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FOUNDATIONS

The logic of distributive nearlattices

Luciano J. González[1]

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Abstract In this paper, we propose a sentential logic naturally associated, in the sense of Abstract Algebraic Logic, with the variety of distributive nearlattices. We show that the class of algebras canonically associated (in the sense of Abstract Algebraic Logic) with this logic is the variety of distributive nearlattices. We also present several properties of this sentential logic.

Keywords Distributive nearlattices · Sentential logic · Gentzen system

1 Introduction

In this article, we define a Gentzen system through some Gentzen-style rules and we consider the sentential logic defined by this Gentzen system. Then, in the framework of the general theory of Abstract Algebraic Logic, we show that the algebraic counterparts of both this Gentzen system and its sentential logic coincide with the variety of distributive nearlattices.

Abstract Algebraic Logic (AAL) studies, in a completely general and abstract way, relations between sentential logics and algebraic semantics. One of the major achievements of AAL was a development of a general method by which a class of algebras can be associated with every sentential logic; this general method can be considered as the result of performing the *Lindenbaum–Tarski process* suitably generalized, using the notion of generalized matrix. For each sentential logic, the class of algebras defined by this method is considered in AAL as its natural algebraic counterpart.

Font and Verdú (1991) studied (see also Font and Jansana 2009, Chapter 5) the relations between the $\{\wedge, \vee\}$-fragment of classical propositional logic and the variety of distributive lattices. They presented a Gentzen system adequate for the $\{\wedge, \vee\}$-fragment of classical proposition logic and proved that the class of algebras naturally associated with this Gentzen system is the variety of distributive lattices. Then, Rebagliato and Verdú (1993) proved that this Gentzen system is very close to the equational consequence relation relative to the variety of distributive lattices; in other words, they have proved that this Gentzen system is *algebraizable* (Rebagliato and Verdú 1995; Font and Jansana 2009) with *equivalent algebraic semantics* the variety of distributive lattices. From these results, Font and Jansana (2009) found that the canonical class of algebras naturally associated with the $\{\wedge, \vee\}$-fragment of classical propositional logic is the variety of distributive lattices.

In Rebagliato and Verdú (1993) (see also Font and Jansana 2009, Chapter 5) proposed is a Gentzen system associated with the variety of lattices. It is proved there that this Gentzen system is algebraizable with equivalent algebraic semantics the variety of lattices and its corresponding class of algebras canonically associated is the variety of lattices. In Font and Jansana (2009), it is proved that the canonical class of algebras associated with the sentential logic defined by this Gentzen system is the variety of lattices.

In the present paper, we use some of the strategies and techniques developed in Font and Verdú (1991), Rebagliato and Verdú (1993) and Font and Jansana (2009) to show that the corresponding classes of algebras canonically associated with both the proposed Gentzen system and its sentential logic coincide with the variety of distributive nearlattices.

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✉ Luciano J. González
lucianogonzalez@exactas.unlpam.edu.ar

1 Fac. de Cs. Exactas y Naturales, Universidad Nacional de La Pampa, 3600 Santa Rosa, Argentina

The concept of (*distributive*) *nearlattice* is a generalization of the notion of implication algebras. The variety of *implication algebras* was introduced and studied by Abbott (1967). This variety is the algebraic counterpart of the $\{\to\}$-fragment of classical propositional logic. An implication algebra (Abbott 1967; Monteiro 1980, see also Abbott 1976; Chajda and Halaš 2005; Chajda et al. 2001) is defined as an algebra with only one binary connective such that satisfies some identities. Abbott (1967) showed that the class of implication algebras can also be defined as the class of join-semilattices with top element for which each principal upset is a Boolean algebra. By weakening the condition that each principal upset is a Boolean algebra to be a (distributive) lattice, we come to the concept of (distributive) nearlattice. That is, a (distributive) nearlattice is a join-semilattice such that each principal upset is a (distributive) lattice. Nearlattices were studied by many Cornish and Hickman (1978), Hickman (1980), Chajda et al. (2007), Chajda and Kolařík (2008), Celani and Calomino (2014), Celani and Calomino (2016), Calomino (2015). The concept of (distributive) nearlattice can be defined equivalently as those algebras with only one ternary connective such that some identities hold (see Hickman 1980); hence, the class of (distributive) nearlattices is a variety.

The paper is organized as follows. Section 2 is devoted to introducing the concepts needed to follow the paper. In Sect. 2.1, we present some notions of Abstract Algebraic Logic; in Sect 2.2, we present the basic facts about nearlattices. In Sect 3, we introduce a definition that will be useful in what follows; we show a new characterization of the distributivity condition on nearlattice using this new definition. The aim of Sect. 4 is to propose a Gentzen system defined by means of a Gentzen calculus and then study the relations between this Gentzen system, its associated sentential logic and the variety of distributive nearlattices. In fact, we will see that the variety of distributive nearlattices is the algebraic counterpart, in the sense of AAL, of this Gentzen system and of its associated sentential logic.

2 Preliminaries

Let X be a non-empty set. We write $X_0 \subseteq_\omega^* X$ to concisely indicate that X_0 is a non-empty finite subset of X.

2.1 Abstract Algebraic Logic

The aim of this part is to fix some terminology and notations in AAL. For more detailed information about AAL, we refer the reader to Czelakowski (2001), Font and Jansana (2009) and Font et al. (2003).

Throughout the whole article, we use the terminology *sentential logic* as an abbreviation of *finitary sentential logic* (also called *deductive system* in AAL).

Let $\mathcal{L}$ be an algebraic language and $\mathcal{S} = \langle Fm, \vdash_\mathcal{S}\rangle$ a sentential logic. Let us denote by $\mathsf{Th}(\mathcal{S})$ the collection of all theories of $\mathcal{S}$ (or $\mathcal{S}$-theories for short).

Let $\mathcal{S}$ be a sentential logic. The *Frege relation* of $\mathcal{S}$, in symbols $\Lambda(\mathcal{S})$, is the interderivability relation, that is, $(\varphi, \psi) \in \Lambda(\mathcal{S})$ if and only if $\varphi \vdash_\mathcal{S} \psi$ and $\psi \vdash_\mathcal{S} \varphi$. The Frege relation of a sentential logic is an equivalence relation, but it is not necessarily a congruence on Fm. A sentential logic $\mathcal{S}$ is said to be *selfextensional* if the Frege relation $\Lambda(\mathcal{S})$ is a congruence on Fm.

Let A be an algebra of the same similarity type as $\mathcal{S}$. A subset $F \subseteq A$ is said to be an $\mathcal{S}$-*filter* of A if and only if for any $\Gamma \cup \{\varphi\} \subseteq Fm$ and any interpretation $h \in \mathrm{Hom}(Fm, A)$,

if $\Gamma \vdash_\mathcal{S} \varphi$ and $h[\Gamma] \subseteq F$ then $h(\varphi) \in F$.

The set of all $\mathcal{S}$-filters on a given algebra A is denoted by $\mathsf{Fi}_\mathcal{S}(A)$; this set is an algebraic closure system. The associated closure operator will be denoted by $\mathrm{Fi}_\mathcal{S}^A$.

Let Fm be the algebra of formulas of a given algebraic similarity type $\mathcal{L}$. For our purpose, we will consider a *sequent* of type $\mathcal{L}$ to be a pair $\langle \Gamma, \varphi\rangle$ where Γ is a finite (possible empty) set of formulas and φ is a formula. As usual, we write $\Gamma \rhd \varphi$ instead of $\langle \Gamma, \varphi\rangle$. Let us denote by $\mathrm{Seq}(\mathcal{L})$ the collection of all sequents, and we consider the set $\mathrm{Seq}^{\mathrm{o}}(\mathcal{L}) := \{\Gamma \rhd \varphi : \Gamma \neq \emptyset\}$. A *Gentzen-style rule* is a pair $\langle X, \Gamma \rhd \varphi\rangle$ where X is a (possible empty) finite set of sequents and $\Gamma \rhd \varphi$ is a sequent. As usual, we shall use the standard fraction notation for Gentzen-style rules:

$$\frac{\Gamma_0 \rhd \varphi_0, \ldots, \Gamma_{n-1} \rhd \varphi_{n-1}}{\Gamma \rhd \varphi} \tag{1}$$

A *substitution instance of a Gentzen-style rule* $\langle X, \Gamma \rhd \varphi\rangle$ is a Gentzen-style rule of the form $\langle \sigma[X], \sigma[\Gamma] \rhd \sigma(\varphi)\rangle$ for some substitution $\sigma \in \mathrm{Hom}(Fm, Fm)$ and where $\sigma[X] := \{\sigma[\Delta] \rhd \sigma(\psi) : \Delta \rhd \psi \in X\}$. A *Gentzen calculus* is a set of Gentzen-style rules. Given a Gentzen calculus $\mathbf{G}$, the notion of a formal proof can be defined as usual. That is, a *proof* in the Gentzen calculus $\mathbf{G}$ from a set of sequents X is a finite sequence of sequents each of which is a substitution instance of a rule of $\mathbf{G}$ or a sequent in X or is obtained by applying a substitution instance of a rule of $\mathbf{G}$ to previous elements in the sequence. A sequent $\Gamma \rhd \varphi$ is *derivable in* $\mathbf{G}$ *from* a set of sequents X if there is a proof in $\mathbf{G}$ from X whose last sequent in the proof is $\Gamma \rhd \varphi$. We express this writing $X \mid\!\sim_{\mathbf{G}} \Gamma \rhd \varphi$.

Definition 2.1 A *Gentzen system of type* ω (resp. of type ω^{o}) is a pair $\mathcal{G} = \langle Fm, \mid\!\sim_\mathcal{G}\rangle$ where $\mid\!\sim_\mathcal{G}$ is a finitary closure operator on the set $\mathrm{Seq}(\mathcal{L})$ (resp. on the set $\mathrm{Seq}^{\mathrm{o}}(\mathcal{L})$)

that is substitution-invariant and which satisfies the following *structural rules:* for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\text{(Axiom)}\ \frac{\emptyset}{\varphi \rhd \varphi} \qquad \text{(Weakening)}\ \frac{\Gamma \rhd \varphi}{\Gamma, \psi \rhd \varphi}$$

$$\text{(Cut)}\ \frac{\Gamma \rhd \varphi \qquad \Gamma, \varphi \rhd \psi}{\Gamma \rhd \psi}$$

We say that a Gentzen system $\mathcal{G} = \langle Fm, \mid\!\sim_{\mathcal{G}}\rangle$ *has a Gentzen-style rule* of type (1) or that (1) is a *Gentzen-style rule of* $\mathcal{G}$ if $\Gamma_0 \rhd \varphi_0, \ldots, \Gamma_{n-1} \rhd \varphi_{n-1} \mid\!\sim_{\mathcal{G}} \Gamma \rhd \varphi$ and we say that a sequent $\Gamma \rhd \varphi$ is a *derivable sequent* of $\mathcal{G}$ when $\emptyset \mid\!\sim_{\mathcal{G}} \Gamma \rhd \varphi$.

Let **G** be a Gentzen calculus with the structural rules of (Axiom), (Weakening) and (Cut). Hence, **G** defines in a standard way the Gentzen system $\mathcal{G}_{\mathbf{G}} = \langle Fm, \mid\!\sim_{\mathbf{G}}\rangle$ (see Font and Jansana 2009; Rebagliato and Verdú 1995).

For any Gentzen system $\mathcal{G}$, we denote by $\mathrm{Seq}(\mathcal{G})$ either $\mathrm{Seq}(\mathcal{L})$ if $\mathcal{G}$ is of type ω or $\mathrm{Seq}^{\mathrm{o}}(\mathcal{L})$ if $\mathcal{G}$ is of type ω^{o}, and we call *sequents of* $\mathcal{G}$ to the elements of $\mathrm{Seq}(\mathcal{G})$.

Definition 2.2 Let $\mathcal{G}$ be a Gentzen system. The *sentential logic defined by* $\mathcal{G}$ is the logic $\mathcal{S}_{\mathcal{G}} = \langle Fm, \vdash_{\mathcal{G}}\rangle$ where the consequence relation $\vdash_{\mathcal{G}}$ is defined as follows: for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathcal{G}} \varphi \iff \text{there is a finite } \Gamma_0 \subseteq \Gamma \text{ such that } \mid\!\sim_{\mathcal{G}} \Gamma \rhd \varphi.$$

If $\mathcal{S}$ is a sentential logic, then we say that a Gentzen system $\mathcal{G}$ is *adequate for* $\mathcal{S}$ when $\mathcal{S}$ is the logic defined by $\mathcal{G}$, i.e., $\mathcal{S} = \mathcal{S}_{\mathcal{G}}$.

Now we are going to present algebraic models for Gentzen systems and sentential logics. Let $\mathcal{L}$ be a fixed but arbitrary algebraic language. A *generalized matrix*, *g-matrix* for short, of similarity type $\mathcal{L}$ is a pair $\langle A, \mathcal{C}\rangle$ where A is an algebra of type $\mathcal{L}$ and $\mathcal{C}$ is an algebraic closure system on A. We denote by C the closure operator associated with $\mathcal{C}$, and we will often identify the g-matrix $\langle A, \mathcal{C}\rangle$ with the pair $\langle A, \mathrm{C}\rangle$. Notice that the closure operator C is finitary, i.e., for all $X \cup \{a\} \subseteq A$, $a \in \mathrm{C}(X)$ implies that there is a finite $X_0 \subseteq X$ such that $a \in \mathrm{C}(X_0)$. The reader should be keep in mind that all logics and g-matrices considered in this paper are finitary, and thus, some general results of AAL are restricted to this assumption.

One of the most interesting aspects of g-matrices is that they can be used in a completely natural way as models of sentential logics and as models of Gentzen systems. This double function of g-matrices allows to relate the algebraic theory of sentential logics to the algebraic theory of Gentzen systems.

Definition 2.3 Let $\langle A, \mathrm{C}\rangle$ be a g-matrix.

- $\langle A, \mathrm{C}\rangle$ is a *g-model* of a sentential logic $\mathcal{S}$ when for all $\Gamma \cup \{\varphi\} \subseteq Fm$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $h(\varphi) \in \mathrm{C}(h[\Gamma])$ for all $h \in \mathrm{Hom}(Fm, A)$. Let us denote the class of all g-models of a sentential logic $\mathcal{S}$ by $\mathbf{GMod}(\mathcal{S})$.
- $\langle A, \mathrm{C}\rangle$ is a *model* of a Gentzen-style rule (1) when for all $h \in \mathrm{Hom}(Fm, A)$, if $h(\varphi_i) \in \mathrm{C}(h[\Gamma_i])$ for all $0 \leq i \leq n-1$, then $h(\varphi) \in \mathrm{C}(h[\Gamma])$.
- $\langle A, \mathrm{C}\rangle$ is a *model* of a Gentzen system $\mathcal{G}$ when it is a model of all Gentzen-style rules of $\mathcal{G}$. The class of all models of $\mathcal{G}$ is denoted by $\mathbf{Mod}(\mathcal{G})$.

Notice that if a g-matrix is a model of all Gentzen-style rules of a Gentzen calculus, then it is a model of the Gentzen system defined by this Gentzen calculus.

The *Frege relation* of a g-matrix $\langle A, \mathrm{C}\rangle$ is defined by:

$$(a, b) \in \Lambda_A \mathrm{C} \iff \mathrm{C}(a) = \mathrm{C}(b)$$

for every $a, b \in A$. The *Tarski congruence* of a g-matrix $\langle A, \mathrm{C}\rangle$ is the largest congruence below the Frege relation of the g-matrix. We denote the Tarski congruence of $\langle A, \mathrm{C}\rangle$ by $\widetilde{\Omega}_A(\mathrm{C})$. A g-matrix is said to be *reduced* when its Tarski congruence is the identity relation.

Let us denote by $\mathbf{GMod}^*(\mathcal{S})$ the class of all reduced g-models of a sentential logic $\mathcal{S}$, and we denote by $\mathbf{Mod}^*(\mathcal{G})$ the class of all reduced models of a Gentzen system $\mathcal{G}$.

We can now introduce the classes of algebras that are considered in AAL as canonically associated with a sentential logic and with a Gentzen system.

Definition 2.4 Let $\mathcal{S}$ be a sentential logic and $\mathcal{G}$ be a Gentzen system. The following classes of algebras are defined:

- $\mathrm{Alg}(\mathcal{S}) := \mathrm{Alg}(\mathbf{GMod}^*(\mathcal{S}))$, class of the algebraic reducts of the reduced g-models of $\mathcal{S}$. These algebras are called $\mathcal{S}$*-algebras*.
- $\mathrm{Alg}(\mathcal{G}) := \mathrm{Alg}(\mathbf{Mod}^*(\mathcal{G}))$, class of the algebraic reducts of the reduced models of $\mathcal{G}$. These algebras are called $\mathcal{G}$*-algebras*.

For a sentential logic $\mathcal{S}$, the class $\mathrm{Alg}(\mathcal{S})$ is not necessarily a variety. An important and useful variety of algebras naturally associated with a sentential logic is defined by: $\mathrm{K}_{\mathcal{S}} := \mathbb{V}(Fm/\widetilde{\Omega}(\mathcal{S}))$ is the variety generated by the algebra $Fm/\widetilde{\Omega}(\mathcal{S})$; this variety is called *the intrinsic variety of* $\mathcal{S}$.

Now we present some known relations between the previous classes of algebras.

Lemma 2.5 *Let* $\mathcal{S}$ *be a sentential logic. Then, the intrinsic variety of* $\mathcal{S}$ *is the variety generated by the class* $\mathrm{Alg}(\mathcal{S})$*, and hence, we have* $\mathrm{Alg}(\mathcal{S}) \subseteq \mathbb{V}(\mathrm{Alg}(\mathcal{S})) = \mathrm{K}_{\mathcal{S}}$.

Lemma 2.6 *(Font and Jansana (2009), Lemma 4.9) Let $\mathcal{G}$ be a Gentzen system and let $\mathcal{S}_{\mathcal{G}}$ be the sentential logic defined by $\mathcal{G}$. Then, every model of $\mathcal{G}$ is a g-model of $\mathcal{S}_{\mathcal{G}}$, and hence,* $\mathrm{Alg}(\mathcal{G}) \subseteq \mathrm{Alg}(\mathcal{S}_{\mathcal{G}})$.

Definition 2.7 A g-matrix $\langle A, \mathcal{C}\rangle$ is called a *full g-model of* a logic $\mathcal{S}$ if and only if $\mathsf{Fi}_{\mathcal{S}}(A/\widetilde{\Omega}_A\mathcal{C}) = \{T/\widetilde{\Omega}_A\mathcal{C} : T \in \mathcal{C}\}$. We denote by $\mathbf{FGMod}(\mathcal{S})$ the class of all full g-models of $\mathcal{S}$.

The notion of *fully selfextensional* logic has several useful characterizations. We present this concept and its characterizations in the next definition.

Definition 2.8 A sentential logic $\mathcal{S}$ is called *fully selfextensional* if one (and all) of the following equivalent conditions hold:

(1) the Frege relation of all full g-models of $\mathcal{S}$ is a congruence;
(2) for every algebra A, the Frege relation of the g-matrix $\langle A, \mathsf{Fi}_{\mathcal{S}}(A)\rangle$ is a congruence on A;
(3) for every $A \in \mathrm{Alg}(\mathcal{S})$, the Frege relation of the g-matrix $\langle A, \mathsf{Fi}_{\mathcal{S}}(A)\rangle$ is the identity relation.

Definition 2.9 Let $\mathcal{G}$ be a Gentzen system and $\mathcal{S}$ a sentential logic. We say that $\mathcal{G}$ is *fully adequate for* $\mathcal{S}$ when one of the following two conditions holds:

(1) $\mathcal{S}$ has theorems, $\mathcal{G}$ is of type ω and $\mathbf{FGMod}(\mathcal{S}) = \mathbf{Mod}(\mathcal{G})$.
(2) $\mathcal{S}$ does not have theorems, $\mathcal{G}$ is of type ω^o and $\mathbf{FGMod}(\mathcal{S}) = \{\langle A, \mathrm{C}\rangle \in \mathbf{Mod}(\mathcal{G}) : \langle A, \mathrm{C}\rangle$ does not have theorems$\}$.

An *equation* of type $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$ for each $\varphi, \psi \in Fm$. The set of all equations will be denoted by $\mathrm{Eq}(Fm)$. Let K be a class of algebras of the same similarity type $\mathcal{L}$. The *equational consequence relative to* K, denoted by $\models_{\mathrm{K}}$, is defined as follows: if $\Theta \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}(Fm)$, then

$$\begin{aligned}\Theta \models_{\mathrm{K}} \varphi \approx \psi \iff & (\forall A \in \mathrm{K})(\forall h \in \mathrm{Hom}(Fm, A))\\ & (h(\alpha) = h(\beta) \text{ for all } \alpha \approx \beta \in \Theta\\ & \text{implies} \quad h(\varphi) = h(\psi)).\end{aligned}$$

2.2 Nearlattices

Here we introduce the main definitions and basic properties of nearlattices. Our main references for the theory of nearlattices are Chajda et al. (2007), Araújo and Kinyon (2011), Hickman (1980), and Calomino (2015). We address the reader to these references for those concepts mentioned in this paper that are not explicit introduced here. Moreover, we assume that the reader is familiar with elementary order-theoretic notions (see, for instance, Davey and Priestley 2002).

The notion of nearlattice can be presented in two different and equivalent ways. They can be defined as join-semilattices that satisfy some property and can be defined as algebras with only one ternary connective satisfying some identities. The two different ways to consider nearlattices are useful for different purposes.

Definition 2.10 An algebra $\langle A, m\rangle$ of type (3) is called a *nearlattice* if the following identities hold:

(P1) $m(x, y, x) = x$,
(P2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$.

Theorem 2.11 (1) *If $\langle A, m\rangle$ is a nearlattice, then the algebra $A_* = \langle A, \vee\rangle$, where*

$$x \vee y := m(x, x, y), \tag{J2}$$

is a join-semilattice such that for every $a \in A$ the principal upset $[a) = \{b \in A : a \leq b\}$ is a lattice with respect to the order induced by $\vee$.

(2) *If $\langle S, \vee\rangle$ is a join-semilattice such that every principal upset is a lattice, then the algebra $S^* = \langle S, m\rangle$ with*

$$m(x, y, z) := (x \vee z) \wedge_z (y \vee z)$$

(where $\wedge_z$ is the meet of $x \vee z$ and $y \vee z$ in $[z)$) is a nearlattice.

(3) *If A is a nearlattice and S is join-semilattice such that every principal upset is a lattice, then $(A_*)^* = A$ and $(S^*)_* = S$.*

This theorem show us that there is a one-to-one correspondence between nearlattices and join-semilattices where every principal upset is a lattice. Hence, the join-semilattices where all principal upsets are lattices can also be called *nearlattices*. We will consider nearlattices as ternary algebras $\langle A, m\rangle$ satisfying the identities (P1) and (P2), and we consider the join operation $\vee$ on A defined as in (J2). Moreover, the partial order $\leq$ on A is determined by $\vee$, i.e., $x \leq y$ if and only if $y = x \vee y = m(x, x, y)$.

Let A be a nearlattice. For every element $a \in A$, we denote the meet in $[a)$ by $\wedge_a$. It should be noted that the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A. Thus, the meet of x and y in $[a)$ coincides with their meet in A for all $x, y \in [a)$, i.e., $x \wedge_a y = x \wedge y$. This should be kept in mind since we will use it without mention.

Definition 2.12 A nearlattice $\langle A, m\rangle$ is called *distributive* if for every $a \in A$, the lattice $\langle [a), \vee, \wedge_a\rangle$ is distributive.

Proposition 2.13 *A nearlattice $\langle A, m\rangle$ is distributive if and only if it satisfies either of the following (equivalent) identities:*

(D1) $m(x, y \vee z, w) = m(x, y, w) \vee m(x, z, w)$;
(D2) $x \vee m(y, z, w) = m(x \vee y, x \vee z, w)$.

We denote by $\mathbb{DN}$ the variety of all distributive nearlattices. Let $\mathbf{2} = \{0, 1\}$ be the *two-element* distributive nearlattice with $0 < 1$. Then, it can be proved that $\mathbb{DN}$ is the variety generated by $\mathbf{2}$, i.e., $\mathbb{DN} = \mathbb{V}(\mathbf{2})$ (Chajda et al. 2007, Corollary 2.7.6).

Definition 2.14 Let $\langle A, m\rangle$ be a nearlattice. Let $I, F \subseteq A$ be non-empty.

(1) I is said to be an *ideal* of A if
 (i) $y \in I$ and $x \leq y$ implies $x \in I$,
 (ii) if $x, y \in I$, then $x \vee y \in I$.
(2) F is said to be a *filter* of A if
 (i) $x \in F$ and $x \leq y$ implies $y \in F$,
 (ii) if $x, y \in F$ and $x \wedge y$ exists in A, then $x \wedge y \in F$.

Let us denote by $\mathsf{Fi}(A)$ the collection of all filters of a nearlattice A. It is easy to check that for every nearlattice A the intersection of any collection of filters is either a filter or an empty set. So, for every non-empty $X \subseteq A$, there exists the least filter containing X; it is denoted by $\mathrm{Fi}_A(X)$. If $X = \{a_1, \ldots, a_n\}$, then we write $\mathrm{Fi}_A(a_1, \ldots, a_n)$ instead of $\mathrm{Fi}_A(\{a_1, \ldots, a_n\})$; and it easy to check that $\mathrm{Fi}_A(a) = [a)$.

A proper ideal I of A is called *prime* when for every $x, y \in A$, if $x \wedge y$ exists and $x \wedge y \in I$, then $x \in I$ or $y \in I$.

Proposition 2.15 *Let A be a nearlattice and $F \subseteq A$ be non-empty. Then, the following conditions are equivalent:*

(1) $F \in \mathsf{Fi}(A)$;
(2) *if $a, b \in F$, then $m(a, b, c) \in F$ for all $c \in A$.*

Proof (1) $\Rightarrow$ (2) Let $a, b \in F$ and $c \in A$. As F is an upset, $a \vee c, b \vee c \in F$, and since F is closed under existing meets, it follows that $m(a, b, c) = (a \vee c) \wedge (b \vee c) \in F$. (2) $\Rightarrow$ (1) Let $a \in F$ and $b \in A$ be such that $a \leq b$. So $b = a \vee b = m(a, a, b) \in F$. Thus, F is an upset. Let $a, b \in F$ be such that $a \wedge b$ exists. By condition (2), we obtain that $a \wedge b = m(a, b, a \wedge b) \in F$. Hence, $F \in \mathsf{Fi}(A)$. □

3 Distributive nearlattices

The aim of this short section is to present a certain system of elements on nearlattices; we state several properties and we prove a new characterization of the distributivity condition on nearlattices.

Definition 3.1 Let $\langle A, m\rangle$ be an algebra of type (3). For each natural number n, we define inductively, for all $a_1, \ldots, a_n, b \in A$, an element $m^{n-1}(a_1, \ldots, a_n, b)$ as follows:

- $m^0(a_1, b) := m(a_1, a_1, b)$ and
- for $n > 1$,

$$m^{n-1}(a_1, \ldots, a_n, b) := m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b).$$

In particular, for a nearlattice $\langle A, m\rangle$, we obtain that $m^0(a_1, b) = a_1 \vee b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$.

The following proposition follows directly by induction, and thus, we omit its proof.

Proposition 3.2 *Let $\langle A, m_A\rangle$ and $\langle B, m_B\rangle$ be algebras of type (3) and let $h \in \mathrm{Hom}(A, B)$. Then,*

$$h(m_A^{n-1}(a_1, \ldots, a_n, b)) = m_B^{n-1}(h(a_1), \ldots, h(a_n), h(b))$$

for all $a_1, \ldots, a_n, b \in A$.

Proposition 3.3 *Let $\langle A, m\rangle$ be a nearlattice, and let $a_1, \ldots, a_{n+1}, a, b \in A$. Then,*

(1) $m^{n-1}(a_1, \ldots, a_n, b) = (a_1 \vee b) \wedge_b \cdots \wedge_b (a_n \vee b)$;
(2) $m^{n-1}(a_1, \ldots, a_n, b) = m^{n-1}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}, b)$ *for every permutation σ of $\{1, \ldots, n\}$;*
(3) $b \leq m^{n-1}(a_1, \ldots, a_n, b)$;
(4) $a \leq m^{n-1}(a_1, \ldots, a_n, b)$ *whenever $a \leq a_i$ for all $i \in \{1, \ldots, n\}$;*
(5) $m^n(a_1, \ldots, a_{n+1}, b) \leq m^{n-1}(a_1, \ldots, a_n, b)$.

Proof (1) We proceed by induction on n. If $n = 1$, then $m^0(a_1, b) = a_1 \vee b$. We suppose that (1) is valid for n. Thus, applying the inductive hypothesis we have

$$\begin{aligned} m^n(a_1, \ldots, a_{n+1}, b) &= m(m^{n-1}(a_1, \ldots, a_n, b), a_{n+1}, b) \\ &= m([(a_1 \vee b) \wedge_b \cdots \wedge_b (a_n \vee b)], a_{n+1}, b) \\ &= ([(a_1 \vee b) \wedge_b \cdots \wedge_b (a_n \vee b)] \vee b) \wedge_b (a_{n+1} \vee b) \\ &= (a_1 \vee b) \wedge_b \cdots \wedge_b (a_n \vee b) \wedge_b (a_{n+1} \vee b). \end{aligned}$$

Properties (2)–(5) are consequences of (1). □

In view of property (2) of the previous proposition, if X is a non-empty finite subset of A and $b \in A$, we denote by $m(X, b)$ the element $m^{n-1}(a_1, \ldots, a_n, b)$ where $a_1, \ldots, a_n$ is an arbitrary enumeration of X.

In the following proposition, we establish a new characterization of the distributivity condition on a nearlattice.

Proposition 3.4 *Let A be a nearlattice. Then, the following conditions are equivalent:*

(1) *A is distributive;*
(2) *for every* $X \subseteq^*_\omega A$ *and* $b \in A$,

$$b \in \mathrm{Fi}_A(X) \quad \textit{implies} \quad b \in \mathrm{Fi}_A(m(X,b)).$$

Proof It should be noted, by property (3) of Proposition 3.3, that $b \in \mathrm{Fi}_A(m(X,b))$ if and only if $b = m(X,b)$. (1) $\Rightarrow$ (2) We assume that A is a distributive nearlattice. Since A is distributive, it follows that (see Calomino 2015, Lema 2.2.9)

$$\mathrm{Fi}_A(X) = \{a \in A : a = b_1 \wedge \cdots \wedge b_k \text{ for some } b_1, \ldots, b_k \in [X)\}.$$

If $b \in \mathrm{Fi}_A(X)$, then there are $b_1, \ldots, b_k \in [X)$ such that $b = b_1 \wedge \cdots \wedge b_k$. Suppose, toward a contradiction, that $m(X,b) \nleq b$. Then, there exists a prime ideal P such that $b \in P$ and $m(X,b) \notin P$ (see Chajda et al. 2007, Corollary 2.7.8). On the one hand, since $b = b_1 \wedge \cdots \wedge b_k \in P$ and P is a prime ideal, it follows that there is $i \in \{1, \ldots, k\}$ such that $b_i \in P$ and then $X \cap P \neq \emptyset$. On the other hand, given that $m(X,b) \notin P$ and $b \in P$, we have $X \cap P = \emptyset$. This is a contradiction. Then, $m(X,b) \leq b$. Hence, $b \in \mathrm{Fi}_A(m(X,b))$.

(2) $\Rightarrow$ (1) Let $a, b, c, d \in A$. Let us prove that $a \vee m(b,c,d) = m(a \vee b, a \vee c, d)$. For this, first we show that $a \vee m(b,c,d) \in \mathrm{Fi}_A(b,c)$. Given that $b \vee d, c \vee d \in \mathrm{Fi}_A(b,c)$ and moreover since $(b \vee d) \wedge (c \vee d)$ exists, it follows that $m(b,c,d) \in \mathrm{Fi}_A(b,c)$. Then, $a \vee m(b,c,d) \in \mathrm{Fi}_A(b,c)$. Thus, by (2), we obtain that

$$\begin{aligned} a \vee m(b,c,d) &= m(b,c,a \vee m(b,c,d)) \\ &= (b \vee a \vee m(b,c,d)) \wedge (c \vee a \vee m(b,c,d)). \end{aligned} \tag{2}$$

Now let us see that $b \vee m(b,c,d) = b \vee d$ and $c \vee m(b,c,d) = c \vee d$. We show only one of these identities, and the other one holds by an analogous argument. Since $m(b,c,d) \leq b \vee d$, it follows that $b \vee m(b,c,d) \leq b \vee d$. On the other hand, given that $b \leq b \vee m(b,c,d)$ and $d \leq m(b,c,d) \leq b \vee m(b,c,d)$, we have $b \vee d \leq b \vee m(b,c,d)$. We thus obtain that $b \vee m(b,c,d) = b \vee d$. Hence, from (2) we obtain

$$\begin{aligned} a \vee m(b,c,d) &= (a \vee b \vee d) \wedge (a \vee c \vee d) \\ &= m(a \vee b, a \vee c, d). \end{aligned}$$

Therefore, A is distributive. □

Remark 3.5 Let A be a distributive nearlattice and let $X \subseteq^*_\omega A$ and $a \in A$. Then, by Propositions 3.4 and 3.3, we have

$$a \in \mathrm{Fi}_A(X) \iff a = m(X,a).$$

4 A Gentzen system associated with $\mathbb{DN}$

The main aim of this section is to propose a Gentzen system $\mathcal{G}_{\mathbb{DN}}$, defined by a Gentzen calculus, on the algebraic language $\mathcal{L} = \{m\}$ such that $\mathrm{Alg}(\mathcal{G}_{\mathbb{DN}}) = \mathbb{DN}$, the sentential logic $\mathcal{S}_{\mathbb{DN}} := \mathcal{S}_{\mathcal{G}_{\mathbb{DN}}}$ be fully selfextensional and $\mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) = \mathbb{DN}$. To this end, it will be necessary to show that $\mathcal{G}_{\mathbb{DN}}$ is algebraizable (in the sense of Rebagliato and Verdú 1995, 1993) with equivalent algebraic semantics the variety of distributive nearlattices and moreover that the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ is fully adequate for $\mathcal{S}_{\mathbb{DN}}$.

Let $\mathcal{L} = \{m\}$ be an algebraic language of type (3). All the algebras and sentential logics consider in this section will be of the type $\mathcal{L}$; for every algebra $\langle A, m\rangle$ of type $\mathcal{L}$, we also consider the binary operation $\vee$ defined by $a \vee b := m(a,a,b)$. Moreover, for notational convenience, we consider the following notations: $\overline{a} := (a_1, \ldots, a_n)$ and $m(\overline{a}, b) := m^{n-1}(a_1, \ldots, a_n, b)$.

Let us first note that the variety $\mathbb{DN}$ cannot be the equivalent algebraic semantics of any algebraizable sentential logic, in the sense of Blok and Pigozzi (1989). This result is obtained by a very similar argument to Font and Verdú (1991), Proposition 2.1, and thus, we omit its proof.

Proposition 4.1 *Neither the variety of distributive nearlattices $\mathbb{DN}$ nor the variety of nearlattices $\mathbb{N}$ can be the equivalent algebraic semantics of any algebraizable sentential logic.*

Now we present the main definition of this section.

Definition 4.2 Let $\mathcal{G}_{\mathbb{DN}} = \langle Fm, \mid\!\sim_{\mathbb{DN}}\rangle$ be the Gentzen system of type ω^{o} defined by the following Gentzen-style rules: the structural rules (Axiom), (Weakening) and (Cut) and the following rules

$$(\vee \rhd)\ \frac{\varphi \rhd \chi \qquad \psi \rhd \chi}{\varphi \vee \psi \rhd \chi}$$

$$(\rhd \vee)\ \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \quad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \psi \vee \varphi}$$

$$(m \rhd)\ \frac{}{m(\varphi,\psi,\chi) \rhd \varphi \vee \chi} \quad \frac{}{m(\varphi,\psi,\chi) \rhd \psi \vee \chi}$$

$$(\rhd m)\ \frac{\Gamma \rhd \varphi \vee \chi \qquad \Gamma \rhd \psi \vee \chi}{\Gamma \rhd m(\varphi,\psi,\chi)}$$

$$(m^n \rhd)\ \frac{\varphi_1, \ldots, \varphi_n \rhd \varphi}{m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi) \rhd \varphi}$$

Let us denote by $\mathcal{S}_{\mathbb{DN}} = \langle Fm, \vdash_{\mathbb{DN}}\rangle$ the sentential logic defined by $\mathcal{G}_{\mathbb{DN}}$.

The following proposition establishes two properties for the sentential logic $\mathcal{S}_{\mathbb{DN}}$. These results will be useful at the end of this section.

Proposition 4.3 *The sentential logic $\mathcal{S}_{\mathbb{DN}}$ has the following properties:*

(1) *for every natural number n and $\varphi_1, \ldots, \varphi_n, \varphi \in Fm$, we have that $\varphi_1, \ldots, \varphi_n \vdash_{\mathbb{DN}} m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi)$;*
(2) *for every algebra A and $F \in \mathrm{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$, if $a_1, \ldots, a_n \in F$, then $m^{n-1}(a_1, \ldots, a_n, a) \in F$ for all $a \in A$.*

Proof (1) By definition, we have

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathbb{DN}} m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi) \iff \\ \iff \;\mid\!\sim_{\mathbb{DN}} \varphi_1, \ldots, \varphi_n \rhd m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi).$$

We proceed by induction on n. If $n = 1$, it is straightforward by (Axiom) and applying the rule $(\rhd \vee)$, because $m^0(\varphi_1, \varphi) = \varphi_1 \vee \varphi$. We suppose that the property is valid for n, that is, $\mid\!\sim_{\mathbb{DN}} \varphi_1, \ldots, \varphi_n \rhd m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi)$ and it is proved for $n+1$ by the following formal proof: let $\Gamma := \{\varphi_1, \ldots, \varphi_n\}$ and $\overline{\varphi} := (\varphi_1, \ldots, \varphi_n)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \rhd m(\overline{\varphi}, \varphi)}\,\text{(I.H.)}}{\Gamma, \varphi_{n+1} \rhd m(\overline{\varphi}, \varphi)}\,\text{(W)}}{\Gamma, \varphi_{n+1} \rhd m(\overline{\varphi}, \varphi) \vee \varphi}\,(\rhd \vee) \qquad \dfrac{\dfrac{\dfrac{}{\varphi_{n+1} \rhd \varphi_{n+1}}\,\text{(A)}}{\varphi_{n+1} \rhd \varphi_{n+1} \vee \varphi}\,(\rhd \vee)}{\Gamma, \varphi_{n+1} \rhd \varphi_{n+1} \vee \varphi}\,\text{(W)}}{\Gamma, \varphi_{n+1} \rhd m(m(\overline{\varphi}, \varphi), \varphi_{n+1}, \varphi)}\,(\rhd m)$$

Hence, property (1) holds for all natural numbers n. Property (2) is an immediate consequence of (1). □

Proposition 4.4 *Let $\langle A, \mathrm{C}\rangle$ be a g-matrix that is a model of the Gentzen system $\mathcal{G}_{\mathbb{DN}}$. Then, $\langle A, \mathrm{C}\rangle$ has the following properties: for all $a, b, c \in A$ and $a_1, \ldots, a_n \in A$,*

(WPD) *$\mathrm{C}(a \vee b) = \mathrm{C}(a) \cap \mathrm{C}(b)$;*

(PD1) *$\mathrm{C}(a \vee a) = \mathrm{C}(a)$;*
(PD2) *$\mathrm{C}(a \vee b) = \mathrm{C}(b \vee a)$;*
(PD3) *$\mathrm{C}(a \vee (b \vee c)) = \mathrm{C}((a \vee b) \vee c)$;*

(N1) *$a \vee c, b \vee c \in \mathrm{C}(m(a, b, c))$;*
(N2) *$a \vee c, b \vee c \in \mathrm{C}(X)$ implies $m(a, b, c) \in \mathrm{C}(X)$, for every non-empty finite $X \subseteq A$;*
(N3) *$\mathrm{C}(a \vee c, b \vee c) = \mathrm{C}(m(a, b, c))$;*
(N4) *$\mathrm{C}(m(a, b, c)) \subseteq \mathrm{C}(c)$;*
(N5) *$\mathrm{C}(m^{n-1}(a_1, \ldots, a_n, b)) \subseteq \mathrm{C}(b)$;*
(N6) *$\mathrm{C}(a_1 \vee b, \ldots, a_n \vee b) = \mathrm{C}(m^{n-1}(a_1, \ldots, a_n, b))$;*
(N7) *$b \in \mathrm{C}(a_1, \ldots, a_n) \iff b \in \mathrm{C}(m(\overline{a}, b))$.*

Proof Property (WPD) is a consequence of the Gentzen-style rules $(\vee \rhd)$ and $(\rhd \vee)$, and properties (PD1)–(PD3) are consequences of (WPD). We also have that properties (N1) and (N2) are immediate consequences of $(m \rhd)$ and $(\rhd m)$, respectively. Then, (N3) follows from (N1) and (N2).

By (N3), we have that $\mathrm{C}(m(a, b, c)) = \mathrm{C}(a \vee c, b \vee c) = \mathrm{C}(\mathrm{C}(a \vee c) \cup \mathrm{C}(b \vee c))$. Then, since $\mathrm{C}(a \vee c) \cup \mathrm{C}(b \vee c) \subseteq \mathrm{C}(c)$, it follows that $\mathrm{C}(m(a, b, c)) \subseteq \mathrm{C}(c)$, and hence, (N4) holds. Property (N5) is a consequence of (N4) and because $m^{n-1}(a_1, \ldots, a_n, b) = m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b)$.

We prove property (N6) by induction on n. If $n = 1$, then $\mathrm{C}(m^0(a_1, b)) = \mathrm{C}(a_1 \vee b)$. Assume that the property holds for n, that is, $\mathrm{C}(a_1 \vee b, \ldots, a_n \vee b) = \mathrm{C}(m^{n-1}(a_1, \ldots, a_n, b))$. Then, by inductive hypothesis and properties (WPD), (N5) and (N3), we have

$$\begin{aligned}
&\mathrm{C}(a_1 \vee b, \ldots, a_{n+1} \vee b) \\
&\quad = \mathrm{C}\left(\mathrm{C}(a_1 \vee b, \ldots, a_n \vee b) \cup \mathrm{C}(a_{n+1} \vee b)\right) \\
&\quad = \mathrm{C}\left(\mathrm{C}(m(\overline{a}, b)) \cup (\mathrm{C}(a_{n+1}) \cap \mathrm{C}(b))\right) \\
&\quad = \mathrm{C}\left([\mathrm{C}(m(\overline{a}, b)) \cup \mathrm{C}(a_{n+1})] \cap \mathrm{C}(m(\overline{a}, b)) \cup \mathrm{C}(b)]\right) \\
&\quad = \mathrm{C}\left([\mathrm{C}(m(\overline{a}, b)) \cup \mathrm{C}(a_{n+1})] \cap \mathrm{C}(b)\right) \\
&\quad = \mathrm{C}(\mathrm{C}(m(\overline{a}, b) \vee b) \cup \mathrm{C}(a_{n+1} \vee b)) \\
&\quad = \mathrm{C}(m(\overline{a}, b) \vee b, a_{n+1} \vee b) \\
&\quad = \mathrm{C}(m(m(\overline{a}, b), a_{n+1}, b)) \\
&\quad = \mathrm{C}(m^n(a_1, \ldots, a_n, a_{n+1}, b)).
\end{aligned}$$

Hence, property (N6) holds.

Notice that the implication from left to right in (N7) is a consequence of the Gentzen-style rule $(m^n \rhd)$. On the other hand, by (N6) and (WPD), we have

$$\begin{aligned}
\mathrm{C}(m(\overline{a}, b)) &= \mathrm{C}\left([\mathrm{C}(a_1) \cup \cdots \cup \mathrm{C}(a_n)] \cap \mathrm{C}(b)\right) \\
&\subseteq \mathrm{C}(a_1, \ldots, a_n).
\end{aligned}$$

Thus, if $b \in \mathrm{C}(m(\overline{a}, b))$, then we obtain $b \in \mathrm{C}(a_1, \ldots, a_n)$. Hence, property (N7) holds. □

The Gentzen-style rules $(\vee \rhd)$ and $(\rhd \vee)$ and their algebraic counterpart (WPD), as also the stronger and more general versions of these, were studied in many works related to several sentential logics, for instance for the $\{\wedge, \vee\}$-fragment of classical logic, the logic of lattices (Font and Jansana 2009; Font and Verdú 1991; Rebagliato and Verdú 1993) and for more general sentential logics (Czelakowski 2001).

Remark 4.5 The Gentzen-style rule $(m^n \rhd)$ cannot be deduced from the Gentzen-style rules $(\vee \rhd)$, $(\rhd \vee)$, $(\rhd m)$ and $(m \rhd)$. Indeed, let A be any non-distributive nearlattice and consider the g-matrix $\langle A, \mathrm{Fi}(A)\rangle$. Since the filters of A are upsets and closed under existing finite meets, and since moreover for all $a, b, c \in A$ we have $m^A(a, b, c) = (a \vee c) \wedge_c (b \vee c)$, it follows that $\langle A, \mathrm{Fi}(A)\rangle$ is a model for the Gentzen-style rules $(\rhd \vee)$, $(\rhd m)$ and $(m \rhd)$. It is straightforward to check that $\mathrm{Fi}_A(a \vee b) = \mathrm{Fi}_A(a) \cap \mathrm{Fi}_A(b)$ for all $a, b \in A$ and hence $\langle A, \mathsf{Fi}(A)\rangle$ is a model for the g-rule $(\vee \rhd)$. Since A is a non-distributive nearlattice, it follows by Proposition 3.4 that the condition $a \in \mathrm{Fi}_A(a_1, \ldots, a_n) \implies a \in \mathrm{Fi}_A(m^{n-1}(a_1, \ldots, a_n, a))$ is not true. Therefore, we have proved that $\langle A, \mathsf{Fi}(A)\rangle$ is a model of the Gentzen-style rules $(\vee \rhd)$, $(\rhd \vee)$, $(\rhd\, m)$ and $(m \rhd)$, but not of $(m^n \rhd)$.

Proposition 4.6 *Let* $\langle A, \mathrm{C}\rangle$ *be a g-matrix satisfying the properties* (WPD), (N1)*,* (N2) *and* (N7)*. Then, we have* $\Lambda_A\mathrm{C} \in \mathrm{Con}_{\mathbb{DN}}(A)$*. That is,* $A/\Lambda_A\mathrm{C} \in \mathbb{DN}$*.*

Proof Recall that $\Lambda_A\mathrm{C} = \{(a, b) \in A^2 : \mathrm{C}(a) = \mathrm{C}(b)\}$. By (WPD), it is easily checked that $\Lambda_A\mathrm{C}$ is a congruence with respect to $\vee$ (recall that the binary term $\vee$ is not a primitive connective), i.e., if $\mathrm{C}(a_1) = \mathrm{C}(a_2)$ and $\mathrm{C}(b_1) = \mathrm{C}(b_2)$, then $\mathrm{C}(a_1 \vee b_1) = \mathrm{C}(a_2 \vee b_2)$. Now we prove that $\Lambda_A\mathrm{C} \in \mathrm{Con}(A)$. Let $a_i, b_i, c_i \in A$ with $i = 1, 2$ be such that $\mathrm{C}(a_1) = \mathrm{C}(a_2)$, $\mathrm{C}(b_1) = \mathrm{C}(b_2)$ and $\mathrm{C}(c_1) = \mathrm{C}(c_2)$. Then, by property (N3) and since $\Lambda_A\mathrm{C}$ is a congruence with respect to $\vee$, it follows that

$$\begin{aligned}
\mathrm{C}(m(a_1, b_1, c_1)) &= \mathrm{C}(a_1 \vee c_1, b_1 \vee c_1)\\
&= \mathrm{C}(\mathrm{C}(a_1 \vee c_1) \cup \mathrm{C}(b_1 \vee c_1))\\
&= \mathrm{C}(\mathrm{C}(a_2 \vee c_2) \cup \mathrm{C}(b_2 \vee c_2))\\
&= \mathrm{C}(m(a_2, b_2, c_2)).
\end{aligned}$$

Now let us show that the algebra $\langle A/\Lambda_A\mathrm{C}, m^*\rangle$, with $m^*(\overline{a}, \overline{b}, \overline{c}) := \overline{m(a, b, c)}$, is a distributive nearlattice. By properties (PD1)–(PD3), we have $\langle A/\Lambda_A\mathrm{C}, \vee^*\rangle$ is a join-semilattice with the order induced by: $\overline{a} \leq \overline{b}$ if and only if $\overline{a} \vee^* \overline{b} = \overline{b}$ if and only if $\mathrm{C}(b) \subseteq \mathrm{C}(a)$. Now, by property (N3), it follows that for every $c \in A$ the upset $[\overline{c}) = \{\overline{a} \in A/\Lambda_A\mathrm{C} : \overline{c} \leq \overline{a}\}$ is a lattice with respect to the order induced by $\vee^*$ and where for all $\overline{a}, \overline{b} \in [\overline{c})$, we have $\overline{a} \wedge_{\overline{c}} \overline{b} = m^*(\overline{a}, \overline{b}, \overline{c})$. We thus obtain that $\langle A/\Lambda_A\mathrm{C}, m^*\rangle$ is a nearlattice. Lastly, we need to show that $A/\Lambda_A\mathrm{C}$ is distributive as nearlattice. Let $a, b, c, d \in A$. We prove that $\overline{a} \vee^* m^*(\overline{b}, \overline{c}, \overline{d}) = m^*(\overline{a} \vee^* \overline{b}, \overline{a} \vee^* \overline{c}, \overline{d})$. That is, let us to prove that $\mathrm{C}(a \vee m(b, c, d)) = \mathrm{C}(m(a \vee b, a \vee c, d))$. First, by (WPD) we have $b \vee d, c \vee d \in \mathrm{C}(b, c)$, and thus, by (N2), we obtain that $m(b, c, d) \in \mathrm{C}(b, c)$. So, by (WPD) again, $a \vee m(b, c, d) \in \mathrm{C}(b, c)$. From this and using property (N7), we have $a \vee m(b, c, d) \in \mathrm{C}(m(b, c, a \vee m(b, c, d)))$, and then, by (N4), we can derive $\mathrm{C}(a \vee m(b, c, d)) = \mathrm{C}(m(b, c, a \vee m(b, c, d)))$. Thus, by (N3) and (PD2)–(PD3), we obtain that

$$\mathrm{C}(a \vee m(b, c, d)) = \mathrm{C}(a \vee b \vee m(b, c, d), a \vee c \vee m(b, c, d)). \tag{3}$$

Let us see that $\mathrm{C}(b \vee m(b, c, d)) = \mathrm{C}(b \vee d)$ and $\mathrm{C}(c \vee m(b, c, d)) = \mathrm{C}(c \vee d)$. By (WPD) and (N3), we have $\mathrm{C}(b \vee m(b, c, d)) = \mathrm{C}(b) \cap \mathrm{C}(b \vee d, c \vee d)$, and since $b \vee d, c \vee d \in \mathrm{C}(d)$, it follows that $\mathrm{C}(b \vee d, c \vee d) \subseteq \mathrm{C}(d)$. Then, $\mathrm{C}(b \vee m(b, c, d)) \subseteq \mathrm{C}(b) \cap \mathrm{C}(d) = \mathrm{C}(b \vee d)$. On the other hand, since $\mathrm{C}(b \vee d) \subseteq \mathrm{C}(b \vee d, c \vee d) = \mathrm{C}(m(b, c, d))$, it follows that $\mathrm{C}(b) \cap \mathrm{C}(b \vee d) \subseteq \mathrm{C}(b) \cap \mathrm{C}(m(b, c, d))$ and then $\mathrm{C}(b \vee d) \subseteq \mathrm{C}(b \vee m(b, c, d))$. Similarly we can show that $\mathrm{C}(c \vee m(b, c, d)) = \mathrm{C}(c \vee d)$. Now, from (3) we have that

$$\begin{aligned}
&\mathrm{C}(a \vee m(b, c, d))\\
&\quad = \mathrm{C}(a \vee b \vee m(b, c, d), a \vee c \vee m(b, c, d))\\
&\quad = \mathrm{C}(\mathrm{C}(a \vee b \vee m(b, c, d)) \cup \mathrm{C}(a \vee c \vee m(b, c, d)))\\
&\quad = \mathrm{C}\left([\mathrm{C}(a) \cap \mathrm{C}(b \vee m(b, c, d))] \cup\right.\\
&\qquad\qquad \left.\cup\ [\mathrm{C}(a) \cap \mathrm{C}(c \vee m(b, c, d))]\right)\\
&\quad = \mathrm{C}\left([\mathrm{C}(a) \cap \mathrm{C}(b \vee d)] \cup [\mathrm{C}(a) \cap \mathrm{C}(c \vee d)]\right)\\
&\quad = \mathrm{C}(a \vee b \vee d, a \vee c \vee d) = \mathrm{C}(m(a \vee b, a \vee c, d)).
\end{aligned}$$

Hence, we have proved that $\langle A/\Lambda_A\mathrm{C}, m^*\rangle$ is a distributive nearlattice, and therefore, $\Lambda_A\mathrm{C} \in \mathrm{Con}_{\mathbb{DN}}(A)$. □

Theorem 4.7 *Let A be an algebra of type* $\mathcal{L} = \{m\}$*. Then, A is the algebraic reduct of a reduced model of* $\mathcal{G}_{\mathbb{DN}}$ *if and only if A is a distributive nearlattice. Therefore,* $\mathrm{Alg}(\mathcal{G}_{\mathbb{DN}}) = \mathbb{DN}$*.*

Proof First, assume that A is the algebraic reduct of a reduced model of $\mathcal{G}_{\mathbb{DN}}$. So, there is a finitary closure operator C on A such that $\langle A, \mathrm{C}\rangle \in \mathbf{Mod}^*(\mathcal{G}_{\mathbb{DN}})$. Then, by Propositions 4.4 and 4.6, we have $\Lambda_A\mathrm{C} \in \mathrm{Con}_{\mathbb{DN}}(A)$. Moreover, since $\langle A, \mathrm{C}\rangle$ is reduced, it follows that $\Lambda_A\mathrm{C} = \widetilde{\Omega}_A\mathrm{C} = \mathrm{Id}_A$, and hence, $A \cong A/\Lambda_A\mathrm{C} \in \mathbb{DN}$.

Now we suppose that A is a distributive nearlattice and we consider the g-matrix $\langle A, \mathsf{Fi}(A)\rangle$. Let us see that $\langle A, \mathsf{Fi}(A)\rangle \in \mathbf{Mod}^*(\mathcal{G}_{\mathbb{DN}})$. Notice that is straightforward to check directly that the g-matrix $\langle A, \mathsf{Fi}(A)\rangle$ is reduced. By Remark 4.5, we know that $\langle A, \mathsf{Fi}(A)\rangle$ is a model of $(\vee \rhd)$, $(\rhd \vee)$, $(\rhd\, m)$ and $(m \rhd)$ and by Proposition 3.4, it is model of $(m^n \rhd)$. Therefore, $\langle A, \mathsf{Fi}(A)\rangle \in \mathbf{Mod}^*(\mathcal{G}_{\mathbb{DN}})$ and thus $A \in \mathrm{Alg}(\mathcal{G}_{\mathbb{DN}})$. □

Now we are going to show that the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ is *algebraizable* with *equivalent algebraic semantics* the variety $\mathbb{DN}$ (see Theorem 4.10). The concept of *algebraizability* is original on the framework of sentential logic and due to Blok and Pigozzi (1989). The development of a theory of the algebraization of Gentzen systems is parallel to that of the algebraization of sentential logics and it was developed in Rebagliato and Verdú (1995) (see also Font and Jansana 2009; Rebagliato and Verdú 1993).

For this purpose, we need the following lemma which is similar to Proposition 3.8 in Rebagliato and Verdú (1993). Let us denote by $\mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$ the algebraic closure system associated with the closure operator $\vdash_{\mathbb{DN}}$. That is, $T \in \mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$ if and only if $T \vdash_{\mathbb{DN}} \Gamma \rhd \varphi$ implies $\Gamma \rhd \varphi \in T$. We also define $\mathsf{Mod}_{\mathbb{DN}}(Fm) :=$

$\{\mathrm{C} : \langle Fm, \mathrm{C}\rangle$ is a model of the Gentzen system $\mathcal{G}_{\mathbb{DN}}\}$. The set $\mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$ is ordered with the inclusion order and we consider on $\mathsf{Mod}_{\mathbb{DN}}(Fm)$ the usual order: $\mathrm{C}_1 \leq \mathrm{C}_2 \iff \mathrm{C}_1(\Gamma) \subseteq \mathrm{C}_2(\Gamma)$ for all $\Gamma \subseteq Fm$.

Lemma 4.8 *The sets $\mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$ and $\mathsf{Mod}_{\mathbb{DN}}(Fm)$, with the corresponding orders, are order isomorphic via the following maps:*

- $f\colon \mathsf{Th}(\mathcal{G}_{\mathbb{DN}}) \to \mathsf{Mod}_{\mathbb{DN}}(Fm)$, *where for every* $T \in \mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$, $f(T) = \mathrm{C}_T$ *is defined by:*

$$\mathrm{C}_T(\Gamma) = \{\varphi \in Fm : \Gamma_0 \rhd \varphi \in T \text{ for some } \Gamma_0 \subseteq^*_\omega \Gamma\}$$

$$\mathrm{C}_T(\emptyset) = \bigcap_{\varphi \in Fm} \mathrm{C}_T(\varphi)$$

- $g : \mathsf{Mod}_{\mathbb{DN}}(Fm) \to \mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$, *for every element* $\mathrm{C} \in \mathsf{Mod}_{\mathbb{DN}}(Fm)$,

$$g(\mathrm{C}) = T_{\mathrm{C}} = \{\Gamma \rhd \varphi \in \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}}) : \varphi \in \mathrm{C}(\Gamma)\}.$$

Corollary 4.9 *For every* $T \in \mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$, *the relation* $\theta_T := \{(\varphi, \psi) \in Fm^2 : \varphi \rhd \psi, \psi \rhd \varphi \in T\}$ *is such that* $\theta_T \in \mathrm{Con}_{\mathbb{DN}}(Fm)$.

Proof By the previous lemma, we have $\theta_T = \{(\varphi, \psi) \in Fm^2 : \mathrm{C}_T(\varphi) = \mathrm{C}_T(\psi)\} = \Lambda(\mathrm{C}_T)$, that is, θ_T is the Frege relation of the g-matrix $\langle Fm, \mathrm{C}_T\rangle$. As we know, by the previous lemma, $\langle Fm, \mathrm{C}_T\rangle$ is a model of $\mathcal{G}_{\mathbb{DN}}$. Then, from Propositions 4.4 and 4.6, we have that $\theta_T \in \mathrm{Con}_{\mathbb{DN}}(Fm)$. □

Now we consider the following translations which allow us to prove that the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{DN}$. Let $\mathsf{sq}\colon \mathrm{Eq}(Fm) \to \mathcal{P}(\mathrm{Seq}(\mathcal{G}_{\mathbb{DN}}))$ be defined by $\mathsf{sq}(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}$ and let $\mathsf{t_m}\colon \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}}) \to \mathcal{P}(\mathrm{Eq}(Fm))$ be defined by $\mathsf{t_m}(\Gamma \rhd \psi) = \{m(\overline{\varphi}, \psi) \approx \psi\}$, where $\overline{\varphi} = (\varphi_1, \dots, \varphi_n)$ if $\Gamma = \{\varphi_1, \dots, \varphi_n\}$. Moreover, sq and $\mathsf{t_m}$ are extended to subsets as usual: $\mathsf{sq}(E) = \bigcup\{\mathsf{sq}(\varphi \approx \psi) : \varphi \approx \psi \in E\}$ and $\mathsf{t_m}(S) = \bigcup\{\mathsf{t_m}(\Gamma \rhd \psi) : \Gamma \rhd \psi \in S\}$ for all $E \subseteq \mathrm{Eq}(Fm)$ and $S \subseteq \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}})$.

Theorem 4.10 *Let* $\{\Gamma_i \rhd \psi_i : i \in I\} \cup \{\Gamma \rhd \psi\} \subseteq \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}})$ *and* $\varphi \approx \psi$ *be an equation. Then,*

(1) $\{\Gamma_i \rhd \psi_i : i \in I\} \vdash_{\mathbb{DN}} \Gamma \rhd \psi \iff$
$\iff \mathsf{t_m}(\{\Gamma_i \rhd \psi_i : i \in I\}) \models_{\mathbb{DN}} \mathsf{t_m}(\Gamma \rhd \psi)$;
(2) $\varphi \approx \psi \models_{\mathbb{DN}} \mathsf{t_m}(\mathsf{sq}(\varphi \approx \psi))$ *and* $\mathsf{t_m}(\mathsf{sq}(\varphi \approx \psi)) \models_{\mathbb{DN}} \varphi \approx \psi$.

Proof First, we denote by $\overline{\varphi}_i$ the sequence $(\varphi_{i1}, \dots, \varphi_{ik_i})$ if $\Gamma_i = \{\varphi_{i1}, \dots, \varphi_{ik_i}\}$ for every $i \in I$ and by $\overline{\varphi}$ the sequence $(\varphi_1, \dots, \varphi_n)$ if $\Gamma = \{\varphi_1, \dots, \varphi_n\}$.

(1) We assume that $\{\Gamma_i \rhd \psi_i : i \in I\} \vdash_{\mathbb{DN}} \Gamma \rhd \psi$ and we prove that $\{m(\overline{\varphi}_i, \psi_i) \approx \psi_i : i \in I\} \models_{\mathbb{DN}} m(\overline{\varphi}, \psi) \approx \psi$. Let $A \in \mathbb{DN}$ and $h \in \mathrm{Hom}(Fm, A)$ be such that $h(m(\overline{\varphi}_i, \psi_i)) = h(\psi_i)$ for all $i \in I$. Since h is a homomorphism, it follows by Proposition 3.2 that $m_A(h[\Gamma_i], h(\psi_i)) = h(\psi_i)$ for all $i \in I$. Then, by Remark 3.5, we have $h(\psi_i) \in \mathrm{Fi}_A(h[\Gamma_i])$ for every $i \in I$. Thus, since $\langle A, \mathsf{Fi}(A)\rangle$ is a reduced model of $\mathcal{G}_{\mathbb{DN}}$, it follows that $h(\psi) \in \mathrm{Fi}_A(h[\Gamma])$. Hence, by Remark 3.5 again and the fact that h is a homomorphism, we obtain $h(m(\overline{\varphi}, \psi)) = h(\psi)$. Therefore, $\{m(\overline{\varphi}_i, \psi_i) \approx \psi_i : i \in I\} \models_{\mathbb{DN}} m(\overline{\varphi}, \psi) \approx \psi$.

Now we assume that $\{m(\overline{\varphi}_i, \psi_i) \approx \psi_i : i \in I\} \models_{\mathbb{DN}} m(\overline{\varphi}, \psi) \approx \psi$. We consider the following element of $\mathsf{Th}(\mathcal{G}_{\mathbb{DN}})$:

$$T := \{\Delta \rhd \chi \in \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}}) : \{\Gamma_i \rhd \psi_i : i \in I\} \vdash_{\mathbb{DN}} \Delta \rhd \chi\}.$$

By Corollary 4.9, we have $Fm/\theta_T \in \mathbb{DN}$. Let $h\colon Fm \to Fm/\theta_T$ be the canonical morphism, i.e., $h(\psi) = \psi/\theta_T$, for every $\psi \in Fm$. Since $\Gamma_i \rhd \psi_i \in T$, it follows by Lemma 4.8 that $\psi_i \in \mathrm{C}_T(\Gamma_i)$ for every $i \in I$, and because $\langle Fm, \mathrm{C}_T\rangle$ is a model of $\mathcal{G}_{\mathbb{DN}}$, we obtain by (N7) of Proposition 4.4 that $\psi_i \in \mathrm{C}_T(m(\overline{\varphi}_i, \psi_i))$ for every $i \in I$. Thus $m(\overline{\varphi}_i, \psi_i) \rhd \psi_i \in T$ for all $i \in I$. By (N5), we also have that $m(\overline{\varphi}_i, \psi_i) \in \mathrm{C}_T(\psi_i)$ and so $\psi_i \rhd m(\overline{\varphi}_i, \psi_i) \in T$ for all $i \in I$. Thus, we have found that for all $i \in I$, $m(\overline{\varphi}_i, \psi_i) \rhd \psi_i, \psi_i \rhd m(\overline{\varphi}_i, \psi_i) \in T$. Hence, $m(\overline{\varphi}_i, \psi_i)/\theta_T = \psi_i/\theta_T$ for all $i \in I$, that is, $h(m(\overline{\varphi}_i, \psi_i)) = h(\psi_i)$ for all $i \in I$. Then, by hypothesis, we have $h(m(\overline{\varphi}, \psi)) = h(\psi)$ and so $m(\overline{\varphi}, \psi)/\theta_T = \psi/\theta_T$. This implies that $m(\overline{\varphi}, \psi) \rhd \psi, \psi \rhd m(\overline{\varphi}, \psi) \in T$ and thus $\psi \in \mathrm{C}_T(m(\overline{\varphi}, \psi))$. By (N7), we obtain $\psi \in \mathrm{C}_T(\Gamma)$ and hence $\Gamma \rhd \psi \in T$. Then, by definition of T, $\{\Gamma_i \rhd \psi_i : i \in I\} \vdash_{\mathbb{DN}} \Gamma \rhd \psi$. Therefore, (1) holds.

(2) Notice that

$$\begin{aligned}\mathsf{t_m}(\mathsf{sq}(\varphi \approx \psi)) &= \mathsf{t_m}(\{\varphi \rhd \psi, \psi \rhd \varphi\})\\ &= \{m^0(\varphi, \psi) \approx \psi, m^0(\psi, \varphi) \approx \varphi\}\\ &= \{\varphi \vee \psi \approx \psi, \psi \vee \varphi \approx \varphi\}.\end{aligned}$$

Then, it is straightforward to check directly that $\varphi \approx \psi \models_{\mathbb{DN}} \mathsf{t_m}(\mathsf{sq}(\varphi \approx \psi))$ and $\mathsf{t_m}(\mathsf{sq}(\varphi \approx \psi)) \models_{\mathbb{DN}} \varphi \approx \psi$. This completes the proof. □

Corollary 4.11 (Font and Jansana 2009, Proposition 4.15) *Let* $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}(Fm)$ *and* $\Gamma \rhd \varphi \in \mathrm{Seq}(\mathcal{G}_{\mathbb{DN}})$. *Then,*

(3) $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathbb{DN}} \varphi \approx \psi \iff$
$\iff \mathsf{sq}(\{\varphi_i \approx \psi_i : i \in I\}) \vdash_{\mathbb{DN}} \mathsf{sq}(\varphi \approx \psi)$;
(4) $\Gamma \rhd \varphi \vdash_{\mathbb{DN}} \mathsf{sq}(\mathsf{t_m}(\Gamma \rhd \varphi))$ *and* $\mathsf{sq}(\mathsf{t_m}(\Gamma \rhd \varphi)) \vdash_{\mathbb{DN}} \Gamma \rhd \varphi$.

The following result is a consequence of the previous corollary and Theorem 4.10. Moreover, its proof is similar to the proof of Proposition 4.18 in Font and Jansana (2009), and thus, we leave the details to the reader.

Corollary 4.12 *The sentential logic $\mathcal{S}_{\mathbb{DN}}$ is selfextensional and the intrinsic variety of $\mathcal{S}_{\mathbb{DN}}$ is $\mathbb{DN}$, i.e., $\mathrm{K}_{\mathcal{S}_{\mathbb{DN}}} = \mathbb{DN}$.*

Now we show that the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ is fully adequate for the sentential logic $\mathcal{S}_{\mathbb{DN}}$. To this end, we use the following useful characterization of the notion of full adequacy.

Proposition 4.13 (Font and Jansana 2009, Proposition 4.12) *Let $\mathcal{G}$ be a Gentzen system and $\mathcal{S}$ be a sentential logic. Then, $\mathcal{G}$ is fully adequate for $\mathcal{S}$ if and only if the following conditions hold:*

(1) $\mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}(\mathcal{G})$*;*
(2) *for every $A \in \mathrm{Alg}(\mathcal{S})$, the g-matrix $\langle A, \mathsf{Fi}_{\mathcal{S}}(A)\rangle$ is the only reduced model of $\mathcal{G}$ (having no theorems, if $\mathcal{S}$ has not) on A; and*
(3) *either $\mathcal{S}$ has theorems and $\mathcal{G}$ is of type ω, or $\mathcal{S}$ has no theorem and $\mathcal{G}$ is of type ω^o.*

Theorem 4.14 *The Gentzen system $\mathcal{G}_{\mathbb{DN}}$ is fully adequate for $\mathcal{S}_{\mathbb{DN}}$.*

Proof Let us see that $\mathcal{G}_{\mathbb{DN}}$ and $\mathcal{S}_{\mathbb{DN}}$ satisfy the conditions (1)–(3) of the previous proposition. Condition (3) is trivial. By Lemma 2.6, we have $\mathrm{Alg}(\mathcal{G}_{\mathbb{DN}}) \subseteq \mathrm{Alg}(\mathcal{S}_{\mathbb{DN}})$ and from Lemma 2.5 and Corollary 4.12, we obtain that $\mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) \subseteq \mathrm{K}_{\mathcal{S}_{\mathbb{DN}}} = \mathbb{DN} = \mathrm{Alg}(\mathcal{G}_{\mathbb{DN}})$. Hence, $\mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) = \mathrm{Alg}(\mathcal{G}_{\mathbb{DN}})$, and thus, condition (1) holds. In order to show (2), let $A \in \mathrm{Alg}(\mathcal{S}_{\mathbb{DN}})$. So $A \in \mathrm{Alg}(\mathcal{G}_{\mathbb{DN}})$. Let $\langle A, \mathcal{C}\rangle$ be any reduced model (having no theorems) of $\mathcal{G}_{\mathbb{DN}}$ and where C denotes the closure operator associated with $\mathcal{C}$. By Propositions 4.4 and 4.6, we have $\Lambda_A\mathcal{C} \in \mathrm{Con}_{\mathbb{DN}}(A)$ and since $\langle A, \mathcal{C}\rangle$ is reduced, it follows that $\Lambda_A\mathcal{C} = \widetilde{\Omega}_A\mathcal{C} = \mathrm{Id}_A$. Next, we prove that $\mathcal{C} = \mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$. By Lemma 2.6, $\langle A, \mathcal{C}\rangle$ is a g-model of $\mathcal{S}_{\mathcal{G}_{\mathbb{DN}}}$ and thus $\mathcal{C} \subseteq \mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$. Let $F \in \mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$. We only need to prove that $\mathrm{C}(F) \subseteq F$. If $F = \emptyset$, then $\mathrm{C}(F) = \mathrm{C}(\emptyset) = \emptyset$, because $\langle A, \mathcal{C}\rangle$ has not theorems. Suppose that $F \neq \emptyset$ and let $a \in \mathrm{C}(F)$. Since C is finitary, there are $a_1, \ldots, a_n \in F$ such that $a \in \mathrm{C}(a_1, \ldots, a_n)$. From properties (N5) and (N7), we obtain that $\mathrm{C}(a) = \mathrm{C}(m^{n-1}(a_1, \ldots, a_n, a))$, and from the fact that $\Lambda_A\mathcal{C} = \widetilde{\Omega}_A\mathcal{C} = \mathrm{Id}_A$, we have $a = m^{n-1}(a_1, \ldots, a_n, a)$. Now since F is an $\mathcal{S}_{\mathbb{DN}}$-filter of A, it follows by Proposition 4.3 that $a = m^{n-1}(a_1, \ldots, a_n, a) \in F$. Then, $F \in \mathcal{C}$, and hence, $\mathcal{C} = \mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$. Therefore, since the three conditions of Proposition 4.13 hold, we obtain that $\mathcal{G}_{\mathbb{DN}}$ is fully adequate for $\mathcal{S}_{\mathbb{DN}}$. □

In the following theorem, we establish some of the main results of this paper.

Theorem 4.15 *The sentential logic $\mathcal{S}_{\mathbb{DN}}$ has the following properties:*

(1) $\mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) = \mathbb{DN}$*;*
(2) $\langle A, \mathrm{C}\rangle \in \mathbf{FGMod}(\mathcal{S}_{\mathbb{DN}})$ *if and only if* $\langle A, \mathrm{C}\rangle$ *does not have theorems and satisfies* (WPD), (N1), (N2) *and* (N7)*;*
(3) *for every $\Gamma = \{\varphi_1, \ldots, \varphi_n\} \subseteq^*_\omega Fm$ and $\psi \in Fm$,*

$$\Gamma \vdash_{\mathbb{DN}} \psi \iff \models_{\mathbb{DN}} m(\overline{\varphi}, \psi) \approx \varphi \iff \models_{\mathbf{2}} m(\overline{\varphi}, \psi) \approx \varphi$$

$\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)$*;*
(4) *if $A \in \mathrm{Alg}(\mathcal{S}_{\mathbb{DN}})$, then $\mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A) = \mathsf{Fi}(A) \cup \{\emptyset\}$;*
(5) *$\mathcal{S}_{\mathbb{DN}}$ is fully selfextensional.*

Proof (1) It is an immediate consequence from the previous theorem and Theorem 4.7.

(2) It follows from Theorem 4.14, Definition 2.9 and Proposition 4.4.

(3) By definition of the logic $\mathcal{S}_{\mathbb{DN}}$ and from Theorem 4.10, we have

$$\Gamma \vdash_{\mathbb{DN}} \psi \iff \;\mid\!\sim_{\mathbb{DN}} \Gamma \rhd \psi \iff \models_{\mathbb{DN}} \mathsf{t_m}(\Gamma \rhd \psi)$$
$$\iff \models_{\mathbb{DN}} m(\overline{\varphi}, \psi) \approx \psi \iff \models_{\mathbf{2}} m(\overline{\varphi}, \psi) \approx \psi.$$

(4) Let $A \in \mathrm{Alg}(\mathcal{S}_{\mathbb{DN}}) = \mathbb{DN}$. We know (see Theorem 4.7) that $\langle A, \mathsf{Fi}(A) \cup \{\emptyset\}\rangle$ is a reduced model of $\mathcal{G}_{\mathbb{DN}}$. Then, since $\mathcal{G}_{\mathbb{DN}}$ is fully adequate for $\mathcal{S}_{\mathbb{DN}}$, it follows by Proposition 4.13 that $\mathsf{Fi}(A) \cup \{\emptyset\} = \mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(A)$.

(5) It follows from Definition 2.8 (1), condition (2) of this theorem and by Proposition 4.6. □

In view of the previous results, we can see that the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ and its sentential logic $\mathcal{S}_{\mathbb{DN}}$ are naturally associated with the variety of distributive nearlattices $\mathbb{DN}$. In fact, we have that the variety $\mathbb{DN}$ is the algebraic counterpart of both the sentential logic $\mathcal{S}_{\mathbb{DN}}$ and the Gentzen system $\mathcal{G}_{\mathbb{DN}}$. Thus, we think that the sentential logic $\mathcal{S}_{\mathbb{DN}}$ deserves to be called *the logic of distributive nearlattices*.

Definition 4.16 A sentential logic $\mathcal{S}$ is called *Fregean* if for any $\Gamma \in \mathsf{Th}(\mathcal{S})$, the relation $\Lambda_{\mathcal{S}}(\Gamma) = \{(\varphi, \psi) \in Fm^2 : \Gamma, \varphi \vdash_{\mathcal{S}} \psi$ and $\Gamma, \psi \vdash_{\mathcal{S}} \varphi\}$ is a congruence on Fm.

Next we prove that the logic $\mathcal{S}_{\mathbb{DN}}$ is Fregean. To this end, we show first that $\mathcal{S}_{\mathbb{DN}}$ is complete with respect to the two-element distributive nearlattice. For the two-element distributive nearlattice $\mathbf{2} = \{0, 1\}$, we can define the sentential logic $\langle Fm, \models_{\mathbf{2}}\rangle$ as usual: for every finite $\Gamma \subseteq Fm$ and $\varphi \in Fm$,

$$\Gamma \models_{\mathbf{2}} \varphi \iff (\forall h \in \mathrm{Hom}(Fm, \mathbf{2}))$$
$$(h[\Gamma] \subseteq \{1\} \implies h(\varphi) = 1)$$

and for all $\Gamma \subseteq Fm$, $\Gamma \vDash_{\mathbf{2}} \varphi$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash_{\mathbf{2}} \varphi$.

Proposition 4.17 *The logic $\mathcal{S}_{\mathbb{DN}}$ is complete with respect to $\langle Fm, \vDash_{\mathbf{2}} \rangle$. That is, for every $\Gamma \cup \{\psi\} \subseteq Fm$, $\Gamma \vdash_{\mathbb{DN}} \psi \iff \Gamma \vDash_{\mathbf{2}} \psi$.*

Proof As the consequence relations $\vdash_{\mathbb{DN}}$ and $\vDash_{\mathbf{2}}$ are finitary, it is enough to show that $\Gamma \vdash_{\mathbb{DN}} \psi \iff \Gamma \vDash_{\mathbf{2}} \psi$ when Γ is finite and non-empty. Suppose first that $\Gamma \vdash_{\mathbb{DN}} \psi$. So, by (3) of Theorem 4.15, we have $\vDash_{\mathbf{2}} m(\overline{\varphi}, \psi) \approx \psi$. Let $h \in \mathrm{Hom}(Fm, \mathbf{2})$ be such that $h[\Gamma] \subseteq \{1\}$. Then, we obtain by Propositions 3.2 and 3.3 that $h(\psi) = h(m(\overline{\varphi}, \psi)) = m(h[\Gamma], h(\psi)) = 1$. Hence, $\Gamma \vDash_{\mathbf{2}} \psi$

Now assume that $\Gamma \vDash_{\mathbf{2}} \psi$ and let $h \in \mathrm{Hom}(Fm, \mathbf{2})$. By Proposition 3.3, we have $h(\psi) \leq m(h[\Gamma], h(\psi))$. Suppose by contradiction that $h(\psi) \neq m(h[\Gamma], h(\psi))$. Thus, $h(\psi) = 0$ and $m(h[\Gamma], h(\psi)) = 1$. Since

$$m(h[\Gamma], h(\psi)) = \bigwedge_{\gamma \in \Gamma} (h(\gamma) \vee h(\psi)),$$

it follows that $h(\gamma) \vee h(\psi) = 1$ for all $\gamma \in \Gamma$. Then, $h(\gamma) = 1$ for all $\gamma \in \Gamma$. Because $\Gamma \vDash_{\mathbf{2}} \varphi$, we obtain that $h(\psi) = 1$, which is a contradiction. Hence, $h(\psi) = m(h[\Gamma], h(\psi))$, and thus, $\vDash_{\mathbf{2}} m(\overline{\varphi}, \psi) \approx \psi$. Therefore, $\Gamma \vdash_{\mathbb{DN}} \psi$. $\square$

It is well known that all two-valued sentential logics are Fregean (Font and Jansana 2009, p. 68). Hence, we have the desired result:

Corollary 4.18 *The sentential logic $\mathcal{S}_{\mathbb{DN}}$ is Fregean.*

We end this article by proving that the logic $\mathcal{S}_{\mathbb{DN}}$ is not protoalgebraic and hence neither algebraizable. The proof follows the idea of the proof of Proposition 2.8 in Font and Verdú (1991), where the authors show that the $\{\wedge, \vee\}$-fragment of the classical propositional logic is not protoalgebraic. The notion of protoalgebraicity can be presented in several equivalent forms, see, for instance, (Czelakowski 2001). We need the following important tool in the theory of sentential logics. For every algebra A, the *Leibniz operator* (Blok and Pigozzi 1989) is the map $\Omega_A \colon \mathcal{P}(A) \to \mathcal{P}(A)$, which associates with any $F \subseteq A$, the greatest congruence relation $\Omega_A F$ of A which is compatible with F.

Proposition 4.19 *The logic $\mathcal{S}_{\mathbb{DN}}$ is not protoalgebraic.*

Proof In order to prove this, we use the following characterization of protoalgebraility: a sentential logic $\mathcal{S}$ is protoalgebraic if and only if for every algebra A, the Leibniz operator Ω_A is monotonic over the set $\mathsf{Fi}_{\mathcal{S}}(A)$. Now we consider the two-element distributive nearlattice $\mathbf{2}$. So $\mathbf{2} \in \mathrm{Alg}(\mathcal{S}_{\mathbb{DN}})$ and then $\mathsf{Fi}_{\mathcal{S}_{\mathbb{DN}}}(\mathbf{2}) = \mathsf{Fi}(\mathbf{2}) \cup \{\emptyset\} = \{\emptyset, \{1\}, \mathbf{2}\}$. Since $\mathrm{Con}(\mathbf{2}) = \{\Delta_{\mathbf{2}}, \nabla_{\mathbf{2}}\}$, it easily follows by the definition of the Leibniz operator Ω_A that $\Omega_A \emptyset = \nabla_{\mathbf{2}}$, $\Omega_A\{1\} = \Delta_{\mathbf{2}}$ and $\Omega_A \mathbf{2} = \nabla_{\mathbf{2}}$. Thus, we have that $\emptyset \subseteq \{1\}$ but $\Omega_A \emptyset \nsubseteq \Omega_A\{1\}$. Hence, $\mathcal{S}_{\mathbb{DN}}$ is not protoalgebraic. $\square$

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