A CATEGORICAL DUALITY FOR SEMILATTICES AND LATTICES

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Abstract. The main aim of this article is to develop a categorical duality between the category of semilattices with homomorphisms and a category of certain topological spaces with certain morphisms. The principal tool to achieve this goal is the notion of irreducible filter. Then, we apply this dual equivalence to obtain a topological duality for the category of bounded lattices and lattice homomorphism. We show that our topological dualities for semilattices and lattices are natural generalizations of the duality developed by Stone for distributive lattices through spectral spaces. Finally, we obtain directly the categorical equivalence between our topological spaces and those presented for Moshier and Jipsen (*Topological duality and lattice expansions, I: A topological construction of canonical extensions.* Algebra Univers. 71 (2014), 109–126.).

1. Introduction

The categorical dualities for ordered algebraic structures through topological spaces arose with the famous work of M. H. Stone [22] developing a categorical duality between the category of Boolean algebras with homomorphisms and the category of compact Hausdorff zero-dimensional spaces (called *Boolean spaces* or *Stone spaces*) with continuous functions. Then, Stone himself extended this duality from Boolean algebras to bounded distributive lattices through spectral spaces and spectral functions ([13]). Some time later, Priestley developed another topological duality from a different approach for the category of distributive lattices employing compact totally

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order-disconnected spaces (called *Priestley spaces*) with continuous monotone functions.

Both topological dualities for distributive lattices are very useful for the study of distributive lattices and they are also a powerful mathematical tool in the study of many non-classical logics having an algebraic semantics based on distributive lattices (see for instance [19, 11]). From there, many generalizations of the topological dualities for distributive lattices following the Stone's approach or the Priestley's approach were obtained for several ordered algebraic structures having an adequate distributivity condition [11, 9, 2, 3, 12, 1, 5, 7, 14, 6].

There are in the literature several categorical dualities for the category of arbitrary bounded lattices following the ideas of Stone or Priestley [23, 17, 16]. It is fair to say that these topological dualities are closer to the Priestley's duality than the Stone one because the topological spaces obtained in [23, 17, 16] are equipped with some additional structure.

Recently, in [20] Moshier and Jipsen developed a topological duality for the category of bounded lattices following an approach different from Stone and Priestley. Their duality is closely related to the concept of canonical extension. But, it has the disadvantage that the dual object of a distributive lattice is neither the dual Stone space nor the dual Priestley space of the distributive lattice. Thus, the duality given by Moshier and Jipsen is neither a generalization of the Stone duality nor of the Priestley duality for distributive lattices. Even in the Boolean case is not a generalization.

Our purpose in this paper is to develop a topological duality for the category of semilattices (lattices) such that the dual objects are topological spaces with no additional structure and in such a way that when restricted to the full subcategory of distributive lattices coincides with the Stone duality.

The paper is organized as follows. In Section 2, we present several notions and results needed for what follows in the paper. In Section 3, we introduce the notion of *S-space*, and we establish the topological duality between the category of semilattices and homomorphisms and the category of S-spaces and meet-relations. Then, we apply this duality to obtain a categorical duality between the category of lattices and homomorphism and the category of L-spaces and L-relations. In Section 4, we show that our topological duality for semilattices (and thus also for lattices) is a generalization of the topological duality developed by Stone for the category of distributive lattices. We also characterize the spectral spaces through the concept of S-space. Finally, in Section 5, we develop directly the categorical equivalence between the category of S-spaces and meet-relations and the category of HMS spaces and F-continuous functions. The HMS spaces are the dual spaces of the semilattices used by Moshier and Jipsen in their topological duality for semilattices.

2. Preliminaries

In this section, we will introduce some definitions and notations. Let X be a set. Let $\mathcal{P}(X)$ be the powerset of X. For a subset Y of X, we shall denote the complement of Y with respect to X by $X \setminus Y$ or simply by Y^c.

Our main references for Order and Lattice Theory are [8, 15]. Let $\langle L, \leq \rangle$ be a poset. A subset $U \subseteq L$ is said to be an *upset* of L if for all $a, b \in L$ such that $a \in U$ and $a \leq b$, we have $b \in U$. Dually, we have the notion of *downset*.

A *meet-semilattice with a greatest element* is an algebra $\langle L, \wedge, 1 \rangle$ of type $(2, 0)$ such that the operation $\wedge$ is idempotent, commutative and associative, and $a \wedge 1 = a$, for all $a \in L$. As usual, the partial order $\leq$ associated with $\wedge$ is defined on L as follows: $a \leq b$ if and only if $a \wedge b = a$. In what follows, for brevity, by *semilattice* we mean a meet-semilattice with a greatest element. A *bounded* semilattice is an algebra $\langle L, \wedge, 0, 1 \rangle$ of type $(2, 0, 0)$ such that $\langle L, \wedge, 1 \rangle$ is a semilattice and $a \wedge 0 = 0$ for all $a \in L$.

A nonempty subset F of a semilattice L is said to be a *filter* if F is an upset, and $a, b \in F$ implies $a \wedge b \in F$. We denote by $\mathrm{Fi}(L)$ the collection of all filters of L. It is well known that $\mathrm{Fi}(L)$ is an algebraic closure system. If we denote by $\mathrm{Fig}(.)$ the closure operator associated with $\mathrm{Fi}(L)$, it is also known that the *filter generated by a subset* $H \subseteq L$ can be characterized as

$$\mathrm{Fig}(H) = \{a \in L : \exists \{h_1, \ldots, h_n\} \subseteq H \text{ s.t. } h_1 \wedge \cdots \wedge h_n \leq a\}.$$

A proper filter P of L is called *irreducible* when for all $F_1, F_2 \in \mathrm{Fi}(L)$, if $P = F_1 \cap F_2$, then $P = F_1$ or $P = F_2$. The set of all irreducible filters of L will be denoted by $\mathrm{X}(L)$.

A nonempty subset I of a semilattice L is said to be an *order-ideal* if it is a downset and, for all $a, b \in I$, there exists $c \in I$ such that $a, b \leq c$.

Theorem 2.1 ([2]). *Let L be a semilattice. Let $F \in \mathrm{Fi}(L)$ and I an order-ideal of L. If $F \cap I = \emptyset$, then there exists $P \in \mathrm{X}(L)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Corollary 2.2. *Let L be a semilattice. Then every proper filter is the intersection of irreducible filters.*

The following lemma is a useful characterization of irreducible filters. It will be important to develop our duality.

Lemma 2.3 ([2])**.** *Let L be a semilattice and let $F \in \mathrm{Fi}(L)$. Then, F is irreducible if and only if for every $a, b \notin F$ there exist $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.*

The following definition arises from the previous lemma.

Definition 2.4. Let L be a semilattice and $F \in \mathrm{Fi}(L)$. A subset I of L is said to be an F-*ideal* if it is a downset, and for all $a, b \in I$, there exist $c \in I$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.

Thus, the previous lemma claims that a filter F is irreducible if and only if F^c is an F-ideal. The following theorem will be crucial for what follows.

Theorem 2.5. *Let L be a semilattice. Let $F \in \mathrm{Fi}(L)$, and let I be an F-ideal. If $F \cap I = \emptyset$, then there exists $P \in \mathrm{X}(L)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let us consider the family

$$\mathcal{F} = \{H \in \mathrm{Fi}(L) : F \subseteq H \text{ and } H \cap I = \emptyset\}.$$

We note that $\mathcal{F} \neq \emptyset$ because $F \in \mathcal{F}$. By the Zorn's lemma, there exists a maximal element P of $\mathcal{F}$. We prove that P is irreducible. Let $F_1, F_2 \in \mathrm{Fi}(L)$ be such that $P = F_1 \cap F_2$. Suppose that $P \neq F_1$ and $P \neq F_2$. Then $F_1, F_2 \notin \mathcal{F}$. Thus $F_1 \cap I \neq \emptyset$ and $F_2 \cap I \neq \emptyset$. So, there exist $a \in F_1 \cap I$ and $b \in F_2 \cap I$. As $a, b \in I$ and I is an F-ideal downset, there exist $f \in F$ and $c \in I$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$. As $f \in F \subseteq P$, we get $f \in F_1 \cap F_2$. So, $a \wedge f \in F_1$ and $b \wedge f \in F_2$. It follows that $c \in F_1 \cap F_2$, which is an absurd. Hence, $P = F_1$ or $P = F_2$. □ □

We close this section introducing some topological concepts. We assume that the reader is familiar with basic topological notions. We refer the reader to [10].

Let $\langle X, \tau \rangle$ be a topological space. We denote the collection of all closed subsets of X by $\mathcal{C}(X)$, and for every $Y \subseteq X$, $\mathrm{cl}(Y)$ denotes the topological closure of Y. The *specialization order* of X is the binary relation $\sqsubseteq$ defined as follows: for all $x, y \in X$,

$$x \sqsubseteq y \iff \forall U \in \tau (x \in U \implies y \in U) \iff x \in \mathrm{cl}(y).$$

It is clear that X is a T_0-space if and only if $\sqsubseteq$ is a partial order.

A closed subset Y of X is said to be *irreducible* if $Y \subseteq A \cup B$ with $A, B \in \mathcal{C}(X)$, then $Y \subseteq A$ or $Y \subseteq B$. A topological space X is called *sober* if it is T_0 and, for every irreducible closed subset A, there is an element $x \in X$ such that $Y = \mathrm{cl}(x)$.

Lemma 2.6 ([21, 1.3.1])**.** *A topological space $\langle X, \tau\rangle$ is sober if and only if it is T_0 and for every completely prime filter $\mathcal{F}$ of the lattice of open subsets of X, there is $x \in X$ such that $\mathcal{F} = \mathrm{N}(x) = \{U \in \tau : x \in U\}$.*

3. Topological dualities

3.1. **S-spaces.** Let X be a nonempty set and let $\mathcal{K} \subseteq \mathcal{P}(X)$. Let us denote by $\tau_{\mathcal{K}}$ the topology on X generated by $\mathcal{K}$. In other words, $\tau_{\mathcal{K}}$ is the smallest topology on X such that $\mathcal{K} \subseteq \tau_{\mathcal{K}}$. It is well known that $\tau_{\mathcal{K}}$ consists of $\emptyset$, X, all finite intersections of $\mathcal{K}$, and all arbitrary unions of these finite intersections. The collection $\mathcal{K}$ is called a subbase for $\tau_{\mathcal{K}}$. We shall say simply that $\langle X, \mathcal{K}\rangle$ is a *topological space*, meaning that $\mathcal{K}$ is a subbase for the smallest topology $\tau_{\mathcal{K}}$ on X such that $\mathcal{K} \subseteq \tau_{\mathcal{K}}$.

Let $\langle X, \mathcal{K}\rangle$ be topological space. We consider the following collection of subsets of X:

$$\mathrm{S}(X) = \{U^c : U \in \mathcal{K}\}.$$

Let $\mathcal{C}_{\mathcal{K}}(X)$ be the closure system on X generated by $\mathrm{S}(X)$. Thus $\mathcal{C}_{\mathcal{K}}(X) = \{\bigcap \mathcal{A} : \mathcal{A} \subseteq \mathrm{S}(X)\}$. We denote by $\mathrm{cl}_{\mathcal{K}}$ the closure operator associated with $\mathcal{C}_{\mathcal{K}}(X)$; that is,

$$\mathrm{cl}_{\mathcal{K}}(Y) = \bigcap \{A \in \mathrm{S}(X) : Y \subseteq A\},$$

for all $Y \subseteq X$. The elements of $\mathcal{C}_{\mathcal{K}}(X)$ will be called *subbasic closed subsets* of X. Notice that $\mathrm{S}(X) \subseteq \mathcal{C}_{\mathcal{K}}(X) \subseteq \mathcal{C}(X)$.

Lemma 3.1. *Let $\langle X, \mathcal{K}\rangle$ be a topological space. Then,*

(1) *for every $Y \subseteq X$, $\mathrm{cl}(Y) \subseteq \mathrm{cl}_{\mathcal{K}}(Y)$;*
(2) *$\mathrm{cl}(x) = \mathrm{cl}_{\mathcal{K}}(x)$, for all $x \in X$.*

Proof. Property (1) is clear because $\mathcal{C}_{\mathcal{K}}(X) \subseteq \mathcal{C}(X)$. Property (2) follows from the fact that $\mathcal{K}$ is a subbase for the topology $\tau_{\mathcal{K}}$. □ □

Let $\langle X, \mathcal{K}\rangle$ be a topological space such that the subbase $\mathcal{K}$ is closed under finite unions and $\emptyset \in \mathcal{K}$. Then $\langle \mathrm{S}(X), \cap, X\rangle$ is a semilattice, and it will be called the *dual semilattice* of $\langle X, \mathcal{K}\rangle$. Now we establish a relationship between the subbasic closed subsets of a topological space $\langle X, \mathcal{K}\rangle$ and the filters of the semilattice $\mathrm{S}(X)$.

Proposition 3.2. *Let $\langle X, \mathcal{K}\rangle$ be a topological space such that $\mathcal{K}$ is a subbase of compact open subsets, and it is closed under finite unions and $\emptyset \in \mathcal{K}$. Then,*

(1) *for every $Y \in \mathcal{C}_{\mathcal{K}}(X)$, $F_Y := \{A \in \mathrm{S}(X) : Y \subseteq A\} \in \mathrm{Fi}(\mathrm{S}(X))$;*
(2) *for every $F \in \mathrm{Fi}(\mathrm{S}(X))$, $Y_F := \bigcap F \in \mathcal{C}_{\mathcal{K}}(X)$;*
(3) *if $Y_1, Y_2 \in \mathcal{C}_{\mathcal{K}}(X)$ and $Y_1 \subseteq Y_2$, then $F_{Y_2} \subseteq F_{Y_1}$;*
(4) *if $F_1, F_2 \in \mathrm{Fi}(\mathrm{S}(X))$ and $F_1 \subseteq F_2$, then $Y_{F_2} \subseteq Y_{F_1}$;*
(5) *$Y = Y_{F_Y}$, for all $Y \in \mathcal{C}_{\mathcal{K}}(X)$;*
(6) *$F = F_{Y_F}$, for all $F \in \mathrm{Fi}(\mathrm{S}(X))$.*

Hence, the posets $\mathcal{C}_{\mathcal{K}}(X)$ and $\mathrm{Fi}(\mathrm{S}(X))$, both ordered by the set theoretic inclusion, are dually isomorphic.

Proof. By the definition of $\mathcal{C}_{\mathcal{K}}(X)$, F_Y and Y_F, it is straightforward to show directly that properties (1)-(5) hold. We prove property (6). Let $F \in \mathrm{Fi}(\mathrm{S}(X))$. We need to show that $F = \{A \in \mathrm{S}(X) : \bigcap F \subseteq A\}$. It is clear that $F \subseteq \{A \in \mathrm{S}(X) : \bigcap F \subseteq A\}$. Now let $A \in \mathrm{S}(X)$ be such that $\bigcap F \subseteq A$. So $A^c \subseteq \bigcup\{B^c : B \in F\}$. Notice that $A^c, B^c \in \mathcal{K}$ for all B^c. Since $A^c \in \mathcal{K}$ is compact, there are $B_1, \ldots, B_n \in F$ such that $A^c \subseteq B_1^c \cup \cdots \cup B_n^c$. Thus $B_1 \cap \cdots \cap B_n \subseteq A$, and since F is a filter of $\mathrm{S}(X)$, it follows that $A \in F$. Then, $\{A \in \mathrm{S}(X) : \bigcap F \subseteq A\} \subseteq F$. Hence $F = F_{Y_F}$. □ □

Definition 3.3. Let $\langle X, \mathcal{K}\rangle$ be a topological space. Let $Y \subseteq X$. We will say that a family $\mathcal{Z} \subseteq \mathrm{S}(X)$ is a *Y-family* if for all $A, B \in \mathcal{Z}$, there exist $H, C \in \mathrm{S}(X)$ such that $Y \subseteq H$, $C \in \mathcal{Z}$, $A \cap H \subseteq C$ and $B \cap H \subseteq C$.

Proposition 3.4. *Let $\langle X, \mathcal{K}\rangle$ be a topological space such that $\mathcal{K}$ is a subbase of compact open subsets, and it is closed under finite unions and $\emptyset \in \mathcal{K}$. Let $Y \in \mathcal{C}_{\mathcal{K}}(X)$. Then, a downset $\mathcal{Z} \subseteq \mathrm{S}(X)$ is a Y-family if and only if it is an F_Y-ideal of $\mathrm{S}(X)$.*

Definition 3.5. An *S-space* is a topological space $\langle X, \mathcal{K}\rangle$ satisfying the following:

(S1) $\langle X, \mathcal{K}\rangle$ is a T_0-space and $X = \bigcup \mathcal{K}$;

(S2) $\mathcal{K}$ is a subbase of compact open subsets, it is closed under finite unions and $\emptyset \in \mathcal{K}$;

(S3) For all $U, V \in \mathcal{K}$, if $x \in U \cap V$, then there exist $W, D \in \mathcal{K}$ such that $x \notin W$, $x \in D$ and $D \subseteq (U \cap V) \cup W$.

(S4) If $Y \in \mathcal{C}_{\mathcal{K}}(X)$ and $\mathcal{Z} \subseteq \mathrm{S}(X)$ is a Y-family such that $Y \cap A^c \neq \emptyset$, for all $A \in \mathcal{Z}$, then $Y \cap \bigcap\{A^c : A \in \mathcal{Z}\} \neq \emptyset$.

Remark 3.6. Let $\langle X, \mathcal{K}\rangle$ be a T_1-space. Then, condition (S3) follows from condition (S2). Indeed, let $U, V \in \mathcal{K}$ and $x \in X$ be such that $x \in U \cap V$. Let $D \in \mathcal{K}$ be such that $x \in D$ (for instance, $D := U$). Since X is a T_1-space, we have that $\{x\}$ is a closed subset of X. Then, by Lemma 3.1, we obtain that $\{x\} = \mathrm{cl}(x) = \mathrm{cl}_{\mathcal{K}}(x) = \bigcap\{A \in \mathrm{S}(X) : x \in A\}$. Thus

$$\{x\}^c = \bigcup\{U \in \mathcal{K} : x \notin U\}.$$

Hence, for every $y \in D \setminus \{x\}$, there is $U_y \in \mathcal{K}$ such that $y \in U_y$ and $x \notin U_y$. Then, $D \setminus \{x\} \subseteq \bigcup\{U_y : y \in D \setminus \{x\}\}$. We obtain that $D \subseteq \bigcup\{U_y : y \in D \setminus \{x\}\} \cup (U \cap V)$. Since $D \in \mathcal{K}$ is compact, it follows that there are $y_1, \ldots, y_n \in D \setminus \{x\}$ such that $D \subseteq (U_{y_1} \cup \cdots \cup U_{y_n}) \cup (U \cap V)$. By (S2), we have that $W := U_{y_1} \cup \cdots \cup U_{y_n} \in \mathcal{K}$. Then $D \subseteq (U \cap V) \cup W$ and $x \notin W$. Hence (S3) holds.

Given a space $\langle X, \mathcal{K}\rangle$ satisfying (S2), recall that $\mathrm{X}(\mathrm{S}(X))$ denotes the collection of all irreducible filters of the semilattice $\langle \mathrm{S}(X), \cap, X\rangle$.

Lemma 3.7. *Let $\langle X, \mathcal{K}\rangle$ be a topological space satisfying conditions* (S1)-(S3). *Then, for every $x \in X$, we have $\{A \in \mathrm{S}(X) : x \in A\} \in \mathrm{X}(\mathrm{S}(X))$.*

Proof. Let $x \in X$. We denote $H_X(x) := \{A \in \mathrm{S}(X) : x \in A\}$. It is clear that $H_X(x)$ is a filter of $\mathrm{S}(X)$. Since $X = \bigcup \mathcal{K}$, we have $H_X(x) \neq \mathrm{S}(X)$. Thus, $H_X(x)$ is a proper filter of $\mathrm{S}(X)$. Now we show that $H_X(x)$ is irreducible. Let us use Lemma 2.3. Let $A, B \notin H_X(x)$. Thus $x \in A^c \cap B^c$ and $A^c, B^c \in \mathcal{K}$. By (S3), there exist $W, D \in \mathcal{K}$ such that $x \notin W$, $x \in D$ and $D \subseteq (A^c \cap B^c) \cup W$. Then, there are $W^c, D^c \in \mathrm{S}(X)$ such that $D^c \notin H_X(x)$, $W^c \in H_X(x)$, $W^c \cap A \subseteq D^c$ and $W^c \cap B \subseteq D^c$. Hence, by Lemma 2.3, we obtain that $H_X(x)$ is an irreducible filter of $\mathrm{S}(X)$. □

In the next result, we obtain an equivalent condition to condition (S4). This will be useful for what follows.

Proposition 3.8. *Let $\langle X, \mathcal{K}\rangle$ be a topological space satisfying conditions* (S1)-(S3). *Then, the following are equivalent.*

(1) *The space* $\langle X, \mathcal{K}\rangle$ *satisfies condition* (S4).
(2) *The map* $H_X : X \to \mathrm{X}(\mathrm{S}(X))$ *defined by*

$$H_X(x) = \{A \in \mathrm{S}(X) : x \in A\}, \text{ for each } x \in X,$$

is onto.

Proof. By Lemma 3.7, we have that H_X is well defined.

(1) $\Rightarrow$ (2) Let $P \in \mathrm{X}(\mathrm{S}(X))$. Consider the set $Y = \bigcap\{A : A \in P\}$. It is clear that $Y \in \mathcal{C}_{\mathcal{K}}(X)$. Consider the family $\mathcal{Z} = \{B \in \mathrm{S}(X) : B \notin P\}$. As P is an irreducible filter of $\mathrm{S}(X)$, it is straightforward to check that $\mathcal{Z}$ is a Y-family. We prove that $Y \cap B^c \neq \emptyset$, for all $B \in \mathcal{Z}$. If there exists $B \in \mathcal{Z}$ such that $Y \subseteq B$, we have that $B^c \subseteq \bigcup\{A^c : A \in P\}$, and since $B^c \in \mathcal{K}$ is compact, it follows that there exist $A_1, \ldots, A_n \in P$ such that $A_1 \cap \cdots \cap A_n \subseteq B$. As P is a filter, we obtain that $B \in P$, which is impossible. Thus, $Y \cap B^c \neq \emptyset$, for all $B \in \mathcal{Z}$.

Then, by condition (S4), we have $Y \cap \bigcap\{B^c : B \in \mathcal{Z}\} \neq \emptyset$. So there exists $x \in Y \cap \bigcap\{B^c : B \in \mathcal{Z}\}$, which implies that $P = H_X(x)$.

(2) $\Rightarrow$ (1) Let $Y \in \mathcal{C}_{\mathcal{K}}(X)$ and $\mathcal{Z} \subseteq \mathrm{S}(X)$ be a Y-family such that $Y \cap B^c \neq \emptyset$, for all $B \in \mathcal{Z}$. We need to prove that $Y \cap \bigcap\{B^c : B \in \mathcal{Z}\} \neq \emptyset$. By Proposition 3.2, we have that $F_Y = \{A \in \mathrm{S}(X) : Y \subseteq A\} \in \mathrm{Fi}(\mathrm{S}(X))$ and $\bigcap F_Y = Y$. Now let $(\mathcal{Z}] = \{A \in \mathrm{S}(X) : A \subseteq B, \text{ for some } B \in \mathcal{Z}\}$. Since $\mathcal{Z}$ is a Y-family, it follows that $(\mathcal{Z}]$ is also a Y-family. By Proposition 3.4, we have that $(\mathcal{Z}]$ is F_Y-ideal. Now let us show that $F_Y \cap (\mathcal{Z}] = \emptyset$. Suppose it is not. Thus there is $A \in F_Y$ and $C \in \mathcal{Z}$ such that $A \subseteq C$. Then, $Y \subseteq C$. This is a contradiction because $Y \cap B^c \neq \emptyset$ for all $B \in \mathcal{Z}$. Now, since $F_Y \cap (\mathcal{Z}] = \emptyset$, it follows by Theorem 2.5 that there is $P \in \mathrm{X}(\mathrm{S}(X))$ such that $F_Y \subseteq P$ and $P \cap (\mathcal{Z}] = \emptyset$. By (2), there is $x \in X$ such that $H_X(x) = P$. Then, since $F_Y \subseteq H_X(x)$ and $(\mathcal{Z}] \cap H_X(x) = \emptyset$, we have that $x \in Y$ and $x \in B^c$, for all $B \in \mathcal{Z}$. Hence $Y \cap \bigcap\{B^c : B \in \mathcal{Z}\} \neq \emptyset$. Therefore, we have proved that the space $\langle X, \mathcal{K}\rangle$ satisfies condition (S4). □ □

3.2. **Representation for semilattices.** Let $\langle L, \wedge, 1\rangle$ be a semilattice. Recall that $\mathrm{X}(L)$ denotes the collection of all irreducible filters of L. We define the map $\sigma : L \to \mathcal{P}(\mathrm{X}(L))$ as follows: $\sigma(a) = \{P \in \mathrm{X}(L) : a \in P\}$, for all $a \in L$. Let $\mathcal{K}_L = \{\sigma(a)^c : a \in L\}$. Notice that for every $a \in L$, $\sigma(a)^c = \mathrm{X}(L) \setminus \sigma(a) = \{P \in \mathrm{X}(L) : a \notin P\}$. It is straightforward to show that, for all $a, b \in L$, $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$, and $\sigma(1) = \mathrm{X}(L)$. Moreover, by Theorem 2.1, we have that $a \leq b$ if and only if $\sigma(a) \subseteq \sigma(b)$, for all $a, b \in L$.

We consider the topological space $\mathbf{X}(L) = \langle \mathrm{X}(L), \mathcal{K}_L \rangle$, which will be called the *dual S-space* of L. Notice that $\mathrm{S}(\mathrm{X}(L)) = \{\sigma(a) : a \in L\}$. Hence, we have the following result, which is straightforward.

Proposition 3.9. *Let $\langle L, \wedge, 1 \rangle$ be a semilattice. Then $\mathbf{X}(L) = \langle \mathrm{X}(L), \mathcal{K}_L \rangle$ is a topological space such that $\mathcal{K}_L$ is a subbase closed under finite unions, and $\emptyset \in \mathcal{K}_L$. Moreover, the map $\sigma \colon L \to \mathrm{S}(\mathrm{X}(L))$ is an isomorphism of semilattices.*

Proposition 3.10. *Let $\langle L, \wedge, 1 \rangle$ be a semilattice. Then, the topological space $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ is an S-space.*

Proof. It is clear that $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ is a T_0-space. Moreover, since every irreducible filter $F \in \mathrm{X}(L)$ is proper, it follows that $\mathrm{X}(L) = \bigcup \mathcal{K}_L$. Hence, $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ satisfies condition (S1).

By the previous proposition, we know that $\mathcal{K}_L$ is closed under finite unions, and $\emptyset \in \mathcal{K}_L$. Let $a \in L$. Assume that $\sigma(a)^c \subseteq \bigcup_{b \in B} \sigma(b)^c$ for some $B \subseteq L$. So $\bigcap_{b \in B} \sigma(b) \subseteq \sigma(a)$. Let $F := \mathrm{Fig}_L(B)$. If $a \notin F$, then by Theorem 2.1 there is $P \in \mathrm{X}(L)$ such that $F \subseteq P$ and $a \notin P$. Thus $P \in \bigcap_{b \in B} \sigma(b)$ and $P \notin \sigma(a)$, which is a contradiction. Hence $a \in F$. Then, there exist $b_1, \dots, b_n \in F$ such that $b_1 \wedge \cdots \wedge b_n \leq a$. Hence $\sigma(b_1) \cap \dots \sigma(b_n) \subseteq \sigma(a)$, and thus $\sigma(a)^c \subseteq \sigma(b_1)^c \cup \cdots \cup \sigma(b_n)^c$. We have proved that any cover of $\sigma(a)^c$ by elements of $\mathcal{K}_L$ has a finite subcover. Then, since $\mathcal{K}_L$ is a subbase for the space $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$, it follows by the Alexander subbase Lemma that $\sigma(a)^c$ is compact. Therefore, the topological space $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ satisfies condition (S2).

Condition (S3) follows from Lemma 2.3.

Finally, we prove that condition (S4) holds. Let $Y \in \mathcal{C}_{\mathcal{K}_L}(\mathrm{X}(L))$ and let $\mathcal{Z} \subseteq \mathrm{S}(\mathrm{X}(L))$ be a Y-family such that $Y \cap \sigma(a)^c \neq \emptyset$, for all $\sigma(a) \in \mathcal{Z}$. Suppose, towards a contradiction, that $Y \cap \bigcap \{\sigma(a)^c : \sigma(a) \in \mathcal{Z}\} = \emptyset$. Since $Y \in \mathcal{C}_{\mathcal{K}_L}(\mathrm{X}(L))$ it follows from Propositions 3.2 and 3.9, that $F = \sigma^{-1}[F_Y]$ is a filter of L. Since $\mathcal{Z}$ is a Y-family of $\mathrm{S}(\mathrm{X}(L))$, it follows that $(\mathcal{Z}] = \{\sigma(b) \in \mathrm{S}(\mathrm{X}(L)) : \sigma(b) \subseteq \sigma(a) \text{ for some } \sigma(a) \in \mathcal{Z}\}$ is an F-ideal downset of $\mathrm{S}(\mathrm{X}(L)$. Thus, $I = \{b \in L : \sigma(b) \in (\mathcal{Z}]\}$ is an F-ideal downset of L. Now, if $I \cap F \neq \emptyset$, then there is $a \in L$ such that $\sigma(a) \in \mathcal{Z}$ and $\sigma(a) \in F_Y$; thus $Y \cap \sigma(a)^c = \emptyset$ with $\sigma(a) \in \mathcal{Z}$, which is a contradiction. Hence, we have $F \cap I = \emptyset$. By Theorem 2.5, there exists $P \in \mathrm{X}(L)$ such that $F \subseteq P$ and $P \cap I = \emptyset$. Then, since $Y = \bigcap F_Y$, it follows that $P \in Y$. Since $P \cap I = \emptyset$, we obtain that $P \in \sigma(a)^c$, for all $a \in I$. Thus $P \in \sigma(a)^c$, for all $\sigma(a) \in \mathcal{Z}$. Hence $P \in Y \cap \bigcap\{\sigma(a)^c : \sigma(a) \in \mathcal{Z}\}$. This completes the proof. □ □

Theorem 3.11 (Topological representation)**.** *Every semilattice L is isomorphic to the dual semilattice $\mathrm{S}(X)$ of an S-space $\langle X, \mathcal{K}\rangle$.*

Let $\langle X, \mathcal{K}\rangle$ be an S-space and consider its dual semilattice $\langle \mathrm{S}(X), \cap, X\rangle$. Now we consider the dual S-space $\mathbf{X}(\mathrm{S}(X)) = \langle \mathrm{X}(\mathrm{S}(X)), \mathcal{K}_{\mathrm{S}(X)}\rangle$ of the semilattice $\mathrm{S}(X)$. Recall that $H_X\colon X \to \mathrm{X}(\mathrm{S}(X))$ is the map defined by

$$H_X(x) = \{A \in \mathrm{S}(X) : x \in A\}$$

for all $x \in X$ (see Lemma 3.7).

Proposition 3.12. *$H_X\colon X \to \mathrm{X}(\mathrm{S}(X))$ is a homeomorphism between the S-spaces $\langle X, \mathcal{K}\rangle$ and $\langle \mathrm{X}(\mathrm{S}(X)), \mathcal{K}_{\mathrm{S}(X)}\rangle$. Moreover $\mathcal{K}_{\mathrm{S}(X)} = \{H_X[U] : U \in \mathcal{K}\}$.*

Proof. We prove the theorem in several steps.

- H_X is onto. It follows from Proposition 3.8.
- H_X is one-to-one. Let $x, y \in X$ and suppose that $H_X(x) = H_X(y)$. Notice that $H_X(x) = \{A \in \mathrm{S}(X) : \mathrm{cl}_{\mathcal{K}}(x) \subseteq A\} = F_{\mathrm{cl}_{\mathcal{K}}(x)}$. Thus $F_{\mathrm{cl}_{\mathcal{K}}(x)} = F_{\mathrm{cl}_{\mathcal{K}}(y)}$ and, by Proposition 3.2, it follows that $\mathrm{cl}_{\mathcal{K}}(x) = \mathrm{cl}_{\mathcal{K}}(y)$. Since X is a T_0-space, it follows that $x = y$.
- H_X is a continuous map. Notice that $\mathcal{K}_{\mathrm{S}(X)} = \{\sigma(A)^c : A \in \mathrm{S}(X)\}$ where $\sigma(A)^c = \{P \in \mathrm{X}(\mathrm{S}(X)) : A \notin P\}$. Let $A \in \mathrm{S}(X)$ and $x \in X$. Then,

(3.1)
$$x \in H_X^{-1}[\sigma(A)^c] \iff H_X(x) \in \sigma(A)^c \iff A \notin H_X(x) \iff x \in A^c.$$

Thus $H_X^{-1}[\sigma(A)^c] = A^c \in \mathcal{K}$. Then, H_X is continuous. By (3.1), we also have proved that H_X is an open map because H_X^{-1} is a bijection between subbasic open subsets. Hence H_X is a homeomorphism. Moreover, from (3.1) we obtain that $\mathcal{K}_{\mathrm{S}(X)} = \{H_X[U] : U \in \mathcal{K}\}$. This completes the proof. □ □

Corollary 3.13. *Let L be a semilattice and $\langle X, \mathcal{K}\rangle$ an S-space. Then $\mathbf{X}(L) = \langle \mathrm{X}(L), \mathcal{K}_L\rangle$ is an S-space, $\mathbf{S}(X) = \langle \mathrm{S}(X), \cap, \mathrm{X}\rangle$ is a semilattice, and $L \cong \mathbf{S}(\mathbf{X}(L))$ and $X \cong \mathbf{X}(\mathbf{S}(X))$.*

3.3. **Duality for semilattices.** Our aim here is to extend the representation of semilattices through S-spaces to a full categorical duality. To this end, we need to specify which are the morphisms between two objects in the respective categories. For semilattices, the morphisms will be the natural ones. Let us denote by $\mathbb{MS}$ the category of semilattices and homomorphisms (that is, maps preserving meets and top element).

Now, the morphisms for S-spaces should be the corresponding ones to the homomorphisms. We shall consider the notion of meet-relation introduced in [2, 4] (in [1] the authors considered the same notion of meet-relation to develop a duality for the category of distributive semilattices). We need to introduce some notations. Let $R \subseteq X_1 \times X_2$ be a relation. For every $x \in X_1$, $R(x) = \{y \in X_2 : (x,y) \in R\}$. For every $y \in X_2$, $R^{-1}(y) = \{x \in X_1 : (x,y) \in R\}$. Let $\Box_R \colon \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ be the map defined as follows: for all $B \subseteq X_2$,

$$\Box_R(B) = \{x \in X_1 : R(x) \subseteq B\}.$$

Definition 3.14. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be S-spaces. A relation $R \subseteq X_1 \times X_2$ is said to be a *meet-relation* if:

(R1) For all $B \in \mathrm{S}(X_2)$, $\Box_R(B) \in \mathrm{S}(X_1)$.
(R2) For every $x \in X_1$, $R(x) \in \mathcal{C}_{\mathcal{K}_2}(X_2)$.

Our definition of meet-relation is slightly different from the one given in [4]. Here we use subbases $\mathcal{K}$ of compact opens, while in [4] the authors work with the collection of all compact opens (which form a base for their topology). Despite this slight difference, all the results about meet-relations given in [4] are still valid here. Even more, the proofs presented in [4] can be performed exactly in the same way here. Thus, we shall omit most of these proofs and refer the reader to [4] (see also [2, 1]).

Definition 3.15. Let $\langle X_i, \mathcal{K}_i\rangle$, with $i = 1, 2, 3$, be S-spaces, and let $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ be meet-relations. The *composition* between R and T is defined by the relation $T * R \subseteq X_1 \times X_3$ as follows: for every $x \in X_1$ and $z \in X_3$,

$$(x,z) \in T * R \iff (\forall D \in \mathrm{S}(X_3))((T \circ R)(x) \subseteq D \implies z \in D)$$

where $T \circ R$ is the usual set-theoretical composition.

Notice that for every $x \in X_1$,

$$(T * R)(x) = \mathrm{cl}_{\mathcal{K}_3}\left((T \circ R)(x)\right).$$

Proposition 3.16. *Let $\langle X_i, \mathcal{K}_i\rangle$, with $i = 1, 2, 3$, be S-spaces, and let $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ be meet-relations. Then, for every $C \in \mathrm{S}(X_3)$, we have*

$$\Box_{T \circ R}(C) = (\Box_R \circ \Box_T)(C) = \Box_{T * R}(C).$$

Hence, $T * R \subseteq X_1 \times X_3$ is also a meet-relation.

Proposition 3.17. *For every S-space $\langle X, \mathcal{K}\rangle$, the dual of the specialization order $\sqsupseteq\ \subseteq X \times X$ is a meet-relation. Moreover, for all meet-relations $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$, $\sqsupseteq_2 * R = R$ and $T * \sqsupseteq_2 = T$.*

Proposition 3.18. *Let $R \subseteq X_1 \times X_2$, $S \subseteq X_2 \times X_3$ and $T \subseteq X_3 \times X_4$ be meet-relations. Then $T * (S * R) = (T * S) * R$.*

Proof. It follows from Proposition 3.16. □ □

From the previous three propositions, we can define the category $\mathbb{SS}$ of S-spaces and meet-relations, where the identity morphism is the dual of the specialization order, and the composition is $*$.

Proposition 3.19. *Let $R \subseteq X_1 \times X_2$ be a meet-relation. Then, the map $\Box_R \colon \mathrm{S}(X_2) \to \mathrm{S}(X_1)$ is a homomorphism of semilattices.*

Recall that for every semilattice L, $\sigma \colon L \to \mathrm{S}(\mathrm{X}(L))$ is the isomorphism given by $\sigma(a) = \{P \in \mathrm{X}(L) : a \in P\}$.

Proposition 3.20. *Let $h \colon L_1 \to L_2$ be a semilattice homomorphism. Then, the relation $R_h \subseteq \mathrm{X}(L_2) \times \mathrm{X}(L_1)$ defined as follows:*

$$(Q, P) \in R_h \iff h^{-1}[Q] \subseteq P,$$

for all $(Q, P) \in \mathrm{X}(L_2) \times \mathrm{X}(L_1)$, is a meet-relation. Moreover, $\sigma_2 \circ h = \Box_{R_h} \circ \sigma_1$.

Proof. Recall that $\mathrm{S}(\mathrm{X}(L_i)) = \{\sigma_i(a) : a \in L_i\}$ and for every $a \in L_1$,

$$\Box_{R_h}(\sigma_1(a)) = \{Q \in \mathrm{X}(L_2) : R_h(Q) \subseteq \sigma_1(a)\}.$$

Let $a \in L_1$ and $Q \in \mathrm{X}(L_2)$. Then,

$$\begin{aligned} Q \in \Box_{R_h}(\sigma_1(a)) &\iff R_h(Q) \subseteq \sigma_1(a) \\ &\iff \forall P \in \mathrm{X}(L_1)(h^{-1}[Q] \subseteq P \implies a \in P) \\ &\iff h(a) \in Q \\ &\iff Q \in \sigma_2(h(a)). \end{aligned}$$

Notice that in the third equivalence we have used that $h^{-1}[Q]$ is a filter of L_1 and Theorem 2.1. Hence $\Box_{R_h}(\sigma_1(a)) = \sigma_2(h(a))$. Thus, R_h satisfies condition (R1) of Definition 3.14. Moreover, we have proved that $\sigma_2 \circ h = \Box_{R_h} \circ \sigma_1$. For every $Q \in \mathrm{X}(L_2)$, we have $R_h(Q) = \bigcap\{\sigma_1(a) : a \in h^{-1}[Q]\}$. Then, $R_h(Q) \in \mathcal{C}_{\mathcal{K}_1}(\mathrm{X}(L_1))$. Thus R_h satisfies condition (R2). Hence, R_h is a meet-relation. □ □

For the next proposition, we need the following auxiliary lemma.

Lemma 3.21. *Let* $R, T \subseteq X_1 \times X_2$*, be meet-relations. If* $\Box_R(B) = \Box_T(B)$ *for all* $B \in \mathrm{S}(X_2)$*, then* $R = T$*.*

Proof. It follows from the fact that for every $x \in X_1$, $R(x), T(x) \in \mathcal{C}_{\mathcal{K}_2}(X_2)$. □ □

Proposition 3.22. *Let* $h\colon L_1 \to L_2$ *and* $g\colon L_2 \to L_3$ *be semilattice homomorphisms. Then,* $R_{g\circ h} = R_h * R_g$*.*

Proof. Since $R_{g\circ h}, R_h * R_g \subseteq \mathrm{X}(L_3) \times \mathrm{X}(L_1)$, it follows that

$$\Box_{R_{g\circ h}}, \Box_{(R_h * R_g)}\colon \mathrm{S}(\mathrm{X}(L_1)) \to \mathrm{S}(\mathrm{X}(L_3)).$$

Let us to prove that $\Box_{R_{g\circ h}} = \Box_{(R_h * R_g)}$. Let $a \in L_1$. By Propositions 3.20 and 3.16, we have

$$\begin{aligned}\Box_{R_{g\circ h}}(\sigma_1(a)) &= \sigma_3((g\circ h)(a)) = (\sigma_3 \circ g)(h(a)) = (\Box_{R_g} \circ \sigma_2)(h(a)) \\ &= \Box_{R_g}((\Box_{R_h} \circ \sigma_1)(a)) = (\Box_{R_g} \circ \Box_{R_h})(\sigma_1(a)) = \Box_{(R_h * R_g)}(\sigma_1(a)).\end{aligned}$$

Hence, by Lemma 3.21, we obtain that $R_{g\circ h} = R_h * R_g$. □ □

Let $\langle X, \mathcal{K}\rangle$ be an S-space. By Proposition 3.12, we know that the map $H_X\colon X \to \mathrm{X}(\mathrm{S}(X))$ defined by $H_X(x) = \{A \in \mathrm{S}(X) : x \in A\}$ is a homeomorphism. We define the relation $R_X \subseteq X \times \mathrm{X}(\mathrm{S}(X))$ as follows:

$$(x, H_X(y)) \in R_X \iff H_X(x) \subseteq H_X(y)$$

for all $x, y \in X$. We also define the relation $R_X^{-1} \subseteq \mathrm{X}(\mathrm{S}(X)) \times X$ as follows:

$$(H_X(y), x) \in R_X^{-1} \iff H_X(y) \subseteq H_X(x)$$

for all $x, y \in X$.

Proposition 3.23. *Let* $\langle X, \mathcal{K}\rangle$ *be an S-space. Then,* R_X *and* R_X^{-1} *are meet-relations. Moreover* $R_X * R_X^{-1} = \sqsupseteq_{\mathrm{X}(\mathrm{S}(X))}$ *and* $R_X^{-1} * R_X = \sqsupseteq_X$*.*

Proof. Notice that for all $x, y \in X$, $H_X(x) \subseteq H_X(y) \iff y \sqsubseteq_X x$, and $\mathcal{K}_{\mathrm{S}(X)} = \{H_X[U] : U \in \mathcal{K}\}$ (Prop. 3.12). Then, it follows that R_X and R_X^{-1} are meet-relations. The identity $R_X * R_X^{-1} = \sqsupseteq_{\mathrm{X}(\mathrm{S}(X))}$ follows from the facts that $\sqsupseteq_{\mathrm{X}(\mathrm{S}(X))} = \subseteq$, and $A \in H_X(x) \iff (R_X \circ R_X^{-1})(H_X(x)) \subseteq H_X[A]$, for every $x \in X$ and $A \in \mathrm{S}(X)$. The identity $R_X^{-1} * R_X = \sqsupseteq_X$ follows straightforward from definitions of $*$ and $\sqsupseteq_X$. □ □

Now, from the results of the previous section and those developed here, we are ready to establish and prove one of the main theorems of this paper.

Theorem 3.24. *The categories* $\mathbb{MS}$ *and* $\mathbb{SS}$ *are dually equivalent.*

Proof. Let us define the corresponding functors. On the one hand, let $\mathbf{X}\colon \mathbb{MS} \to \mathbb{SS}$ be defined as follows: for every semilattice L, $\mathbf{X}(L) = \langle \mathrm{X}(L), \mathcal{K}_L\rangle$, and for every homomorphism $h\colon L_1 \to L_2$, $\mathbf{X}(h) = R_h \subseteq \mathbf{X}(L_2) \times \mathbf{X}(L_1)$. By Propositions 3.10 and 3.20, we have that $\mathbf{X}$ is well defined. Since $R_{\mathrm{id}_L} = \sqsupseteq_{\mathrm{X}(L)} = \subseteq$, where $\mathrm{id}_L\colon L \to L$ is the identity homomorphism, and from Proposition 3.22, it follows that $\mathbf{X}$ is a contravariant functor.

On the other hand, let $\mathbf{S}\colon \mathbb{SS} \to \mathbb{MS}$ be defined as follows: for every S-space $\langle X, \mathcal{K}\rangle$, $\mathbf{S}(X) = \langle \mathrm{S}(X), \cap, X\rangle$, and for every meet-relation $R \subseteq X_1 \times X_2$, $\mathbf{S}(R) = \Box_R\colon \mathbf{S}(X_2) \to \mathbf{S}(X_1)$. By Proposition 3.19, it is clear that $\mathbf{S}$ is well defined. For every S-space X, $\Box_{\sqsupseteq_X} = \mathrm{id}_{\mathrm{S}(X)}$. Thus, by Proposition 3.16, $\mathbf{S}$ is a contravariant functor.

Now we need to define the corresponding natural transformations. For every semilattice L, we consider the isomorphism $\sigma\colon L \to \mathbf{S}(\mathbf{X}(L))$. By Proposition 3.20, we have for every semilattice homomorphism $h\colon L_1 \to L_2$ that $\sigma_2 \circ h = \mathbf{S}(\mathbf{X}(h)) \circ \sigma_1$. For every S-space X, we consider the isomorphism (in the category $\mathbb{SS}$) $R_X \subseteq X \times \mathbf{X}(\mathbf{S}(X))$ (Prop. 3.23). Let $R \subseteq X_1 \times X_2$ be a meet-relation. It follows that for every $x \in X_1$, $(R_{X_2} \circ R)(x) = (R_{\Box_R} \circ R_{X_1})(x)$. Then, we have $R_{X_2} * R = R_{\Box_R} * R_{X_1}$. Therefore, the result follows. □ □

3.4. **Topological duality for bounded lattices.** Let $\mathbb{BL}$ be the category of bounded lattices and lattice homomorphisms preserving bounds. We shall restrict the functor $\mathbf{X}$ from $\mathbb{MS}$ to $\mathbb{BL}$ to obtain an equivalence between the category $\mathbb{BL}$ and some subcategory of $\mathbb{SS}$.

Definition 3.25. An *L-space* is an S-space $\langle X, \mathcal{K}\rangle$ satisfying the following conditions:

(L1) $X \in \mathcal{K}$;
(L2) for all $U, V \in \mathcal{K}$, $\bigcup\{W \in \mathcal{K} : W \subseteq U \cap V\} \in \mathcal{K}$.

Let $\langle X, \mathcal{K}\rangle$ be an S-space. Recall that $\mathcal{C}_{\mathcal{K}}(X)$ is the closure system on X generated by $\mathrm{S}(X)$. Thus, $\langle \mathcal{C}_{\mathcal{K}}(X), \cap, \veebar, \emptyset, X\rangle$ is a (complete) lattice, where for all $Y_1, Y_2 \in \mathcal{C}_{\mathcal{K}}(X)$, $Y_1 \veebar Y_2 = \mathrm{cl}_{\mathcal{K}}(Y_1 \cup Y_2)$.

Proposition 3.26. *Let $\langle X, \mathcal{K}\rangle$ be an L-space. Then $\langle \mathrm{S}(X), \cap, \veebar, \emptyset, X\rangle$ is a sublattice of $\langle \mathcal{C}_{\mathcal{K}}(X), \cap, \veebar, \emptyset, X\rangle$.*

Proof. It is clear that $\emptyset \in \mathrm{S}(X)$. From condition (L2) we have, for all $A, B \in \mathrm{S}(X)$, that $\bigcap\{C \in \mathrm{S}(X) : A \cup B \subseteq C\} \in \mathrm{S}(X)$. Thus $A \veebar B = C_{\mathcal{K}}(A \cup B) = \bigcap\{C \in \mathrm{S}(X) : A \cup B \subseteq C\}$. Then $A \veebar B \in \mathrm{S}(X)$, for all $A, B \in \mathrm{S}(X)$. Hence $\mathrm{S}(X)$ is sublattice of $\mathcal{C}_{\mathcal{K}}(X)$. □ □

Let $\langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Consider the dual S-space $\langle \mathrm{X}(L), \mathcal{K}_L\rangle$ of the semilattice reduct $\langle L, \wedge, 1\rangle$. Recall that the semilattice isomorphism $\sigma\colon L \to \mathrm{S}(\mathrm{X}(L)$ is given by $\sigma(a) = \{P \in \mathrm{X}(L) : a \in P\}$.

Proposition 3.27. *Let $\langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Then $\langle \mathrm{X}(L), \mathcal{K}_L\rangle$ is an L-space, and moreover $\sigma\colon L \to \mathrm{S}(\mathrm{X}(L))$ is a lattice isomorphism from $\langle L, \wedge, \vee, 0, 1\rangle$ onto $\langle \mathrm{S}(\mathrm{X}(L)), \cap, \veebar, \emptyset, \mathrm{X}\rangle$.*

Proof. Since $\sigma(0) = \emptyset$, it follows that $X = \sigma(0)^c \in \mathcal{K}_L$. Thus, (L1) is satisfied. Let $a, b \in L$. Let us prove that $\sigma(a \vee b) = \sigma(a) \veebar \sigma(b)$. By definition, $\sigma(a) \veebar \sigma(b) = C_{\mathcal{K}_L}(\sigma(a) \cup \sigma(b)) = \bigcap\{\sigma(c) \in \mathrm{S}(\mathrm{X}(L)) : \sigma(a) \cup \sigma(b) \subseteq \sigma(c)\}$. On the one hand, since σ is order-preserving, it follows that $C_{\mathcal{K}_L}(\sigma(a) \cup \sigma(b)) \subseteq \sigma(a \vee b)$. On the other hand, let $c \in L$ be such that $\sigma(a) \cup \sigma(b) \subseteq \sigma(c)$. Since σ is an order-embedding, it follows that $a, b \leq c$. Thus $a \vee b \leq c$. Then $\sigma(a \vee b) \subseteq \sigma(c)$. Hence $\sigma(a \vee b) \subseteq \sigma(a) \veebar \sigma(b)$. We have proved that $\sigma(a \vee b) = \sigma(a) \veebar \sigma(b)$. Let now $\sigma(a)^c, \sigma(b)^c \in \mathcal{K}_L$. Since $\bigcap\{\sigma(c) \in \mathrm{S}(\mathrm{X}(L)) : \sigma(a) \cup \sigma(b) \subseteq \sigma(c)\} = \sigma(a \vee b) \in \mathrm{S}(\mathrm{X}(L))$, it follows that $\bigcup\{\sigma(c)^c \in \mathcal{K}_L : \sigma(c)^c \subseteq \sigma(a)^c \cap \sigma(b)^c\} \in \mathcal{K}_L$. Hence, condition (L2) holds. Therefore, $\langle \mathrm{X}(L), \mathcal{K}_L\rangle$ is an L-space, and σ is a lattice isomorphism. □ □

Corollary 3.28. *Let $\langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice, and $\langle X, \mathcal{K}\rangle$ an L-space. Then, $\langle \mathrm{X}(L), \mathcal{K}_L\rangle$ is an L-space, $\langle \mathrm{S}(X), \cap, \veebar, \emptyset, X\rangle$ is a bounded lattice, and $\langle L, \wedge, \vee, 0, 1\rangle \cong \langle \mathrm{S}(\mathrm{X}(L)), \cap, \veebar, \emptyset, X\rangle$ and $\langle X, \mathcal{K}\rangle \cong \langle \mathrm{X}(\mathrm{S}(X)), \mathcal{K}_{\mathrm{S}(L)}\rangle$.*

Definition 3.29. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be L-spaces. A relation $R \subseteq X_1 \times X_2$ is called an *L-relation* if it is a meet-relation and satisfies the following:

(R3) $R(x) \neq \emptyset$, for all $x \in X_1$.
(R4) $\Box_R\,(\mathrm{cl}_{\mathcal{K}_2}(B_1 \cup B_2)) \subseteq \mathrm{cl}_{\mathcal{K}_1}(\Box_R(B_1) \cup \Box_R(B_2))$, for all $B_1, B_2 \in \mathrm{S}(X_2)$.

Proposition 3.30. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be L-spaces, and let $R \subseteq X_1 \times X_2$ be an L-relation. Then, $\Box_R\colon \mathrm{S}(X_2) \to \mathrm{S}(X_1)$ is a lattice homomorphism preserving bounds.*

Proof. We need only to prove that $\Box_R(\emptyset) = \emptyset$ and $\Box_R(B_1 \veebar B_2) = \Box_R(B_1) \veebar \Box_R(B_2)$, for all $B_1, B_2 \in \mathrm{S}(X_2)$. By (R3), we have $\Box_R(\emptyset) = \{x \in X_1 : R(x) \subseteq \emptyset\} = \emptyset$. Let $B_1, B_2 \in \mathrm{S}(X_2)$. Since $\Box_R$ is order-preserving, it follows that $\Box_R(B_1) \veebar \Box_R(B_2) \subseteq \Box_R(B_1 \veebar B_2)$. The inverse inclusion follows straightforward from (R4). Indeed, by (R4), we have

$$\Box_R(B_1 \veebar B_2) = \Box_R\left(\mathrm{cl}_{\mathcal{K}_2}(B_1 \cup B_2)\right) \subseteq \\ \subseteq \mathrm{cl}_{\mathcal{K}_1}\left(\Box_R(B_1) \cup \Box_R(B_2)\right) = \Box_R(B_1) \veebar \Box_R(B_2).$$

□ □

Proposition 3.31. *Let L_1 and L_2 be bounded lattices. If $h\colon L_1 \to L_2$ is a lattice homomorphism preserving bounds, then $R_h \subseteq \mathrm{X}(L_2) \times \mathrm{X}(L_1)$ is an L-relation.*

Proof. By Proposition 3.20, we know that $R_h \subseteq \mathrm{X}(L_2) \times \mathrm{X}(L_1)$ is a meet-relation. Let $Q \in \mathrm{X}(L_2)$. Since h is a lattice homomorphism preserving bounds, we have $h^{-1}[Q]$ is a proper filter of L_1. Thus, by Theorem 2.1, there exists $P \in \mathrm{X}(L_1)$ such that $h^{-1}[Q] \subseteq P$. Then $R_h(Q) \neq \emptyset$. Hence R_h satisfies condition (R3). In order to show that R_h satisfies condition (R4), recall that $\mathrm{S}(\mathrm{X}(L_i)) = \{\sigma_i(a) : a \in L_i\}$ and $\Box_{R_h}\colon \mathrm{S}(\mathrm{X}(L_1)) \to \mathrm{S}(\mathrm{X}(L_2))$. Then, by Proposition 3.27 and Proposition 3.20, we have for all $a, b \in L_1$ that

$$\begin{aligned}
\Box_{R_h}\Big(\mathrm{cl}_{\mathcal{K}_{L_1}}(\sigma_1(a) \cup \sigma_1(b))\Big) &= \Box_{R_h}\left(\sigma_1(a) \veebar \sigma_1(b)\right) \\
&= \Box_{R_h}\left(\sigma_1(a \vee b)\right) \\
&= \sigma_2(h(a \vee b)) \\
&= \sigma_2(h(a) \vee h(b)) \\
&= \sigma_2(h(a)) \veebar \sigma_2(h(b)) \\
&= \Box_{R_h}(\sigma_1(a)) \veebar \Box_{R_h}(\sigma_1(b)) \\
&= \mathrm{cl}_{\mathcal{K}_{L_2}}\left(\Box_{R_h}(\sigma_1(a)) \cup \Box_{R_h}(\sigma_1(b))\right)
\end{aligned}$$

Hence, R_h satisfies (R4). □ □

From the categorical duality already established together with Propositions 3.30 and 3.31 and Proposition 3.16 it can be proved that the composition $T * R \subseteq X_1 \times X_3$ of two L-relations $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ is again an L-relation. Moreover, it follows directly that for every L-space X, the meet-relation $\sqsupseteq \subseteq X \times X$ is an L-relation. Hence, we can define the subcategory $\mathbb{LS}$ of $\mathbb{SS}$ formed by L-spaces and L-relations.

Theorem 3.32. *The categories* $\mathbb{BL}$ *and* $\mathbb{LS}$ *are dually equivalent.*

Proof. By Corollary 3.28 and Propositions 3.30 and 3.31, we can consider the restrictions of the functors $\mathbf{X}$ and $\mathbf{S}$ given in Theorem 3.24 to the subcategories $\mathbb{BL}$ and $\mathbb{LS}$, respectively. Therefore, it follows by Theorem 3.24 that the categories $\mathbb{BL}$ and $\mathbb{LS}$ are dually equivalent under the functors $\mathbf{X}\colon \mathbb{BL} \to \mathbb{LS}$ and $\mathbf{S}\colon \mathbb{LS} \to \mathbb{BL}$. □ □

4. DS-spaces and spectral spaces

In this section, we shall characterize the spectral spaces (the dual objects of bounded distributive lattices under the Stone's duality) through the S-spaces. To achieve this, we first characterize the DS-spaces (the dual objects of semilattices under the Celani's duality) through the S-spaces.

Given a set X and a family $\mathcal{K}$ of subsets of X, recall that $\langle X, \mathcal{K}\rangle$ and $\langle X, \tau_{\mathcal{K}}\rangle$ denote a topological space where $\tau_{\mathcal{K}}$ is the topology generated by the subbase $\mathcal{K}$.

A semilattice $\langle L, \wedge, 1\rangle$ is said to be *distributive* ([15]) if for all $a, b_0, b_1 \in L$ such that $b_0 \wedge b_1 \leq a$, there exist $a_0, a_1 \in L$ such that $b_0 \leq a_0$, $b_1 \leq a_1$ and $a = a_0 \wedge a_1$. If $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice, then L is distributive (as lattice) if and only if $\langle L, \wedge, 1\rangle$ is distributive (as semilattice). A filter F of a semilattice L is said to be *prime* if for all $F_1, F_2 \in \mathrm{Fi}(L)$, $F_1 \cap F_2 \subseteq F$ implies $F_1 \subseteq F$ or $F_2 \subseteq F$. It is clear that every prime filter is an irreducible filter. The primer filters of a distributive semilattice (lattice) are used to build up the dual topological space of the semilattice (lattice) under the Grätzer's duality (Stone's duality).

Proposition 4.1 ([2])**.** *Let* $\langle L, \wedge, 1\rangle$ *be a semilattice. The following are equivalent.*

(1) *L is distributive.*
(2) *The lattice* $\mathrm{Fi}(L)$ *is distributive.*
(3) *The irreducible filters of L coincide with the prime filters of L.*

Let L be a distributive semilattice. Recall that $\mathrm{X}(L)$ is the collection of all irreducible filters of L. By the previous proposition, $\mathrm{X}(L)$ is also the collection of all prime filters of L. Recall also that $\sigma(a)^c = \{P \in \mathrm{X}(L) : a \notin P\}$, for every $a \in L$, and $\mathcal{K}_L = \{\sigma(a)^c : a \in L\}$. Thus $\langle \mathrm{X}(L), \mathcal{K}_L\rangle$ is the dual S-space of L.

From the distributivity of L can be proved that $\mathcal{K}_L$ coincide with the collection of all compact open subsets, it is a base for the topology $\tau_{\mathcal{K}_L}$ and the space $\langle \mathrm{X}(L), \tau_{\mathcal{K}_L} \rangle$ is sober, see [2] and [4]. Moreover, if L is a bounded distributive lattice, then $\mathcal{K}_L$ is closed under finite intersections.

Definition 4.2 ([4])**.** A topological space $\langle X, \tau \rangle$ is called a *DS-space* if:

(DS1) The set of all compact open subsets $\mathcal{KO}(X)$ of X is a base for τ.
(DS2) The space $\langle X, \tau \rangle$ is sober.

Definition 4.3 ([18, pp. 43])**.** A topological space $\langle X, \tau \rangle$ is called *spectral* if:

(Sp1) The set of all compact open subsets $\mathcal{KO}(X)$ of X is a base for τ that is closed under finite intersections and $X \in \mathcal{KO}(X)$.
(Sp2) The space $\langle X, \tau \rangle$ is sober.

From the previous observations we have the following.

Corollary 4.4. *If L is a distributive semilattice, then the S-space $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ is the dual DS-space of L. If L is a bounded distributive lattice, then the S-space $\langle \mathrm{X}(L), \mathcal{K}_L \rangle$ is the dual spectral space of L.*

Remark 4.5. It is straightforward that a topological space $\langle X, \tau \rangle$ is spectral if and only if it is a DS-space and $\mathcal{KO}(X)$ is closed under finite intersections.

In order to prove the main result of this section we need the following. Let $\langle X, \mathcal{K} \rangle$ be a topological space satisfying condition (S2) (see page 5). Recall that $\langle \mathcal{C}_{\mathcal{K}}(X), \cap, \veebar \rangle$ is a lattice. Let $Y \in \mathcal{C}_{\mathcal{K}}(X)$ (a subbasic closed subset of X). We shall say that Y is $\mathcal{K}$-*irreducible* if for all $Y_1, Y_2 \in \mathcal{C}_{\mathcal{K}}(X)$, $Y = Y_1 \veebar Y_2$ implies that $Y = Y_1$ or $Y = Y_2$. By Proposition 3.2 the following lemma is clear.

Lemma 4.6. *Let $\langle X, \mathcal{K} \rangle$ be a topological space satisfying condition* (S2)*. If $Y \in \mathcal{C}_{\mathcal{K}}(X)$, then Y is $\mathcal{K}$-irreducible if and only if the filter F_Y of* $\mathrm{S}(X)$ *is irreducible.*

Proposition 4.7. *Let $\langle X, \mathcal{K} \rangle$ be a topological space satisfying conditions* (S1)–(S3)*. Then, the following are equivalent.*

(1) *$\langle X, \mathcal{K} \rangle$ satisfies condition* (S4)*.*
(2) *For every $Y \in \mathcal{C}_{\mathcal{K}}(X)$, if Y is $\mathcal{K}$-irreducible, then there exists $x \in X$ such that* $\mathrm{cl}(x) = Y$*.*

Proof. (1) $\Rightarrow$ (2) Let $Y \in \mathcal{C}_{\mathcal{K}}(X)$ be $\mathcal{K}$-irreducible. Thus F_Y is an irreducible filter of $\mathrm{S}(X)$. Then, by Proposition 3.8, there is $x \in X$ such that $F_Y =$

$H_X(x) = \{A \in \mathrm{S}(X) : x \in A\}$. Hence $Y = \bigcap F_Y = \bigcap H_X(x) = \mathrm{cl}_{\mathcal{K}}(x) = \mathrm{cl}(x)$.

(2) ⇒ (1) Let us prove that the map $H_X \colon X \to \mathrm{X}(\mathrm{S}(X))$ is onto. Let $F \in \mathrm{X}(\mathrm{S}(X))$. By Proposition 3.2, there is $Y \in \mathcal{C}_{\mathcal{K}}(X)$ such that $F = F_Y$. Thus, by Lemma 4.6, Y is $\mathcal{K}$-irreducible. Then, there is $x \in X$ such that $\mathrm{cl}_{\mathcal{K}}(x) = \mathrm{cl}(x) = Y$. Hence $H_X(x) = F_Y = F$. We have proved that H_X is onto. From Proposition 3.8, $\langle X, \mathcal{K}\rangle$ satisfies condition (S4). □ □

Notice that the previous proposition shows that condition (S4) of Definition 3.5 of S-space generalises sobriety.

Theorem 4.8. *Let $\langle X, \mathcal{K}\rangle$ be a topological space. The following are equivalent.*

(1) *$\langle X, \tau_{\mathcal{K}}\rangle$ is a DS-space and $\mathcal{K} = \mathcal{KO}(X)$.*
(2) *$\langle X, \mathcal{K}\rangle$ is a S-space such that $\mathcal{K}$ is a base for $\tau_{\mathcal{K}}$.*

Proof. (1) ⇒ (2) Assume that $\langle X, \tau_{\mathcal{K}}\rangle$ is a DS-space and $\mathcal{K} = \mathcal{KO}(X)$. It is clear that the space $\langle X, \mathcal{K}\rangle$ satisfies conditions (S1)–(S3). Let us show that condition (2) of the previous proposition holds. Since $\mathcal{K} = \mathcal{KO}(X)$ and $\mathcal{KO}(X)$ is a base for the topology $\tau_{\mathcal{K}}$, it follows that $\mathcal{C}_{\mathcal{K}}(X) = \mathcal{C}(X)$. Moreover, for all $Y_1, Y_2 \in \mathcal{C}_{\mathcal{K}}(X)$, we have $Y_1 \veebar Y_2 = Y_1 \cup Y_2$. Thus, $Y \in \mathcal{C}_{\mathcal{K}}(X)$ is $\mathcal{K}$-irreducible if and only if Y is irreducible (as a closed subset). Then, since $\langle X, \tau_{\mathcal{K}}\rangle$ is sober, it follows that for every $\mathcal{K}$-irreducible $Y \in \mathcal{C}_{\mathcal{K}}(X)$ there is $x \in X$ such that $\mathrm{cl}_{\mathcal{K}}(x) = \mathrm{cl}(x) = Y$. Hence, by Proposition 4.7, the space $\langle X, \mathcal{K}\rangle$ satisfies condition (S4).

(2) ⇒ (1) Now we assume that $\langle X, \mathcal{K}\rangle$ is a S-space and $\mathcal{K}$ is a base for $\tau_{\mathcal{K}}$. Since $\mathcal{K}$ is a base for the topology $\tau_{\mathcal{K}}$ of compact open subsets that is closed under finite unions, it follows that $\mathcal{K} = \mathcal{KO}(X)$. Then, $\mathcal{C}_{\mathcal{K}}(X) = \mathcal{C}(X)$. Thus, $Y \in \mathcal{C}(X)$ is irreducible (as a closed subset) if and only if Y is $\mathcal{K}$-irreducible. Hence, condition (2) of Proposition 4.7 implies that the space $\langle X, \tau_{\mathcal{K}}\rangle$ is sober. Therefore, $\langle X, \tau_{\mathcal{K}}\rangle$ is a DS-space. □ □

Corollary 4.9. *A topological space $\langle X, \tau_{\mathcal{K}}\rangle$ is spectral if and only if $\langle X, \mathcal{K}\rangle$ is a S-space and $\mathcal{K}$ is a base for $\tau_{\mathcal{K}}$ that is closed under finite intersections.*

5. The representation by Moshier and Jipsen

In [20] the authors develop a topological duality for semilattices and bounded lattices. In order to obtain the dual space of a semilattice L, they use the collection of all filters of L instead of only the irreducible ones.

In this section, we will establish directly the equivalence between the spaces given in [20] and the S-spaces and L-spaces. We begin presenting the representation for semilattices and lattices developed in [20].

Let $\langle X, \tau \rangle$ be a T_0-space. Recall that $\sqsubseteq$ denotes the specialization order of X. A nonempty subset $F \subseteq X$ is said to be a *filter* of X if F is an upset with respect to $\sqsubseteq$ and, for all $x, y \in F$, there exists $z \in F$ such that $z \sqsubseteq x, y$.

Let us denote by $\mathrm{KOF}(X)$ the collection of all compact open filters of X.

Lemma 5.1 ([20])**.** *Let $\langle X, \tau \rangle$ be a topological space. The compact filters of X are exactly the principal upsets $[x) = \{y \in X : x \sqsubseteq y\}$ of X.*

An element a of a topological space X is called *finite* is $[a) = \{x \in X : a \sqsubseteq x\}$ is open. Thus, from the above lemma, it is clear that there is an order-reversing bijection between $\mathrm{KOF}(X)$ and the set of all finite elements. Thus, for every $U \in \mathrm{KOF}(X)$, there is a finite element $a \in X$ such that $U = [a) = \{x \in X : a \sqsubseteq x\}$.

Definition 5.2. A topological space $\langle X, \tau \rangle$ is said to be an *HMS space* if:

(H1) $\mathrm{KOF}(X)$ forms a base that is closed under finite intersection and $X \in \mathrm{KOF}(X)$;
(H2) X is a sober space.

Let $\langle L, \wedge, 1 \rangle$ be a semilattice. For every $a \in L$, let $\varphi(a) = \{F \in \mathrm{Fi}(L) : a \in F\}$. Then, it is straightforward to check directly that $\mathcal{B}_L = \{\varphi(a) : a \in L\}$ is a base for a topology τ_L on $\mathrm{Fi}(L)$. The space $\langle \mathrm{Fi}(L), \tau_L \rangle$ will be the dual of the semilattice L. Now given an HMS space X, it is clear that $\langle \mathrm{KOF}(X), \cap, X \rangle$ is a semilattice, and it will be the dual of X. Consider the maps $\varphi \colon L \to \mathrm{KOF}(\mathrm{Fi}(L))$, and $\theta \colon X \to \mathrm{Fi}(\mathrm{KOF}(L))$ defined by $\theta(x) = \{U \in \mathrm{KOF}(X) : x \in U\}$.

Theorem 5.3 ([20])**.** *Let $\langle L, \wedge, 1 \rangle$ be a semilattice, and let $\langle X, \tau \rangle$ be an HMS space. Then, $\langle \mathrm{Fi}(L), \tau_L \rangle$ is an HMS space and $\langle \mathrm{KOF}(X), \cap, X \rangle$ is a semilattice. Moreover, $\varphi \colon L \to \mathrm{KOF}(\mathrm{Fi}(L))$ is an isomorphism, and $\theta \colon X \to \mathrm{Fi}(\mathrm{KOF}(L))$ is a homeomorphism.*

5.1. **From S-spaces to HMS spaces.** Let $\langle X, \mathcal{K} \rangle$ be an S-space. Let $\mathcal{H}_{\mathcal{K}}(X) = \{\bigcup \mathcal{U} : \mathcal{U} \subseteq \mathcal{K}\}$. That is, $\mathcal{H}_{\mathcal{K}}(X)$ is the collection of all those subsets of X that are arbitrary unions of members of $\mathcal{K}$. Notice that $\langle \mathcal{H}_{\mathcal{K}}(X), \bigwedge, \bigcup \rangle$ is a complete lattice, where $\bigwedge H_i = \bigcup \{U \in \mathcal{K} : U \subseteq \bigcap H_i\}$, for all $H_i \in \mathcal{H}_{\mathcal{K}}(X)$.

For every $U \in \mathcal{K}$, we define $\Psi_U = \{H \in \mathcal{H}_{\mathcal{K}}(X) : U \subseteq H\}$. We consider the topology τ_{HMS} on $\mathcal{H}_{\mathcal{K}}(X)$ generated by the family $\{\Psi_U : U \in \mathcal{K}\}$. We will prove in several steps that $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS} \rangle$ is an HMS space.

Proposition 5.4. *$\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS} \rangle$ is a T_0-space.*

Proof. Let $H_1, H_2 \in \mathcal{H}_{\mathcal{K}}(X)$ be such that $H_1 \neq H_2$. Assume that there is $x \in X$ such that $x \in H_1$ and $x \notin H_2$. Since $H_1 \in \mathcal{H}_{\mathcal{K}}(X)$, it follows that there is $U \in \mathcal{K}$ such that $x \in U \subseteq H_1$. Then, $H_1 \in \Psi_U$ and $H_2 \notin \Psi_U$. Hence $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS} \rangle$ is a T_0-space. □ □

Proposition 5.5. *The family $\{\Psi_U : U \in \mathcal{K}\}$ is a base for $\mathcal{H}_{\mathcal{K}}(X)$ and closed under finite intersection.*

Proof. Let $U_1, U_2 \in \mathcal{K}$. Since $\mathcal{K}$ is closed under finite unions, we have $U_1 \cup U_2 \in \mathcal{K}$. Then, $\Psi_{U_1 \cup U_2} = \Psi_{U_1} \cap \Psi_{U_2}$. □ □

Since $\{\Psi_U : U \in \mathcal{K}\}$ is a base for $\mathcal{H}_{\mathcal{K}}(X)$, it follows that the specialization order of the space $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS} \rangle$ coincide with the set-theoretical inclusion. That is, for all $H, J \in \mathcal{H}_{\mathcal{K}}(X)$, $H \sqsubseteq J \iff H \subseteq J$. Hence, since $\mathcal{K} \subseteq \mathcal{H}_{\mathcal{K}}(X)$, we have that for every $U \in \mathcal{K}$, $\Psi_U = [U) = \{H \in \mathcal{H}_{\mathcal{K}}(X) : U \sqsubseteq H\} = \{H \in \mathcal{H}_{\mathcal{K}}(X) : U \subseteq H\}$. Then, the following is clear.

Proposition 5.6. *For all $U \in \mathcal{K}$, Ψ_U is a compact filter of $\mathcal{H}_{\mathcal{K}}(X)$.*

We have proved that $\{\Psi_U : U \in \mathcal{K}\} \subseteq \mathrm{KOF}(\mathcal{H}_{\mathcal{K}}(X))$. Now we prove that the above inclusion is actually an equality.

Proposition 5.7. $\mathrm{KOF}(\mathcal{H}_{\mathcal{K}}(X)) = \{\Psi_U : U \in \mathcal{K}\}$.

Proof. Let $\mathcal{U} \in \mathrm{KOF}(\mathcal{H}_{\mathcal{K}}(X))$. Since $\mathcal{U}$ is a compact filter of $\mathcal{H}_{\mathcal{K}}(X)$, it follows by Lemma 5.1 that there is $H \in \mathcal{H}_{\mathcal{K}}(X)$ such that $\mathcal{U} = [H) = \{J \in \mathcal{H}_{\mathcal{K}}(X) : H \subseteq J\}$. Since $H \in \mathcal{U}$ and $\mathcal{U}$ is open, there is $U \in \mathcal{K}$ such that $H \in \Psi_U \subseteq \mathcal{U}$. It follows that $H = U$. Hence $\mathcal{U} = [U) = \Psi_U$. □ □

Proposition 5.8. *The space $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS} \rangle$ is sober.*

Proof. Let $\mathcal{F}$ be a completely prime filter of the lattice of open subsets of the space $\mathcal{H}_{\mathcal{K}}(X)$. We need to prove that there exists an element $H \in \mathcal{H}_{\mathcal{K}}(X)$ such that $\mathcal{F} = \mathrm{N}(H) = \{\mathcal{U} \in \tau_{HMS} : H \in \mathcal{U}\}$. Let $D := \{U \in \mathcal{K} : \Psi_U \in \mathcal{F}\}$. Let $H = \bigcup D \in \mathcal{H}_{\mathcal{K}}(X)$.

Let $\mathcal{U} \in \mathrm{N}(H)$. Since $\mathcal{U}$ is an open of $\mathcal{H}_{\mathcal{K}}(X)$, there is $U \in \mathcal{K}$ such that $H \in \Psi_U \subseteq \mathcal{U}$. So $U \subseteq H = \bigcup D$. Since U is a compact open subset of

X, it follows that there are $U_1, \ldots, U_n \in D$ such that $U \subseteq U_1 \cup \cdots \cup U_n$. Since $\mathcal{K}$ is closed under finite unions and $\mathcal{F}$ is a filter of the lattice of open subsets of $\mathcal{H}_{\mathcal{K}}(X)$, we obtain that $U_1 \cup \cdots \cup U_n \in D$. Let $V := U_1 \cup \cdots \cup U_n$. Thus $U \subseteq V$ and $\Psi_V \in \mathcal{F}$. So $\Psi_V \subseteq \Psi_U \subseteq \mathcal{U}$. Hence, since $\mathcal{F}$ is an upset of $\mathcal{H}_{\mathcal{K}}(X)$ with respect to the specialization order $\sqsubseteq = \subseteq$, we have $\mathcal{U} \in \mathcal{F}$. Therefore, $\mathrm{N}(H) \subseteq \mathcal{F}$.

Let now $\mathcal{U} \in \mathcal{F}$. Since $\mathcal{U}$ is open, it follows that $\mathcal{U} = \bigcup_{i\in I} \Psi_{U_i}$ for some $U_i \in \mathcal{K}$. Then, given that $\mathcal{F}$ is completely prime, $\Psi_{U_i} \in \mathcal{F}$ for some $i \in I$. Thus $U_i \in D$. It follows that $U_i \subseteq \bigcup D = H$. Thus $H \in \Psi_{U_i} \subseteq \mathcal{U}$, and then we have $H \in \mathcal{U}$. Hence $\mathcal{U} \in \mathrm{N}(H)$. Therefore $\mathcal{F} \subseteq \mathrm{N}(H)$.

Hence, $\mathrm{N}(H) = \mathcal{F}$. Therefore, the space $\mathcal{H}_{\mathcal{K}}(X)$ is sober. □ □

Thus, putting all these results together, we have proved the following.

Theorem 5.9. *For every S-space $\langle X, \mathcal{K}\rangle$, the space $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS}\rangle$ is an HMS space.*

Notice that to prove the previous theorem is enough that the space $\langle X, \mathcal{K}\rangle$ satisfies only condition (S2). We will use in the next subsections that the space $\langle X, \mathcal{K}\rangle$ is an S-space.

Remark 5.10. Let $\langle X, \mathcal{K}\rangle$ be an S-space. Recall that the dual semilattice of $\langle X, \mathcal{K}\rangle$ is $\mathrm{S}(X) = \{U^c : U \in \mathcal{K}\}$. The dual semilattice of the HMS space $\langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS}\rangle$ is $\mathrm{KOF}(\mathcal{H}_{\mathcal{K}}(X)) = \{\Psi_U : U \in \mathcal{K}\}$. Now it is straightforward to check that the semilattices $\mathrm{S}(X)$ and $\mathrm{KOF}(\mathcal{H}_{\mathcal{K}}(X))$ are isomorphic under the map $U^c \mapsto \Psi_U$.

5.2. **From HMS spaces to S-spaces.** Let $\langle X, \tau\rangle$ be an HMS space. By [20, Lem. 3.1], we know that $\langle X, \sqsubseteq\rangle$ is a complete lattice. We denote by $\sqcap$ and $\sqcup$ the meet and join of X, respectively. Let us denote by $\mathcal{M}(X)$ the set of all meet-irreducible elements of the lattice $\langle X, \sqcap, \sqcup\rangle$.

By [20, Theo. 3.7], the map $\theta\colon X \to \mathrm{Fi}(\mathrm{KOF}(X))$ defined as $\theta(x) = \{U \in \mathrm{KOF}(X) : x \in U\}$ is a homeomorphism. Then, it is clear that $x \sqsubseteq y \iff \theta(x) \subseteq \theta(y)$. Thus, θ is a lattice isomorphism from $\langle X, \sqsubseteq\rangle$ onto $\langle \mathrm{Fi}(\mathrm{KOF}(X)), \subseteq\rangle$.

Lemma 5.11. *Let $\langle X, \tau\rangle$ be an HMS space. Then, for every $x \in X$ that is not the top, we have $x = \sqcap\{y \in \mathcal{M}(X) : x \sqsubseteq y\}$.*

Proof. It follows from Corollary 2.2. □ □

Lemma 5.12. *Let $\langle X, \tau\rangle$ be an HMS space. Then, for every $x \in X$ and $U \in \mathrm{KOF}(X)$, $x \in U \iff [x) \cap \mathcal{M}(X) \subseteq U$.*

Proof. Recall that the opens are upsets regarding $\sqsubseteq$. So, $x \in U$ implies $[x) \cap \mathcal{M}(X) \subseteq U$. Now suppose that $[x) \cap \mathcal{M}(X) \subseteq U$. Since $U \in \mathrm{KOF}(X)$, there is $a \in X$ such that $U = [a)$. By Lemma 5.11, $a \sqsubseteq \sqcap\{y \in \mathcal{M}(X) : x \sqsubseteq y\} = x$. Thus $x \in U$. □ □

Proposition 5.13. *Let $\langle X, \tau\rangle$ be an HMS space. For all $U, V \in \mathrm{KOF}(X)$, we have*

$$U \cap \mathcal{M}(X) \subseteq V \cap \mathcal{M}(X) \iff U \subseteq V.$$

Proof. Let $U, V \in \mathrm{KOF}(X)$. The implication from right to left is trivial. Assume that $U \cap \mathcal{M}(X) \subseteq V \cap \mathcal{M}(X)$. Let $x \in U$. By Lemma 5.11, we have $x = \sqcap\{y \in \mathcal{M}(X) : x \sqsubseteq y\}$. Since U is an upset with respect to $\sqsubseteq$, it follows that $\{y \in \mathcal{M}(X) : x \sqsubseteq y\} \subseteq U$. Thus $\{y \in \mathcal{M}(X) : x \sqsubseteq y\} \subseteq V$. Given that $V \in \mathrm{KOF}(X)$, there is $a \in X$ such that $V = [a)$. Then, we get $x = \sqcap\{y \in \mathcal{M}(X) : x \sqsubseteq y\} \in [a) = V$. Hence $U \subseteq V$. □ □

Lemma 5.14. *Let $\langle X, \tau\rangle$ be an HMS space. Then, every finite element $a \in X$ is a compact element of the lattice $\langle X, \sqsubseteq\rangle$.*

Proof. Let $a \in X$ be finite. Suppose that $a \sqsubseteq \bigsqcup_{i \in I} x_i$. Since θ is a lattice isomorphism from X onto $\mathrm{Fi}(\mathrm{KOF}(X))$, it follows that

$$\theta(a) \subseteq \theta\left(\bigsqcup x_i\right) = \bigvee \theta(x_i) = \mathrm{Fig}_{KOF(X)}\left(\bigcup \theta(x_i)\right).$$

As a is finite, we have $[a) \in \theta(a)$. Thus $[a) \in \mathrm{Fig}_{\mathrm{KOF}(X)}(\bigcup \theta(x_i))$. Then, there are $i_1, \dots, i_n \in I$ such that $[x_{i_1}) \cap \cdots \cap [x_{i_n}) \subseteq [a)$. Hence $a \sqsubseteq x_{i_1} \sqcup \cdots \sqcup x_{i_n}$. Therefore a is a compact element of the lattice X. □ □

Definition 5.15. Let $\langle X, \tau\rangle$ be an HMS space. We define a topology on $\mathcal{M}(X)$ generated by the family $\mathcal{K}_{\mathcal{M}(X)} = \{U^c \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X)\}$.

Notice that U^c means $X \setminus U$. Thus $U^c \cap \mathcal{M}(X) = \mathcal{M}(X) \setminus U$. From now on, $\langle X, \tau\rangle$ will be an HMS space and $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)}\rangle$ will be the topological space defined as above.

Since the space $\langle X, \tau\rangle$ is T_0, it follows that $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)}\rangle$ is also a T_0-space.

Proposition 5.16. *The family $\mathcal{K}_{\mathcal{M}(X)}$ is a subbase of compact open subsets, it is closed under finite unions and $\emptyset \in \mathcal{K}_{\mathcal{M}(X)}$.*

Proof. By definition, it is obvious that $\mathcal{K}_{\mathcal{M}(X)}$ is a subbase for the space $\mathcal{M}(X)$. Since $X \in \mathrm{KOF}(X)$ and $\mathrm{KOF}(X)$ is closed under finite intersection, it follows that $\emptyset \in \mathcal{K}_{\mathcal{M}(X)}$ and $\mathcal{K}_{\mathcal{M}(X)}$ is closed under finite unions. Let $U \in \mathrm{KOF}(X)$. Let us prove that $U^c \cap \mathcal{M}(X)$ is a compact subset of the space $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)} \rangle$. Suppose that

$$U^c \cap \mathcal{M}(X) \subseteq \bigcup_{i \in I} U_i^c \cap \mathcal{M}(X).$$

Then, $\bigcap_{i \in I} U_i \cap \mathcal{M}(X) \subseteq U \cap \mathcal{M}(X)$. As $U, U_i \in \mathrm{KOF}(X)$, for all $i \in I$, then $U = [a)$ and $U_i = [a_i)$, for some finite elements $a, a_i \in X$, $\forall i \in I$. Thus $\bigcap_{i \in I} [a_i) \cap \mathcal{M}(X) \subseteq [a) \cap \mathcal{M}(X)$. By Lemma 5.11, it follows that $a \sqsubseteq \bigsqcup_{i \in I} a_i$. Since a is a finite element, we have by Lemma 5.14 that there are $i_1, \ldots, i_n \in I$ such that $a \sqsubseteq a_{i_1} \sqcup \cdots \sqcup a_{i_n}$. Thus $[a_{i_1}) \cap \cdots \cap [a_{i_n}) \subseteq [a)$. That is, $U_{i_1} \cap \cdots \cap U_{i_n} \subseteq U$. Then

$$U^c \cap \mathcal{M}(X) \subseteq (U_{i_1}^c \cap \mathcal{M}(X)) \cup \cdots \cup (U_{i_n}^c \cap \mathcal{M}(X)).$$

Hence $U^c \cap \mathcal{M}(X)$ is a compact subset of the space $\mathcal{M}(X)$. □ □

Proposition 5.17. *The space $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)} \rangle$ satisfies condition* (S3) *of Definition 3.5.*

Proof. Given that $\theta\colon X \to \mathrm{Fi}(\mathrm{KOF}(X))$ is a lattice isomorphism, notice that for every $y \in \mathcal{M}(X)$, $\theta(y)$ is an irreducible filter of the semilattice $\mathrm{KOF}(X)$. Let now $U, V \in \mathrm{KOF}(X)$ and $y \in (U^c \cap \mathcal{M}(X)) \cap (V^c \cap \mathcal{M}(X))$. Thus, $U, V \notin \theta(y)$. By Lemma 2.3, there are $W, D \in \mathrm{KOF}(X)$ such that $D \notin \theta(y)$, $W \in \theta(y)$ and, $U \cap W \subseteq D$ and $V \cap W \subseteq D$. Thus

$$D^c \cap \mathcal{M}(X) \subseteq [(U^c \cap \mathcal{M}(X)) \cap (V^c \cap \mathcal{M}(X))] \cup (W^c \cap \mathcal{M}(X))$$

and, $y \in D^c \cap \mathcal{M}(X)$ and $y \notin W^c \cap \mathcal{M}(X)$. □ □

Remark 5.18. From the above results, we know that the space $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)} \rangle$ satisfies conditions (S1)-(S3). Since $\mathcal{K}_{\mathcal{M}(X)} = \{(X \setminus U) \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X)\}$, it is clear that

$$\begin{aligned} \mathrm{S}(\mathcal{M}(X)) &= \{\mathcal{M}(X) \setminus [(X \setminus U) \cap \mathcal{M}(X)] : U \in \mathrm{KOF}(X)\} \\ &= \{U \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X)\}. \end{aligned}$$

Hence, by Proposition 5.13, it follows that the two semilattices $\langle \mathrm{KOF}(X), \cap \rangle$ and $\langle \mathrm{S}(\mathcal{M}(X)), \cap \rangle$ are isomorphic.

Proposition 5.19. *The space $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)} \rangle$ satisfies condition* (S4)*.*

Proof. In order to prove that the space $\mathcal{M}(X)$ satisfies condition (S4), we are going to appeal to Proposition 3.8. Recall that the map $H_{\mathcal{M}(X)}\colon \mathcal{M}(X) \to \mathrm{X}\left(\mathrm{S}\left(\mathcal{M}(X)\right)\right)$ is defined by $H_{\mathcal{M}(X)}(y) = \{A \in \mathrm{S}\left(\mathcal{M}(X)\right) : y \in A\}$. We need to prove that $H_{\mathcal{M}(X)}$ is onto. Let $P \in \mathrm{X}(\mathrm{S}(\mathcal{M}(X)))$. Consider $\widehat{P} = \{U \in \mathrm{KOF}(X) : U \cap \mathcal{M}(X) \in P\}$. Since P is an irreducible filter of the semilattice $\mathrm{S}(\mathcal{M}(X))$, it follows that $\widehat{P}$ is an irreducible filter of the semilattice $\mathrm{KOF}(X)$. Given that $\theta\colon X \to \mathrm{Fi}(\mathrm{KOF}(X))$ is a lattice isomorphism, we get that there is $y \in \mathcal{M}(X)$ such that $\theta(y) = \widehat{P}$. Thus, it follows that $H_{\mathcal{M}(X)}(y) = P$. □ □

From the above results we can conclude the following.

Theorem 5.20. *For every HMS space $\langle X, \tau\rangle$, $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)}\rangle$ is an S-space.*

5.3. **The equivalence between S-spaces and HMS spaces.** Let $\langle X, \mathcal{K}\rangle$ be an S-space. Recall that its dual HMS space is $\langle \mathcal{H}(X), \tau_{HMS}\rangle$ where $\mathcal{H}(X) = \{\bigcup \mathcal{U} : \mathcal{U} \subseteq \mathcal{K}\}$, $\mathrm{KOF}(\mathcal{H}(X)) = \{\Psi_U : U \in \mathcal{K}\}$ and $\Psi_U = \{H \in \mathcal{H}(X) : U \subseteq H\}$. Now the dual S-space of $\mathcal{H}(X)$ is $\langle \mathcal{M}(\mathcal{H}(X)), \mathcal{K}_{\mathcal{M}(\mathcal{H}(X))}\rangle$ where $\mathcal{K}_{\mathcal{M}(\mathcal{H}(X))} = \{\Psi_U^c \cap \mathcal{M}(\mathcal{H}(X)) : U \in \mathcal{K}\}$.

Proposition 5.21. *Let $\langle X, \mathcal{K}\rangle$ be an S-space. Then, the function $\Gamma_X\colon X \to \mathcal{M}(\mathcal{H}(X))$ defined by $\Gamma_X(x) = \bigcup\{U \in \mathcal{K} : x \notin U\}$ is a homeomorphism such that $\mathcal{K}_{\mathcal{M}(\mathcal{H}(X))} = \{\Gamma_X[U] : U \in \mathcal{K}\}$.*

Proof. Let us take into account the following. (i) $H_X\colon X \to \mathrm{X}(\mathrm{S}(X))$ given by $H_X(x) = \{A \in \mathrm{S}(X) : x \in A\}$ is a homeomorphism. (ii) The map $\bigcap\colon \mathrm{Fi}(\mathrm{S}(X)) \to \mathcal{C}_{\mathcal{K}}(X)$ given by $Y_F = \bigcap F \in \mathcal{C}_{\mathcal{K}}(X)$ is a dual isomorphism (Prop. 3.2). (iii) The map $(.)^c\colon \mathcal{C}_{\mathcal{K}}(X) \to \mathcal{H}(X)$ given by $Y^c = \bigcup\{U \in \mathcal{K} : Y \subseteq U^c\}$ is a dual isomorphism.

From (ii) and (iii), we obtain that $(.)^c \circ \bigcap\colon \mathrm{X}(\mathrm{S}(X)) \to \mathcal{M}(\mathcal{H}(X))$ is a bijective function. Then, it follows by (i) that the map $(.)^c \circ \bigcap \circ H_X\colon X \to \mathcal{M}(\mathcal{H}(X))$ is a bijection, and $\Gamma_X(x) = ((.)^c \circ \bigcap \circ H_X)(x)$, for all $x \in X$. Moreover, for every $U \in \mathcal{K}$, we have $\Gamma_X^{-1}[\Psi_U^c \cap \mathcal{M}(\mathcal{H}(X))] = U$. The result follows. □ □

Corollary 5.22. *Let $\langle X, \mathcal{K}\rangle$ be an S-space. Then, the relation $\eta_X \subseteq X \times \mathcal{M}(\mathcal{H}(X))$ defined by $(x, \Gamma_X(x)) \in \eta_X \iff \Gamma_X(x) \subseteq \Gamma_X(y)$ is an isomorphism of the category* **S**.

Let $\langle X, \tau\rangle$ be an HMS space. Recall that its dual S-space is $\langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)}\rangle$ where $\mathcal{M}(X)$ is the set of meet-irreducible elements of $\langle X, \sqcap, \sqcup\rangle$ and $\mathcal{K}_{\mathcal{M}(X)} =$

$\{U^c \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X)\}$. Now, the dual HMS space of $\mathcal{M}(X)$ is $\langle \mathcal{H}(\mathcal{M}(X)), \tau_{HMS}\rangle$ where $\mathcal{H}(\mathcal{M}(X))$ is the collection of all arbitrary unions of $\mathcal{K}_{\mathcal{M}(X)}$, and thus $\mathrm{KOF}(\mathcal{H}(\mathcal{M}(X))) = \{\Psi_{U^c \cap \mathcal{M}(X)} : U \in \mathrm{KOF}(X)\}$.

Proposition 5.23. *Let $\langle X, \tau\rangle$ be an HMS space. Then, the map $\Delta_X : X \to \mathcal{H}(\mathcal{M}(X))$ defined by*

$$\Delta_X(x) = \bigcup\{U^c \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X) \text{ and } x \in U\}$$

is a homeomorphism such that $\Delta_X[\mathrm{KOF}(X)] = \mathrm{KOF}(\mathcal{H}(\mathcal{M}(X))$.

Proof. Let $x_1, x_2 \in X$. Assume that $\Delta_X(x_1) = \Delta_X(x_2)$. Suppose that $x_1 \not\sqsubseteq x_2$. So, by Lemma 5.11, there is $y \in \mathcal{M}(X)$ such that $x_2 \sqsubseteq y$ and $x_1 \not\sqsubseteq y$. Thus $y \in \Delta_X(x_1)$ and $y \notin \Delta_X(x_2)$, a contradiction. Hence Δ_X is injective. We prove that Δ_X is onto. Let $H \in \mathcal{H}(\mathcal{M}(X))$. Thus $H = \bigcup\{U^c \cap \mathcal{M}(X) : U \in \mathrm{KOF}(X) \text{ and } U^c \cap \mathcal{M}(X) \subseteq H\}$. Let $F_H = \{U \in \mathrm{KOF}(X) : U^c \cap \mathcal{M}(X) \subseteq H\}$. It follows that $F_H \in \mathrm{Fi}(\mathrm{KOF}(X))$. Then, by Theorem 5.3, there is $x \in X$ such that $F_H = \theta(x) = \{U \in \mathrm{KOF}(X) : x \in U\}$. Hence $\Delta_X(x) = H$. Thus Δ_X is onto. Now, it is not hard to show that for every $U \in \mathrm{KOF}(X)$, $U = \Delta_X^{-1}[\Psi_{U^c \cap \mathcal{M}(X)}]$. Thus, since the compact open filters form bases for the spaces X and $\mathcal{H}(\mathcal{M}(X))$, we obtain that Δ_X is continuous and open. This completes the proof. □ □

We now turn our attention to morphisms. Recall that $\mathbb{SS}$ denotes the category of S-spaces and meet-relations.

Let X_1 and X_2 be HMS spaces. A function $f : X_1 \to X_2$ is said to be *F-continuous* ([20]) if $\{f^{-1}[V] : V \in \mathrm{KOF}(X_2)\} \subseteq \mathrm{KOF}(X_1)$. Let us denote by $\mathbb{HMS}$ the category of HMS spaces and F-continuous functions.

Let X_1 and X_2 be HMS spaces and $f : X_1 \to X_2$ an F-continuous map. We define the relation $R_f \subseteq \mathcal{M}(X_1) \times \mathcal{M}(X_2)$ as follows:

$$(y_1, y_2) \in R_f \iff f(y_1) \sqsubseteq y_2$$

for all $y_1 \in \mathcal{M}(X_1)$ and $y_2 \in \mathcal{M}(X_2)$. Notice that for every $y \in \mathcal{M}(X_1)$, $R_f(y) = \{y_2 \in \mathcal{M}(X_2) : f(y) \sqsubseteq y_2\} = [f(y)) \cap \mathcal{M}(X_2)$. Thus, by Lemma 5.11, $\sqcap(R_f(y)) = f(y)$.

Proposition 5.24. *R_f is a meet-relation.*

Proof. We need to prove conditions (R1) and (R2) of Definition 3.14. Let $B \in \mathrm{S}(\mathcal{M}(X_2))$. So, there is $V \in \mathrm{KOF}(X_2)$ such that $B = V \cap \mathcal{M}(X_2)$ (see Remark 5.18). Since $V \in \mathrm{KOF}(X_2)$ and f is F-continuous, we have

$f^{-1}[V] \in \mathrm{KOF}(X_1)$, and thus $f^{-1}[V] \cap \mathcal{M}(X_1) \in \mathrm{S}(\mathcal{M}(X_1))$. Let us show that $\Box_{R_f}(V \cap \mathcal{M}(X_2)) = f^{-1}[V] \cap \mathcal{M}(X_1)$. Recall that $\Box_{R_f}(V \cap \mathcal{M}(X_2)) = \{y \in \mathcal{M}(X_1) : R_f(y) \subseteq V \cap \mathcal{M}(X_2)\}$. By Lemma 5.12, we have

$$y \in \Box_{R_f}(V \cap \mathcal{M}(X_2)) \iff R_f(y) \subseteq V \iff f(y) \in V \iff y \in f^{-1}[V] \cap \mathcal{M}(X_1).$$

Hence (R1) holds. Moreover, it is straightforward to show that for every $y \in \mathcal{M}(X_1)$, $R_f(y) = \bigcap\{B \in \mathrm{S}(\mathcal{M}(X_2)) : R_f(y) \subseteq B\}$. Hence (R2) holds. □ □

From the proof of Proposition 5.24, we have that for every F-continuous map $f\colon X_1 \to X_2$, $\Box_{R_f}(V \cap \mathcal{M}(X_2)) = f^{-1}[V] \cap \mathcal{M}(X_1)$, for all $V \in \mathrm{KOF}(X_2)$.

Proposition 5.25. *Let $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_3$ be F-continuous maps. Then $R_{g\circ f} = R_g * R_f$.*

Proof. Since $R_{g\circ f}$ and $R_g * R_f$ are meet-relations, it is enough by Lemma 3.21 to show that $\Box_{R_{g\circ f}} = \Box_{R_g * R_f}$. Moreover, by Proposition 3.16, we have $\Box_{R_g * R_f} = \Box_{R_f} \circ \Box_{R_f}$. Thus, it is enough to show that $\Box_{R_{g\circ f}} = \Box_{R_f} \circ \Box_{R_g}$. Recall $\mathrm{S}(\mathcal{M}(X_3)) = \{W \cap \mathcal{M}(X_3) : W \in \mathrm{KOF}(X_3)\}$. Let $W \in \mathrm{KOF}(X_3)$. Then,

$$\begin{aligned}\Box_{R_f}\left(\Box_{R_g}(W \cap \mathcal{M}(X_3))\right) &= \Box_{R_f}\left(g^{-1}[W] \cap \mathcal{M}(X_2)\right)\\ &= (g\circ f)^{-1}[W] \cap \mathcal{M}(X_1) = \Box_{R_{g\circ f}}(W \cap \mathcal{M}(X_3)). \quad \Box\end{aligned}$$

□

Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be S-spaces and $R \subseteq X_1 \times X_2$ a meet-relation. We define the map $f_R\colon \mathcal{H}(X_1) \to \mathcal{H}(X_2)$ as follows:

$$f_R(H) = \bigcup\{V \in \mathcal{K}_2 : R^{-1}[V] \subseteq H\},$$

for every $H \in \mathcal{H}(X_1)$ and where $R^{-1}[V] = \{x \in X_1 : (\exists y \in V)\,(x, y) \in R\}$.

Remark 5.26. Let $V \in \mathcal{K}_2$. So $V^c \in \mathrm{S}(X_2)$. Since R is a meet-relation, it follows that $\Box_R(V^c) \in \mathrm{S}(X_1)$. Thus $\Box_R(V^c)^c \in \mathcal{K}_1$. Moreover, it is easy to check that $R^{-1}[V] = \Box_R(V^c)^c$. Hence, $R^{-1}[V] \in \mathcal{K}_1$, for all $V \in \mathcal{K}_2$.

Proposition 5.27. *The map $f_R\colon \mathcal{H}(X_1) \to \mathcal{H}(X_2)$ is F-continuous.*

Proof. Recall that $\mathrm{KOF}(\mathcal{H}(X_2)) = \{\Psi_V : V \in \mathcal{K}_2\}$ and $\Psi_V = \{H \in \mathcal{H}(X_2) : V \subseteq H\}$. Let $V \in \mathcal{K}_2$. We know that $R^{-1}[V] \in \mathcal{K}_1$, and thus $\Psi_{R^{-1}[V]} \in \mathrm{KOF}(\mathcal{H}(X_1))$. Let us show that $f_R^{-1}[\Psi_V] = \Psi_{R^{-1}[V]}$. Let $H \in \mathcal{H}(X_1)$.

Notice that $H \in f_R^{-1}[\Psi_V] \iff V \subseteq f_R(H)$. Thus, it is clear that $\Psi_{R^{-1}[V]} \subseteq f_R^{-1}[\Psi_V]$. Suppose now that $H \in f_R^{-1}[\Psi_V]$. So $V \subseteq f_R(H)$. Since V is compact, there are $V_1, \ldots, V_n \in \mathcal{K}_2$ such that $R^{-1}[V_1] \cup \cdots \cup R^{-1}[V_n] \subseteq H$ and $V \subseteq V_1 \cup \cdots \cup V_n$. Then $R^{-1}[V] \subseteq H$. Thus $H \in \Psi_{R^{-1}[V]}$. Hence $f_R^{-1}[\Psi_V] = \Psi_{R^{-1}[V]} \in \mathrm{KOF}(\mathcal{H}(X_1))$. □ □

Proposition 5.28. *Let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be meet-relations. Then, $f_{S*R} = f_S \circ f_R$.*

Proof. Let $H \in \mathcal{H}(X_1)$. By definition, we have $f_{S*R}(H) = \bigcup\{W \in \mathcal{K}_3 : (S*R)^{-1}[W] \subseteq H\}$ and $(f_S \circ f_R)(H) = \{W \in \mathcal{K}_3 : S^{-1}[W] \subseteq f_R(H)\}$. First notice that $S^{-1}[W] \subseteq f_R(H) \iff (S \circ R)^{-1}[W] \subseteq H$. Now by Remark 5.26 and Proposition 3.16, we have on the one hand

$$(S * R)^{-1}[W] = \Box_{S*R}(W^c)^c = (\Box_R \circ \Box_S)(W^c)^c = \Box_R(\Box_S(W^c))^c,$$

and the other hand

$$(S \circ R)^{-1}[W] = R^{-1}[S^{-1}[W]] = \Box_R(S^{-1}[W]^c)^c = \Box_R(\Box_S(W^c))^c.$$

Thus $(S * R)^{-1}[W] = (S \circ R)^{-1}[W]$, for all $W \in \mathcal{K}_3$. Hence $f_{S*R}(H) = (f_S \circ f_R)(H)$, for all $H \in \mathcal{H}(X_1)$. □ □

In order to prove the next proposition we need to note the following. By what we have proved in Proposition 5.27, for every meet-relation $R \subseteq X_1 \times X_2$, $f_R^{-1}[\Psi_V] = \Psi_{R^{-1}[V]}$, for all $V \in \mathcal{K}_2$.

Proposition 5.29. *Let $R \subseteq X_1 \times X_2$ be a meet-relation. Then, $\eta_{X_2} * R = R_{f_R} * \eta_{X_1}$.*

Proof. From Lemma 3.21 and Proposition 3.16, it is enough to show that $\Box_{\eta_{X_1}} \circ \Box_{R_{f_R}} = \Box_R \circ \Box_{\eta_{X_2}}$. Since $R_{f_R} \subseteq \mathcal{M}(\mathcal{H}(X_1)) \times \mathcal{M}(\mathcal{H}(X_2))$, it follows that

$$\Box_{R_{f_R}} \colon \mathrm{S}(\mathcal{M}(\mathcal{H}(X_2))) \to \mathrm{S}(\mathcal{M}(\mathcal{H}(X_1))).$$

Recall that $\mathrm{S}(\mathcal{M}(\mathcal{H}(X_i))) = \{\Psi_U : U \in \mathcal{K}_i\}$.

Let $V \in \mathcal{K}_2$. We need to prove that

$$(\Box_{\eta_{X_1}} \circ \Box_{R_{f_R}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2))) = (\Box_R \circ \Box_{\eta_{X_2}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2))).$$

Let $x \in (\Box_{\eta_{X_1}} \circ \Box_{R_{f_R}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$. Thus

$$\eta_{X_1}(x) \subseteq \Box_{R_{f_R}}(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2))) = f_R^{-1}[\Psi_V] \cap \mathcal{M}(\mathcal{H}(X_1)) = \Psi_{R^{-1}[V]} \cap \mathcal{M}(\mathcal{H}(X_1)).$$

We have to show that $R(x) \subseteq \Box_{\eta_{X_2}}(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$. Let $y \in R(x)$. Now we need to show that $\eta_{X_2}(y) \subseteq \Psi_V \cap \mathcal{M}(\mathcal{H}(X_2))$. Let $z \in X_2$ be such that

$\Gamma_{X_2}(z) \in \eta_{X_2}(y)$. Thus $\Gamma_{X_2}(y) \subseteq \Gamma_{X_2}(z)$. Then $z \sqsubseteq y$. We have to prove that $\Gamma_{X_2}(z) \in \Psi_V$. Notice that $\Gamma_{X_2}(z) \in \Psi_V \iff V \subseteq \Gamma_{X_2}(z) \iff z \notin V$. Now, since $\Gamma_{X_1}(x) \in \eta_{X_1}(x) \subseteq \Psi_{R^{-1}[V]}$, it follows that $R^{-1}[V] \subseteq \Gamma_{X_1}(x)$. Thus $x \notin R^{-1}[V]$. That is $R(x) \cap V = \emptyset$. Since $y \in R(x)$, we have $y \notin V$. Hence, we can conclude that $R(x) \subseteq \Box_{\eta_{X_2}}(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$, which implies that $x \in (\Box_R \circ \Box_{\eta_{X_2}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$.

Now let $x \in (\Box_R \circ \Box_{\eta_{X_2}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$. Then, it follows that $R(x) \subseteq \Box_{\eta_{X_2}}(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$. We need to show that $\eta_{X_1}(x) \subseteq \Psi_{R^{-1}[V]} \cap \mathcal{M}(\mathcal{H}(X_1))$. Let $x' \in X_1$ be such that $\Gamma_{X_1}(x') \in \eta_{X_1}(x)$. We need to prove that $\Gamma_{X_1}(x') \in \Psi_{R^{-1}[V]}$. Notice that $\Gamma_{X_1}(x') \in \Psi_{R^{-1}[V]} \iff R^{-1}[V] \subseteq \Gamma_{X_1}(x') \iff x' \notin R^{-1}[V]$. Suppose that $x' \in R^{-1}[V]$. Since $\Gamma_{X_1}(x') \in \eta_{X_1}(x)$, we have $\Gamma_{X_1}(x) \subseteq \Gamma_{X_1}(x')$. Then $x' \sqsubseteq x$. Now since $x' \in R^{-1}[V] \in \mathcal{K}_1$, it follows that $x \in R^{-1}[V]$. Thus, there is $y \in V$ such that $y \in R(x)$. By hypothesis, we have $\eta_{X_2}(y) \subseteq \Psi_V$. Given that $\Gamma_{X_2}(y) \in \eta_{X_2}(y) \subseteq \Psi_V$, we have $V \subseteq \Gamma_{X_2}(y)$. Thus $y \notin V$, which is a contradiction. Hence

$$\eta_{X_1}(x) \subseteq \Psi_{R^{-1}[V]} \cap \mathcal{M}(\mathcal{H}(X_1)) = \Box_{R_{f_R}}(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2))).$$

Then, $x \in (\Box_{\eta_{X_1}} \circ \Box_{R_{f_R}})(\Psi_V \cap \mathcal{M}(\mathcal{H}(X_2)))$. □ □

Finally, we are ready to define the functors between the category $\mathbb{SS}$ of S-spaces and the category $\mathbb{HMS}$ of HMS spaces, and prove the main result of this section.

Let $\mathbf{H}\colon \mathbb{SS} \to \mathbb{HMS}$ be defined as follows:

- for every S-space $\langle X, \mathcal{K}\rangle$, $\mathbf{H}(X) = \langle \mathcal{H}_{\mathcal{K}}(X), \tau_{HMS}\rangle$;
- for every meet-relation $R \subseteq X_1 \times X_2$, $\mathbf{H}(R) = f_R\colon \mathcal{H}(X_1) \to \mathcal{H}(X_2)$.

Let $\mathbf{M}\colon \mathbb{HMS} \to \mathbb{SS}$ be defined as follows:

- for every HMS space $\langle X, \tau\rangle$, $\mathbf{M}(X) = \langle \mathcal{M}(X), \mathcal{K}_{\mathcal{M}(X)}\rangle$;
- for every F-continuous map $f\colon X_1 \to X_2$, $\mathbf{M}(f) = R_f \subseteq \mathcal{M}(X_1) \times \mathcal{M}(X_2)$.

Theorem 5.30. *$\mathbf{H}\colon \mathbb{SS} \to \mathbb{HMS}$ and $\mathbf{M}\colon \mathbb{HMS} \to \mathbb{SS}$ establish an equivalence between the categories $\mathbb{SS}$ and $\mathbb{HMS}$.*

Proof. By Theorem 5.9 and Proposition 5.27, we have that $\mathbf{H}$ is well defined. Recall that the identity morphism in $\mathbb{SS}$ is $\sqsupseteq_X$, for every S-space X. Thus $f_{\sqsupseteq_X}(H) = \bigcup\{U \in \mathcal{K} : (\sqsupseteq_X)^{-1}[U] \subseteq H\} = H$, for all $H \in \mathcal{H}(X)$. Then, $\mathbf{H}(\sqsupseteq_X) = \mathrm{id}_{\mathcal{H}(X)}$, for every S-space X. Hence, by Proposition 5.28, we have that $\mathbf{H}$ is a functor.

By Theorem 5.20 and Proposition 5.24, we have that $\mathbf{M}$ is well defined. Notice that for every HMS space X, $\mathbf{M}(\mathrm{id}_X) = R_{\mathrm{id}_X} = \sqsupseteq_{\mathcal{M}(X)}$. Then, by Proposition 5.25, $\mathbf{M}$ is a functor.

From Corollary 5.22, and Propositions 5.29 and 5.23, the corresponding natural transformations are clear. This completes the proof. □ □

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