $\wedge$-morphism if and only if $h^{-1}[G] \in \mathsf{Fi}_\wedge(P)$ for all $G \in \mathsf{Fi}_\wedge(Q)$.*

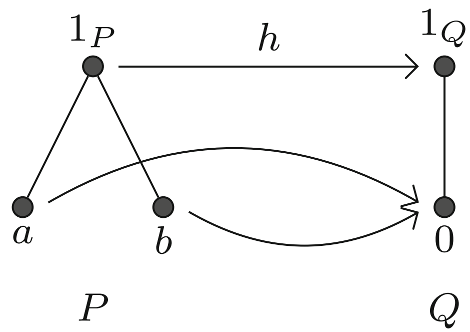

Fig. 1 An ℓu-morphism that is not a down-directed morphism

Proof First, assume that h is a $\wedge$-morphism and let $G \in \mathsf{Fi}_\wedge(Q)$. Notice that if P has a top element 1_P, then $1_P \in h^{-1}[G]$ and thus $h^{-1}[G]$ is non-empty whenever P has a top element. Now, since h is order preserving and G is an up-set of Q, it follows that $h^{-1}[G]$ is an up-set of P. Let $a_1, \ldots, a_n \in h^{-1}[G]$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. Then, $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$ and $h(a_1), \ldots, h(a_n) \in G$. So, since G is a $\wedge$-filter, we have $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n) \in G$. Hence, $a_1 \wedge \cdots \wedge a_n \in h^{-1}[G]$. Therefore, $h^{-1}[G] \in \mathsf{Fi}_\wedge(P)$.

Conversely, suppose that $h^{-1}[G] \in \mathsf{Fi}_\wedge(P)$ for all $G \in \mathsf{Fi}_\wedge(Q)$. Suppose that P has a top element 1_P. Then $1_P \in F$ for all $F \in \mathsf{Fi}_\wedge(P)$. Thus $h(1_P) \in G$ for all $G \in \mathsf{Fi}_\wedge(Q)$. Hence $h(1_P)$ is the top element of Q. Let us show that h is order preserving. Let $a, b \in P$ be such that $a \leq b$. Since $a \in h^{-1}[\uparrow h(a)] \in \mathsf{Fi}_\wedge(P)$, it follows that $b \in h^{-1}[\uparrow h(a)]$. Then, $h(a) \leq h(b)$. Now let $a_1, \ldots, a_n \in P$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. Since h is order preserving, we have $h(a_1 \wedge \cdots \wedge a_n) \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. Let $y \in Q$ be such that $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. So, $h(a_1), \ldots, h(a_n) \in \uparrow y \in \mathsf{Fi}_\wedge(Q)$. Then, $a_1, \ldots, a_n \in h^{-1}[\uparrow y] \in \mathsf{Fi}_\wedge(P)$ and this implies $a_1 \wedge \cdots \wedge a_n \in h^{-1}[\uparrow y]$. Hence, $y \leq h(a_1 \wedge \cdots \wedge a_n)$. That is, we proved that $h(a_1 \wedge \cdots \wedge a_n)$ is the greatest lower bound of $\{h(a_1), \ldots, h(a_n)\}$, i.e., $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Therefore, h is a $\wedge$-morphism. □

Proposition 5.11 *Let P and Q be posets and let $h\colon P \to Q$ be a map. If h is an ℓu-morphism, then h is a $\wedge$-morphism.*

Proof We assume that $h\colon P \to Q$ is an ℓu-morphism. Let $a_1, \ldots, a_n \in P$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. Since h is order preserving, we have $h(a_1 \wedge \cdots \wedge a_n) \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. Let $y \in Q$ be such that $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. So, $y \in \{h(a_1), \ldots, h(a_n)\}^{\ell}$. Since $a_1 \wedge \cdots \wedge a_n \in \{a_1, \ldots, a_n\}^{\ell u}$ and h is an ℓu-morphism, $h(a_1 \wedge \cdots \wedge a_n) \in \{h(a_1), \ldots, h(a_n)\}^{\ell u}$, whereupon $y \leq h(a_1 \wedge \cdots \wedge a_n)$. Hence, we have shown that $h(a_1 \wedge \cdots \wedge a_n)$ is the greatest lower bound of $\{h(a_1), \ldots, h(a_n)\}$, i.e., $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Therefore, h is a $\wedge$-morphism. □

Example 5.12 Figure 2 shows a map h from a non-meet-order distributive poset P to a meet-order distributive poset Q. It is straightforward to check that h is a $\wedge$-morphism but is not an ℓu-morphism.

The following proposition is an immediate consequence from Proposition 4.3 and by Propositions 5.5 and 5.10.

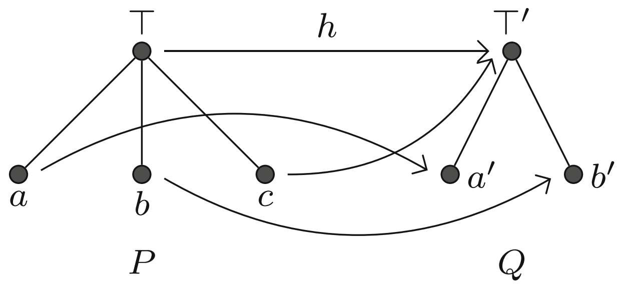

Fig. 2 A $\wedge$-morphism that is not an ℓu-morphism

Proposition 5.13 *Let P and Q be meet-order distributive posets. Let $h\colon P \to Q$ be a map. Then, h is an ℓu-morphism if and only if h is a $\wedge$-morphism.*

We summarise in Fig. 3 the hierarchy of all morphisms between posets considered in this section.

6 The Free Distributive Meet-Semilattice Extension

In this section, we shall prove that every meet-order distributive poset can be extended to a distributive meet-semilattice enjoying a universal property. Then, we show that the category of distributive meet-semilattices with top element and meet-homomorphisms preserving top element is a reflective subcategory of the category of meet-order distributive posets with top element and ℓu-morphisms.

From now on, all meet-semilattices are considered to have a top element and meet-homomorphisms between meet-semilattices with top element are considered to preserve top elements. We will use the terminology "meet-semilattice" as an abbreviation of

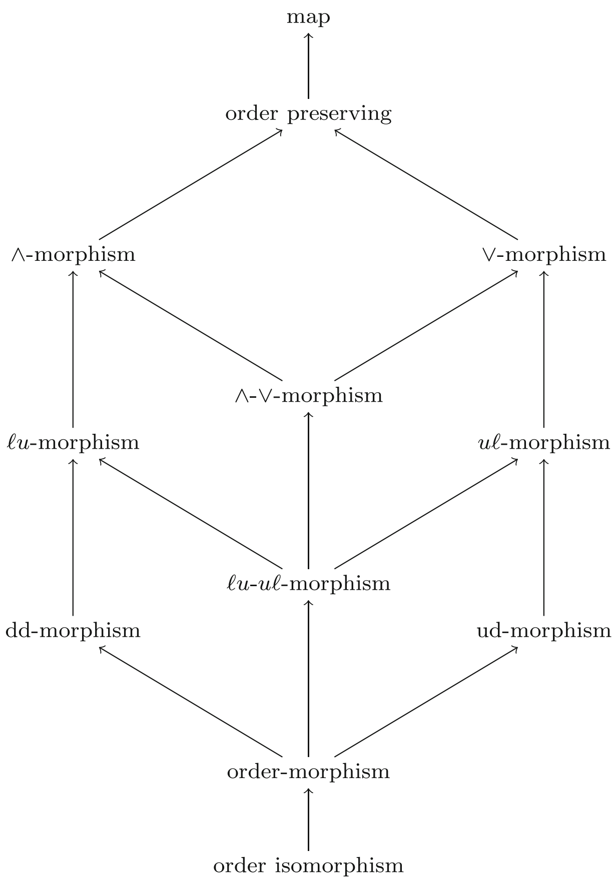

Fig. 3 Hierarchy of morphisms between posets

"meet-semilattice with top element" and "meet-homomorphism" as an abbreviation of "meet-homomorphism preserving top element".

Recall the definition of distributive meet-semilattice, see Definition 2.1.

Definition 6.1 Let P be a poset. A *free distributive meet-semilattice extension of* P is a pair $\langle M, e\rangle$ such that M is a distributive meet-semilattice and $e\colon P \to M$ is an ℓu-$u\ell$-embedding satisfying the following universal property: if $\langle L, \wedge\rangle$ is a distributive meet-semilattice and $f\colon P \to L$ is an ℓu-$u\ell$-embedding, then there is a unique meet-embedding $h\colon M \to L$ such that $h \circ e = f$.

First, we establish a very nice characterisation of when a pair $\langle M, e\rangle$ is a free distributive meet-semilattice extension of a poset P. It should be noted that if $\langle M, \wedge\rangle$ is a meet-semilattice, then $A^{\ell u} = \uparrow(\bigwedge A)$ for every $A \subseteq_\omega M$.

Theorem 6.2 *Let P be a poset. A pair $\langle M, e\rangle$ is a free distributive meet-semilattice extension of P if and only if M is a distributive meet-semilattice and $e\colon P \to M$ is a map such that:*

(E1) *e is an ℓu-ℓu-embedding;*

(E2) *$e[P]$ is finitely meet-dense on M (that is, for every $x \in M$ there is $A \subseteq_\omega P$ such that $x = \bigwedge e[A]$).*

Proof We assume first that $\langle M, e\rangle$ is a free distributive meet-semilattice extension of P. We only need to show that condition (E2) holds. Let L be the sub-meet-semilattice of M generated by $e[P]$, i.e., $L = \{x \in M : (\exists A \subseteq_\omega P)(x = \bigwedge e[A]\}$. It is clear that the map $\widehat{e}\colon P \to L$ defined by $\widehat{e}(a) = e(a)$ is an ℓu-$u\ell$-embedding. Then, there exists a meet-embedding $h\colon M \to L$ such that $h \circ e = \widehat{e}$. Now, let us show that $e[P]$ is finitely meet-dense on M. Let $x \in M$. So $h(x) \in L$ and thus there is $A \subseteq_\omega P$ such that

$$h(x) = \bigwedge e[A] = \bigwedge \widehat{e}[A] = \bigwedge (h \circ e)[A] = h(\bigwedge e[A]).$$

Since h is injective, it follows that $x = \bigwedge e[A]$.

Conversely, suppose that $\langle M, e\rangle$ satisfies conditions (E1) and (E2). Let $\langle L, \wedge\rangle$ be a distributive meet-semilattice and $f\colon P \to L$ an ℓu-$u\ell$-embedding. We define $h\colon M \to L$ as follows: for every $x \in M$, $h(x) = \bigwedge f[A]$ when $x = \bigwedge e[A]$ for some $A \subseteq_\omega P$. First, we show that h is well defined. Let $A, B \subseteq_\omega P$ and suppose that $\bigwedge e[A] = \bigwedge e[B]$. So $\bigwedge e[A] \leq e(b)$ for all $b \in B$ and then $e(b) \in \uparrow(\bigwedge e[A])$ for all $b \in B$. Since e is an ℓu-embedding, it follows that $b \in A^{\ell u}$ for all $b \in B$. Since f is an ℓu-morphism, we obtain that $f(b) \in \uparrow(\bigwedge f[A])$ for all $b \in B$. Then $\bigwedge f[A] \leq \bigwedge f[B]$. Similarly, we have $\bigwedge f[B] \leq \bigwedge f[A]$ and thus $\bigwedge f[A] = \bigwedge f[B]$. Hence h is well defined. With a similar argument to the previous one, we can prove that h is injective. Moreover, it is straightforward to prove directly that h is a meet-homomorphism and $h \circ e = f$. Now, we show that h is unique. Suppose that $g\colon M \to L$ is a meet-embedding such that $g \circ e = f$. Let $x \in M$. Then, there is $A \subseteq_\omega P$ such that $x = \bigwedge e[A]$. Thus, we have

$$h(x) = \bigwedge f[A] = \bigwedge g[e[A]] = g(\bigwedge e[A]) = g(x).$$

Hence, $h = g$. This completes the proof. $\square$

Proposition 6.3 *Let P be a poset. If there exists a free distributive meet-semilattice extension $\langle M, e\rangle$ of P, then P is meet-order distributive.*

Proof Let $\langle M, e\rangle$ be a free distributive meet-semilattice extension of P and let $a, a_1, \ldots, a_n \in P$ be such that $a \in \{a_1, \ldots, a_n\}^{\ell u}$. Since e is an ℓu-morphism, it follows that $e(a) \in \uparrow(e(a_1) \wedge \cdots \wedge e(a_n))$. So $e(a_1) \wedge \cdots \wedge e(a_n) \leq e(a)$. Since M is a distributive meet-semilattice, we have that there exist $x_1, \ldots, x_n \in M$ such that $e(a) = x_1 \wedge \cdots \wedge x_n$ and $e(a_i) \leq x_i$ for all $i = 1, \ldots, n$. Now, by condition (E2), we have that for each $i = 1, \ldots, n$ there exists $A_i \subseteq_\omega P$ such that $x_i = \bigwedge e[A_i]$. Then $e(a) = \bigwedge e[A_1] \wedge \cdots \wedge \bigwedge e[A_n] = \bigwedge e[\bigcup_{i=1}^n A_i]$. We thus obtain that $e(a) \in \uparrow(\bigwedge e[\bigcup_{i=1}^n A_i])$ and then, since e is an ℓu-embedding, it follows that $a \in (\bigcup_{i=1}^n A_i)^{\ell u}$. We also have that $e(a) \leq e(b)$ for all $b \in \bigcup_{i=1}^n A_i$. As e is an order-embedding, $a \leq b$ for all $b \in \bigcup_{i=1}^n A_i$ and then $a \in (\bigcup_{i=1}^n A_i)^{\ell}$. So, we have obtained that $a \in (\bigcup_{i=1}^n A_i)^{\ell u}$ and $a \in (\bigcup_{i=1}^n A_i)^{\ell}$; and this implies that $a = \bigwedge(\bigcup_{i=1}^n A_i)$. Moreover, for each $i = 1, \ldots, n$ we have $e(a_i) \leq x_i = \bigwedge e[A_i] \leq e(b)$ for all $b \in A_i$ and thus for each $i = 1, \ldots, n$, $a_i \leq b$ for all $b \in A_i$. Therefore, P is meet-order distributive. □

It follows by a standard categorical argument that the free distributive meet-semilattice extension of a poset P, if it exists, is unique up to isomorphism. That is:

Proposition 6.4 *Let P be a poset and let $\langle M, e\rangle$ and $\langle M', e'\rangle$ be free distributive meet-semilattice extensions of P. Then there is an isomorphism $h\colon M \to M'$ such that $e' = h \circ e$.*

We show next that there exists a free distributive meet-semilattice extension for every meet-order distributive poset. Let P be a poset. We consider the meet-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ as the dual of the join-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$. That is, $F_1 \wedge_d F_2 := F_1 \vee F_2$ for all $F_1, F_2 \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$. Thus, the order $\leq_d$ associated with $\wedge_d$ on $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$, is given by $F_1 \leq_d F_2 \iff F_2 \subseteq F_1$.

Theorem 6.5 *Let P be a meet-order distributive poset. Then, $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is the free distributive meet-semilattice extension of P.*

Proof By Proposition 4.5, it is clear that the meet-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is distributive. Let $e\colon P \to \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be the map defined by $e(a) = \uparrow a$ for each $a \in P$. Let us show that conditions (E1) and (E2) are satisfied for $\langle\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle, e\rangle$.

Let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$. So, $F = A^{\ell u}$ for some $A \subseteq_\omega P$. Then, we have $F = \bigvee_{a\in A} \uparrow a = \bigvee_{a\in A} e(a) = \bigwedge_d e[A]$. Thus, $e[P]$ is finitely meet-dense on $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ and hence condition (E2) holds. It is straightforward to show that e is an order-embedding. In order to show that e is an ℓu-$u\ell$-morphism, let $A \subseteq_\omega P$ and $b \in P$. First, we assume $b \in A^{\ell u}$ and we prove that $e(b) \in e[A]^{\ell u}$. So, let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be such that $F \leq_d e(a)$ for all $a \in A$. So, $e(a) \subseteq F$ for all $a \in A$; this implies that $A \subseteq F$. Then, since F is a Frink filter, $b \in F$. Thus $e(b) \subseteq F$ and consequently $F \leq_d e(b)$. Then, $e(b) \in e[A]^{\ell u}$ and hence e is an ℓu-morphism. Now, we show that e is a ℓu-morphism. So, assume $b \in A^{u\ell}$ and let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be such that $e(a) \leq_d F$ for all $a \in A$. Then, we have $F \subseteq \bigcap_{a\in A} \uparrow a \subseteq \uparrow b$; that is, $e(b) \leq_d F$. Thus, $e(b) \in e[A]^{u\ell}$. This proves that e is a $u\ell$-morphism. Hence, we have proved that e is an ℓu-ℓu-embedding and thus condition (E1) holds. Therefore $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is the free distributive meet-semilattice extension of P. □

Hereinafter, let us denote by $\langle M(P), e_P\rangle$ or simply by $M(P)$ the free distributive meet-semilattice extension of a meet-order distributive poset P. We drop the subscript on e_P when confusion is unlikely. Notice that if P has a top (bottom) element 1_P (0_P), then $e(1_P)$ ($e(0_p)$) is the top (bottom) element of $M(P)$.

Proposition 6.6 ([12]) *Let P be a meet-order distributive poset. If P is a join-semilattice, then $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ is a sub-lattice of $\mathsf{Fi}_{\mathsf{F}}(P)$.*

Corollary 6.7 *If P is a meet-order distributive join-semilattice, then the free distributive meet-semilattice extension of P is a distributive lattice.*

We close this section showing a categorical result. More precisely, we prove the existence of an adjoint (Theorem 6.12). We start with the following results.

Proposition 6.8 *Let P and Q be meet-order distributive posets. If $h\colon P \to Q$ is an ℓu-morphism, then there is a unique meet-homomorphism $M(h)\colon M(P) \to M(Q)$ such that $e_Q \circ h = M(h) \circ e_P$. Moreover, if h is an ℓu-embedding, then $M(h)$ is a meet-embedding.*

Proof By condition (E2), for every $x \in M(P)$ there is $A \subseteq_\omega P$ such that $x = \bigwedge e_P[A]$. So, we define $M(h)\colon M(P) \to M(Q)$ as follows: for every $x \in M(P)$, $M(h)(x) = \bigwedge(e_Q \circ h)[A]$ if $x = \bigwedge e_P[A]$ for some $A \subseteq_\omega P$. By a similar argument used in the proof of Theorem 6.2, we have that $M(h)$ is well defined. It is straightforward to show that $M(h)$ is a meet-homomorphism and satisfies $e_Q \circ h = M(h) \circ e_P$. If $k\colon M(P) \to M(Q)$ is a meet-homomorphism such that $e_Q \circ h = k \circ e_P$, then

$$M(h)(\bigwedge e_P[A]) = \bigwedge(e_Q \circ h)[A] = \bigwedge(k \circ e_P)[A] = k(\bigwedge e_P[A])$$

for all $A \subseteq_\omega P$. Hence, $k = M(h)$. Finally, by a similar argument used to prove that h is well defined, we can prove that $M(h)$ is injective whenever h is an order embedding. □

Remark 6.9 It is straightforward to check that $M(h)\colon M(P) \to M(Q)$ preserves top elements whenever P and Q have top elements.

Proposition 6.10 *Let P, Q and R be meet-order distributive posets and let $h\colon P \to Q$ and $g\colon Q \to R$ be ℓu-morphisms. Then, $M(g \circ h) = M(g) \circ M(h)$. Moreover, if $\mathrm{id}_P\colon P \to P$ is the identity map, then $M(\mathrm{id}_P) = \mathrm{id}_{M(P)}$.*

Proof We know that the composition $g \circ h\colon P \to R$ is an ℓu-morphism. Then, by Proposition 6.8, $M(g \circ h)\colon M(P) \to M(R)$ is the unique meet-homomorphism such that $e_R \circ (g \circ h) = M(g \circ h) \circ e_P$. By Proposition 6.8 again, we have that $M(h)\colon M(P) \to (Q)$ and $M(g)\colon M(Q) \to M(R)$ are meet-homomorphisms such that $e_Q \circ h = M(h) \circ e_P$ and $e_R \circ g = M(g) \circ e_Q$. Then, we have

$$\begin{aligned} e_R \circ (g \circ h) &= (e_R \circ g) \circ h = (M(g) \circ e_Q) \circ h = M(g) \circ (e_Q \circ h) \\ &= M(g) \circ (M(h) \circ e_P) = (M(g) \circ M(h)) \circ e_P. \end{aligned}$$

Hence $M(g \circ h) = M(g) \circ M(h)$. Moreover, since $e_P \circ \mathrm{id}_P = \mathrm{id}_{M(P)} \circ e_P$, it follows that $\mathrm{id}_{M(P)} = M(\mathrm{id}_P)$. □

Let us denote by $\mathbb{MDP}$ the category formed by all meet-order distributive posets with a top element and all ℓu-morphisms. It should be clear that the composition of morphisms in this category is the usual set-theoretical composition of functions and the identity morphism for an object of $\mathbb{MDP}$ is the identity map. We also consider the category whose objects are all distributive meet-semilattices with a top element and morphisms are all meet-homomorphisms preserving top elements. We denote this category by $\mathbb{DMS}$. From Propositions 6.8 and 6.10, the map $M(-)$ sending every meet-order distributive poset P to its free distributive meet-semilattice extension $M(P)$ extends to a functor $\mathbf{M}\colon \mathbb{MDP} \to \mathbb{DMS}$.

Notice that if M is a distributive meet-semilattice, then the free distributive meet-semilattice extension of M is (up to isomorphism) M. Thus, we have an immediate consequence of Proposition 6.8.

Corollary 6.11 *Let P be a meet-order distributive poset and let L be a distributive meet-semilattice. If $h\colon P \to L$ is an ℓu-morphism, then there exists a unique meet-homomorphism $M(h)\colon M(P) \to L$ such that $h = M(h) \circ e_P$. Moreover, if h is an ℓu-embedding, then $M(h)$ is a meet-embedding.*

Recall that every meet-semilattice M can be defined as a poset where the greatest lower bound exists for every pair of elements of M. Then, we can consider the category $\mathbb{DMS}$ as a full subcategory of $\mathbb{MDP}$, and thus we can define the inclusion functor $\mathbf{U}\colon \mathbb{DMS} \to \mathbb{MDP}$. Then, by Corollary 6.11, we obtain:

Theorem 6.12 *The functor $\mathbf{M}\colon \mathbb{MDP} \to \mathbb{DMS}$ is a left adjoint for the functor $\mathbf{U}$ and therefore the category $\mathbb{DMS}$ is a reflective subcategory of the category $\mathbb{MDP}$.*

Remark 6.13 A free distributive lattice extension for a meet-order distributive poset P can be obtained from the free distributive meet-semilattice extension $M(P)$ of P and from the free distributive lattice extension of the distributive meet-semilattice $M(P)$ developed by Bezhanishvily and Jansana in [2]. We refer the reader to [7] for more details about this topic.

7 The Connection Between the Frink Filters of a Meet-Order Distributive Poset and the Filters of its Free Distributive Meet-Semilattice Extension

Throughout of this section, we will consider that all posets and all meet-semilattices have top elements. The extension of the results of this section to the case without top element is quite direct and can be seen in [7].

From now on, let P be a meet-order distributive poset with a top element and $\langle M, e\rangle$ its free distributive meet-semilattice extension. We recall that for a subset $X \subseteq P$, $\mathrm{Fig}_{\mathrm{F}}(X)$ denotes the Frink filter of P generated by X. Similarly, we denote by $\mathsf{Fi}(M)$ the lattice of all filters of M and by Fig(.) the closure operator associated with the closure system $\mathsf{Fi}(M)$.

Proposition 7.1 *If F is a Frink filter of P, then $e^{-1}\left[\mathrm{Fig}(e[F])\right] = F$.*

Proof Let F be a Frink filter of P. Since $e[F] \subseteq \mathrm{Fig}(e[F])$, we obtain $F \subseteq e^{-1}[\mathrm{Fig}(e[F])]$. Let now $a \in e^{-1}[\mathrm{Fig}(e[F])]$. Thus, there are $a_1, \ldots, a_n \in F$ such that $e(a_1) \wedge \cdots \wedge e(a_n) \leq$

$e(a)$. Since e is an ℓu-embedding, it follows that $a \in \{a_1, \dots, a_n\}^{\ell u}$ and then $a \in F$. Hence $e^{-1}\left[\mathrm{Fig}(e[F])\right] = F$. □

Proposition 7.2 *If G is a filter of M, then $e^{-1}[G]$ is a Frink filter of P and $G = \mathrm{Fig}(e[e^{-1}[G]])$.*

Proof Let G be a filter of M. It is clear by Proposition 5.5 that $e^{-1}[G] \in \mathsf{Fi_F}(P)$. Now, it is also clear that $\mathrm{Fig}(e[e^{-1}[G]]) \subseteq G$. Let $x \in G$. By (E2), $x = e(a_1) \wedge \cdots \wedge e(a_n)$ for some $a_1, \dots, a_n \in P$. Since G is an up-set, it follows that $e(a_i) \in G$ for all $i \in \{1, \dots, n\}$. Then, $a_i \in e^{-1}[G]$ for all $i \in \{1, \dots, n\}$; which implies that $e(a_i) \in \mathrm{Fig}(e[e^{-1}[G]])$ for all $i \in \{1, \dots, n\}$. Hence, we have $x = e(a_1) \wedge \cdots \wedge e(a_n) \in \mathrm{Fig}(e[e^{-1}[G]])$. Therefore, $G = \mathrm{Fig}(e[e^{-1}[G]])$. □

We now consider the maps $\alpha \colon \mathsf{Fi_F}(P) \to \mathsf{Fi}(M)$ and $\beta \colon \mathsf{Fi}(M) \to \mathsf{Fi_F}(P)$ defined as follows:

$$\alpha(F) = \mathrm{Fig}(e[F]) \quad \text{and} \quad \beta(G) = e^{-1}[G]$$

for every $F \in \mathsf{Fi_F}(P)$ and for every $G \in \mathsf{Fi}(M)$, respectively.

Theorem 7.3 *Let P be a meet-order distributive poset and M its free distributive meet-semilattice extension. Then, the map $\alpha \colon \mathsf{Fi_F}(P) \to \mathsf{Fi}(M)$ establishes a lattice isomorphism from the lattice of Frink filters of P onto the lattice of filters of M, whose inverse is the map $\beta \colon \mathsf{Fi}(M) \to \mathsf{Fi_F}(P)$.*

Proof Let $F_1, F_2 \in \mathsf{Fi_F}(P)$. Then, by Proposition 7.1, we have

$$F_1 \subseteq F_2 \iff \mathrm{Fig}(e[F_1]) \subseteq \mathrm{Fig}(e[F_2]) \iff \alpha(F_1) \subseteq \alpha(F_2).$$

Thus we obtain that α is an order-embedding. By Proposition 7.2, it is clear that α is an onto map. Hence, α is an order isomorphism and therefore is a lattice isomorphism. Moreover, from Propositions 7.1 and 7.2, we obtain that β is the inverse map of α. □

8 Some Conclusions

We have shown that for every meet-order distributive poset there exists its free distributive meet-semilattice extension, see Theorem 6.5. In particular, if either P is a finite meet-order distributive poset or a meet-order distributive join-semilattice, then its distributive meet-semilattice extension $M(P)$ is a distributive lattice (see Corollaries 4.6 and 6.7, respectively). Thus, a Priestley-type duality may be developed for the class of finite meet-order distributive posets or the class of meet-order distributive join-semilattices. Similar results in the setting of distributive meet-semilattices can be found in [1, 2].

The free distributive meet-semilattice extension M of a (finite) meet-order distributive poset P could be used to study the equivalence relations $\theta \cap P$ on P, where $\theta \in \mathrm{Con}(M)$, as a generalisation of the concept of congruences in Lattice Theory. For some studies concerning equivalence relations on posets generalising the concept of congruence for lattices, we refer the reader to [10, 15].

References

1. Bezhanishvili, G., Jansana, R.: Duality for Distributive and Implicative Semi-Lattices, Preprints of University of Barcelona. Research Group in non.classical logics (2008)
2. Bezhanishvili, G., Jansana, R.: Priestley style duality for distributive meet-semilattices. Stud. Logica. **98**(1-2), 83–122 (2011)
3. Cornish, W., Hickman, R.: Weakly distributive semilattices. Acta Math. Hungar. **32**(1), 5–16 (1978)
4. Davey, B., Priestley, H.: Introduction to Lattices and Order, 2nd edn. Printed in the United Kingdom at the University Press, Cambridge (2002)
5. David, E., Erné, M.: Ideal completion and stone representation of ideal-distributive ordered sets. Topology Appl. **44**(1), 95–113 (1992)
6. Frink, O.: Ideals in partially ordered sets. Amer. Math. Monthly **61**(4), 223–234 (1954)
7. González, L.: Topological Dualities and Completions for (Distributive) Partially Ordered Sets. Ph.D. Thesis, Universidad de Barcelona (2015). http://hdl.handle.net/10803/314382
8. González, L., Jansana, R.: A spectral-style duality for distributive posets. Order **35**, 321–347 (2018). https://doi.org/10.1007/s11083-017-9435-2
9. Grätzer, G.: Lattice Theory: Foundation. Springer Science & Business Media (2011)
10. Halaš, R.: Congruences on posets. In: Contributions to General Algebras, Proceedings of the Vienna Conference, vol. 12, pp. 195–210. Verlag Johannes Heyn, Vienna 1999 (2000)
11. Hickman, R.: Join algebras. Comm. Algebra **8**(17), 1653–1685 (1980)
12. Hickman, R.: Mildly distributive semilattices. J. Aust. Math. Soc. **36**(3), 287–315 (1984)
13. Hickman, R., Monro, G.: Distributive partially ordered sets. Fund. Math. **120**, 151–166 (1984)
14. Morton, W.: Canonical extensions of posets. Algebra Univers. **72**(2), 167–200 (2014)
15. Shum, K.P., Zhu, P., Kehayopulu, N.: III -homomorphisms and III-congruences on posets. Discret. Math. **308**(21), 5006–5013 (2008)