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A topological duality for posets

Luciano J. González and Ramon Jansana

Abstract. In this paper, we present a topological duality for partially ordered sets. We use the duality to give a topological construction of the canonical extension of a poset, and we also topologically represent the quasi-monotone maps, that is, maps from a finite product of posets to a poset that are order-preserving or order-reversing in each coordinate.

1. Introduction

The theory of topological duality arose mainly with M.H. Stone's work [11] in the mid-thirties of the twentieth century when he developed a duality between Boolean algebras and a class of topological spaces, later known as Stone spaces. In the subsequent paper [12], Stone generalizes the previous duality for Boolean algebras to show that the category of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category of spectral spaces and spectral maps. Both topological categories, Stone spaces and spectral spaces, are subcategories of the category of all topological spaces and continuous maps. Another classical duality, related to Stone's, is given by H.A. Priestley in [10] between the category of bounded distributive lattices and certain ordered topological spaces, which are known as Priestley spaces. Unlike Stone's duality, Priestley spaces are equipped with an additional partial order on the points in the space.

Lattice Theory, mainly developed by the work of G. Birkhoff in the mid-thirties of last century, is fundamental in the study of many ordered algebraic structures and also with regard to the classes of algebras that are associated with certain logics. Moreover, Lattice Theory is also important in other branches of mathematics such as Algebra, Computer Science, Domain Theory, etc. In the literature, there are several topological dualities for bounded lattices, for instance in [13], [5], [4] and [8]. In this last paper, Moshier and

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Jipsen give a topological duality for bounded lattices and a topological duality for meet-semilattices with top element in a way that the corresponding dual categories are subcategories of the category of topological spaces. Thus, we can say that their dualities follow the line of Stone's duality. In [9], Moshier and Jipsen use the duality developed in [8] to give, in a topological framework, a characterization of lattice expansions. These are lattices with additional operations that are order-preserving or order-reversing in each coordinate.

A partially ordered set (poset, for short) is a non-empty set with a binary relation that satisfies the properties of reflexivity, antisymmetry and transitivity. Posets form a large and general class of ordered structures that encompasses that of lattices. That is, a lattice can be seen as a poset in which the greatest lower bound and the least upper bound exist for every pair of elements. From this point of view we can observe the importance of studying posets in general and trying to develop for them analogous results to those obtained in Lattice Theory. This quest has been pursued by many; to name a few, we can highlight the works of M. Erné (see for example [2] and the references therein) and recently the extension to posets given in [1] of the theory of the canonical extension of a lattice.

In this paper, we develop a topological duality for posets. A fundamental concept to build our duality is the notion of filter of a poset. The notion of filter of a poset that we take is that of down-directed up-set. We intend that the dual category of the posets (with the maps that are order-preserving and such that the inverse image of a filter is a filter as the morphisms) form a subcategory of the category of topological spaces, and that our duality generalizes the duality given by Moshier and Jipsen for bounded lattices.

The dual spaces of posets will be the sober spaces $\langle X, \tau \rangle$ with the property that the compact open filters of X with respect to the specialization order form a base for the topology τ. We call these spaces P-spaces. The duals of the morphisms between posets of our category will be the continuous functions with the property that the inverse image of a compact open filter is a compact open filter. We will call such functions F-continuous maps.

The paper is organized as follows. In Section 2, we introduce the basic concepts related to posets and topological spaces we need. In Section 3, we review the concepts of Scott space and sober space. Sections 4 and 5 are devoted first to the representation of the posets by means of P-spaces and then to the duality between the category of posets and the category of P-spaces with F-continuous maps. In Section 6, we apply our duality to obtain a topological proof of the existence of the canonical extension of a poset as defined in [1]. This is the parallel result, but with a different kind of proof, to the topological proof of the existence of the canonical extension of a lattice provided in [8]. Section 7 analyses the dual space of the dual poset of a given poset P. Sections 8 and 9 deal with the topological representation of quasi-monotone maps between posets by maps between their duals, and with related issues. Finally, in Section 10, we specialize our duality to a duality between meet-semilattices and characterize the dual spaces. In this way, we obtain by further specializing to meet-semilattices with a top element the duality obtained in [8].

2. Preliminaries

In this section, we provide the basic facts that we need in the paper. Let P be a poset. A subset X of P is an *up-set* if for all $a, b \in P$ such that $a \in X$ and $a \leq b$, it holds that $b \in X$. Dually, we say that a subset X of P is a *down-set* if $b \in X$ and $a \leq b$ implies $a \in X$. Let $a \in P$. We define $\uparrow a := \{x \in P : a \leq x\}$; this set is called the *principal up-set generated by* a. Dually, we write $\downarrow a := \{x \in P : x \leq a\}$. We say that a subset A of P is *up-directed* if for every $a, b \in A$, there exists $c \in A$ such that $a \leq c$ and $b \leq c$, and we say that it is *down-directed* if for every $a, b \in A$, there exists $c \in A$ such that $c \leq a$ and $c \leq b$. A non-empty subset of P is called a *filter* if it is a down-directed up-set and it is called an *ideal* if it is an up-directed down-set. We denote by $\mathsf{Fi}(P)$ the collection of all filters of P and by $\mathsf{Id}(P)$ the collection of all its ideals.

Let $X = \langle X, \tau \rangle$ be a T_0 topological space. The *specialization order* $\sqsubseteq$ of X is defined as follows: $a \sqsubseteq b \iff \forall U \in \tau (a \in U \implies b \in U)$. We denote by $\sqcup$ and $\sqcap$ the join and meet respectively of the poset $\langle X, \sqsubseteq \rangle$ when they exist. For any $x \in X$, we let $N^o(x) := \{U \in \tau : x \in U\}$ denote the set of open neighbourhoods of x. So for all $a, b \in X$, $a \sqsubseteq b \iff N^o(a) \subseteq N^o(b)$.

Any other notions about partially ordered sets that we use on a T_0-space refer to the poset $\langle X, \sqsubseteq \rangle$. For instance, if U is an open subset of the T_0-space X, then U is an up-set of X, that is, if $a \sqsubseteq b$ and $a \in U$, then $b \in U$. Dually, any closed set is a down-set.

Let $\langle X, \tau \rangle$ be a T_0-space. An element $a \in X$ is said to be *finite* if $\uparrow a$ is an open subset of X, and we let $\mathsf{Fin}(X) := \{a \in X : \uparrow a \text{ is an open subset of } X\}$. We denote by $\mathsf{OF}(X)$ the family of all open filters of the space X, that is, $F \in \mathsf{OF}(X)$ if and only if F is a filter of $\langle X, \sqsubseteq \rangle$ and $F \in \tau$. Also, we define $\mathsf{KOF}(X)$ as the family of all compact open filters of X. It can be proved that all compact filters of the space X are of the form $\uparrow x$ for some $x \in X$ (that is therefore finite) and hence we obtain that $\mathsf{KOF}(X) = \{\uparrow a : a \in \mathsf{Fin}(X)\}$ (for a proof of these two facts see [8]).

3. Scott spaces and sober spaces

In this section, we present the definitions of, and basic facts about, Scott and sober spaces. The contents are well known, and so we leave the details to the reader. References for Scott spaces are [7] and [14], and for sober spaces, [6] and [7].

The Scott topology arises in a natural way by means of posets. Here we choose to give an abstract definition of Scott spaces and we show the intrinsic connection with posets.

Definition 3.1. A topological space $\langle X, \tau \rangle$ is said to be *Scott* if:

(1) X is T_0;
(2) for every subset U of X, U is open if and only if U is an up-set and it is inaccessible by up-directed joins (with respect to $\sqsubseteq$). That is, for each up-directed $D \subseteq X$, if $\bigsqcup^{\uparrow} D \in U$, then $U \cap D \neq \emptyset$.

In the previous definition, by $\bigsqcup^{\uparrow} D \in U$ we mean that D is an up-directed subset and the join of D in the poset $\langle X, \sqsubseteq \rangle$ exists and belongs to U. We keep in mind this convention throughout the paper.

Example 3.2. Let $\langle P, \leq \rangle$ be a poset. The *Scott topology* on P determined by the order $\leq$ is the collection τ_P of all subsets U of P that are up-sets and inaccessible by up-directed joins with respect to $\leq$. So it is clear that $\langle P, \tau_P \rangle$ is a Scott space. Moreover, $\leq$ is its specialization order $\sqsubseteq$.

Proposition 3.3. *Let X and Y be Scott spaces and $f \colon X \to Y$ a function. Then, f is continuous if and only if f preservers up-directed joins, i.e., $f(\bigsqcup^{\uparrow} D) = \bigsqcup^{\uparrow} f[D]$.*

We denote by $\mathbb{P}^{\Uparrow}$ the category whose objects are all posets and whose morphisms are all the functions between posets that preserve up-directed joins and by $\mathbb{TOP}(S)$ we denote the category of all Scott spaces and all continuous functions between them.

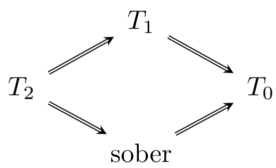
Proposition 3.4. *The categories $\mathbb{P}^{\Uparrow}$ and $\mathbb{TOP}(S)$ are isomorphic via the following functors:*

(1) $\Gamma \colon \mathbb{P}^{\Uparrow} \to \mathbb{TOP}(S)$ *where*
   - $\Gamma(P) := \langle P, \tau_P \rangle$ *for every poset P;*
   - *for every morphism $f \colon P \to Q$ of $\mathbb{P}^{\Uparrow}$, $\Gamma(f) \colon \Gamma(P) \to \Gamma(Q)$ is given by $\Gamma(f) = f$.*

(2) $\Delta \colon \mathbb{TOP}(S) \to \mathbb{P}^{\Uparrow}$ *where*
   - $\Delta(X) := \langle X, \sqsubseteq \rangle$ *for every Scott space X;*
   - *for every morphism $f \colon X \to Y$ of $\mathbb{TOP}(S)$, $\Delta(f) \colon \Delta(X) \to \Delta(Y)$ is defined by $\Delta(f) = f$.*

Definition 3.5. A topological space $\langle X, \tau \rangle$ is *sober* if X is T_0 and for every completely prime filter $\mathcal{F}$ of the lattice of open subsets of X, there exists an element $x \in X$ such that $\mathcal{F} = \{U \in \tau : x \in U\} = N^o(x)$.

The sobriety condition for topological spaces is a kind of separation axiom where its position in the separation hierarchy is



Let X be a topological space and $A \subseteq X$. We say that A is *irreducible* if for all closed subsets B, C of X, $A \subseteq B$ or $A \subseteq C$ whenever $A \subseteq B \cup C$. The following two propositions are useful for working with sober spaces.

Proposition 3.6. *A topological space X is sober if and only if each closed irreducible subset of X is of the form $\downarrow x$ for a unique point x.*

Proposition 3.7. *Let X be a sober space.*

(1) *Every up-directed subset D of X has a join $\bigsqcup^{\uparrow} D$.*
(2) *If U is an open subset of X, then U is inaccessible by up-directed joins.*
(3) *Every continuous function f between sober spaces preserves up-directed joins, that is, $f(\bigsqcup^{\uparrow} D) = \bigsqcup^{\uparrow} f[D]$.*

4. Topological representation of posets

In this section, we present a topological representation theorem for posets via a particular class of topological spaces. These topological spaces are Scott spaces built by means of posets. Our main purpose in this part is to generalize the topological representation for lattices and meet-semilattices given in [8] to posets. For this, we apply the underlying idea in [8] in a more general context.

Let P be a poset. Let us consider the poset $\langle \mathsf{Fi}(P), \subseteq \rangle$ and define the topological space $\langle \mathsf{Fi}(P), \tau_{\mathsf{Fi}(P)} \rangle$ where $\tau_{\mathsf{Fi}(P)}$ is the Scott topology of the poset $\mathsf{Fi}(P)$ (see Example 3.2). For short, we write $X_P := \langle \mathsf{Fi}(P), \tau_{\mathsf{Fi}(P)} \rangle$. It should be noted that the specialization order $\sqsubseteq$ of the space X_P is the order of inclusion $\subseteq$. That is, for all $F, G \in \mathsf{Fi}(P)$, $F \sqsubseteq G$ if and only if $F \subseteq G$. For every $a \in P$, we define the set $\varphi_a := \{F \in \mathsf{Fi}(P) : a \in F\}$.

The proofs of the following two propositions are similar to the ones given for the analogous facts in the case of meet-semilattices with a top element in [8]. We give more details here than in [8], because we work in the more general setting of posets.

Proposition 4.1. *The family $\{\varphi_a : a \in P\}$ is a base for the space X_P.*

Proof. We prove this proposition in two steps. Let $a \in P$. It is clear that φ_a is an up-set (of the poset $\langle \mathsf{Fi}(P), \subseteq \rangle$). Let $\mathcal{A}$ be an up-directed collection of filters of P and suppose that $\bigvee^{\uparrow} \mathcal{A} \in \varphi_a$. Since $\mathcal{A}$ is an up-directed family of filters of P with respect to the inclusion order, we have that $\bigvee^{\uparrow} \mathcal{A} = \bigcup \mathcal{A}$. So $a \in \bigcup \mathcal{A}$. This implies that there exists $F \in \mathcal{A}$ such that $a \in F$, and thus $F \in \varphi_a$. Then $\varphi_a \cap \mathcal{A} \neq \emptyset$. Hence, φ_a is a Scott open of the space X_P.

Now we will prove that the family $\{\varphi_a : a \in P\}$ is a base for the space X_P. Let $U \subseteq \mathsf{Fi}(P)$ be a Scott open set of X_P and let $F \in U$. Let us take the set $\mathcal{D} := \{\uparrow a : a \in F\}$. $\mathcal{D}$ is an up-directed subset of $\mathsf{Fi}(P)$ because F is a filter of P. So $F = \bigcup \mathcal{D} = \bigvee^{\uparrow} \mathcal{D} \in U$ and, since U is Scott open, we obtain $U \cap \mathcal{D} \neq \emptyset$. Hence, there is $a \in F$ such that $\uparrow a \in U$. This implies that $F \in \varphi_a \subseteq U$. □

Proposition 4.2. *For every $a \in P$, φ_a is a compact open filter of X_P.*

Proof. Let $a \in P$. From the previous proposition, we know that φ_a is open. Moreover, clearly, φ_a is a filter of X_P. Now let us prove that φ_a is compact. Let $\{U_i : i \in I\}$ be a family of open subsets of X_P and suppose that $\varphi_a \subseteq \bigcup_{i \in I} U_i$. Since $\uparrow a \in \varphi_a$, $\uparrow a \in \bigcup_{i \in I} U_i$. So for some $i \in I$, $\uparrow a \in U_i$. As U_i is open and the specialization order in X_P is $\subseteq$, we have $\varphi_a \subseteq U_i$. □

We provide a characterization of the posets P whose space X_P is compact. In particular, it turns out that if P has a top element, then the space X_P is compact.

Proposition 4.3. *The space X_P is compact if and only if the set of maximal elements of P is finite and for every $a \in P$ there exists a maximal element $b \in P$ such that $a \leq b$.*

Proof. Suppose that the set $\max(P)$ of maximal elements of P is finite and for every $a \in P$, there exists a maximal element $b \in P$ such that $a \leq b$. Let us consider a cover $\{\varphi_a : a \in Z\}$ of X_P by basic open sets. Let $b \in \max(P)$. Then $\{b\}$ is a filter of P. Therefore, $\{b\} \in \varphi_a$ for some $a \in Z$, and then $a = b$. It follows that $\max(P) \subseteq Z$. Now, since by assumption for every $a \in P$ there exists $b \in \max(P)$ such that $a \leq b$, we obtain that $\{\varphi_b : b \in \max(P)\}$ is a finite subcover of $\{\varphi_a : a \in Z\}$. We conclude that X_P is compact.

Conversely, assume that X_P is compact and the set of maximal elements of P is infinite or there exists $a \in P$ such that for no $b \in \max(P)$, $a \leq b$. If $\max(P)$ is infinite, then $\{\varphi_b : b \in \max(P)\} \cup \{\varphi_a : \forall b \in \max(P),\ a \nleq b\}$ is a cover of X_P without any finite subcover. Now suppose $a_0 \in P$ is such that for no $b \in \max(P)$ is $a_0 \leq b$. Then there exists a strictly increasing infinite chain $a_0 < a_1 < \cdots < a_n < a_{n+1} < \cdots$. So $\{\varphi_{a_n} : n \in \omega\} \cup \{\varphi_a : \forall n \in \omega,\ a \nleq a_n\}$ is a cover of X_P and has no finite subcover. □

Proposition 4.2 tells us that $\{\varphi_a : a \in P\} \subseteq \mathsf{KOF}(X_P)$. We next prove that all compact open filters of the space X_P are of the form φ_a for some $a \in P$. Then applying Proposition 4.1, we have that $\mathsf{KOF}(X_P)$ is a base for X_P.

Proposition 4.4. *For every compact open filter U of X_P, there is $a \in P$ such that $U = \varphi_a$.*

Proof. Let $U \in \mathsf{KOF}(X_P)$. Since U is a compact filter, it is of the form $U = \{G \in \mathsf{Fi}(P) : F \subseteq G\}$ for some $F \in \mathsf{Fi}(P)$. Let $\mathcal{D} := \{\uparrow a : a \in F\}$, and so $\mathcal{D}$ is an up-directed family of filters of P. Then $\bigvee \mathcal{D} = \bigcup \mathcal{D} = F \in U$. As U is a Scott open, $U \cap \mathcal{D} \neq \emptyset$. Thus, there exists $a \in F$ such that $\uparrow a \in U$. Then we obtain that $F = \uparrow a$, and hence

$$U = \{G \in \mathsf{Fi}(P) : \uparrow a \subseteq G\} = \{G \in \mathsf{Fi}(P) : a \in G\} = \varphi_a. \qquad \square$$

Therefore, bringing together the above results, we have that $\mathsf{KOF}(X_P) = \{\varphi_a : a \in P\}$. Let us consider on $\mathsf{KOF}(X_P)$ the inclusion order $\subseteq$. We can now present the main result of this section, namely the representation theorem for posets.

Theorem 4.5. *Let P be a poset. The map $\varphi\colon P \to \mathsf{KOF}(X_P)$, defined by $\varphi(a) = \varphi_a$ for all $a \in P$, is an order isomorphism.*

Proof. By Proposition 4.4, the map φ is onto. Let $a, b \in P$ and $a \neq b$. Suppose $a \not\leq b$. So $\uparrow a \in \varphi_a \setminus \varphi_b$, and hence φ is injective. Let $a, b \in P$. Then we have

$$a \leq b \iff \forall F \in \mathsf{Fi}(P)(a \in F \Longrightarrow b \in F) \iff \varphi_a \subseteq \varphi_b \iff \varphi(a) \subseteq \varphi(b).$$

Therefore, φ is an order isomorphism from P onto $\mathsf{KOF}(X_P)$. $\square$

Remark 4.6. Let P be a poset and $a, b, c \in P$. Note that $\varphi_c = \varphi_a \cap \varphi_b$ if and only if the greatest lower bound of a, b exists and is c. Thus, in P the greatest lower bound of any two elements exists if and only if $U \cap V \in \mathsf{KOF}(X_P)$ for every $U, V \in \mathsf{KOF}(X_P)$. Also note that P has a top element if and only if $X_P \in \mathsf{KOF}(X_P)$.

5. Duality

In the first part of this section, we define the topological spaces that will be the duals of the posets in the categorical duality that we want to establish. These spaces should be an abstract characterization of the spaces X_P constructed by means of posets P as in the previous section. A topological space X dual to a poset should be such that $\mathsf{KOF}(X)$ is a base for the space. Moreover, we observe that the spaces X_P have very nice properties with respect to the specialization order. Since our duality is a kind of Stone duality, it is natural to expect that the spaces we consider will be sober. We begin by giving the following definition.

Definition 5.1. A topological space $\langle X, \tau \rangle$ is a *P-space* if it satisfies the following conditions:

(P1) X is a sober space;
(P2) $\mathsf{KOF}(X)$ is a base for τ.

The notion of P-space is a direct generalization of the notion of HMS-space introduced in [8]. HMS-spaces are duals of meet-semilattices with a top element. We will discuss HMS-spaces in Section 10. The following proposition, which is a characterization of P-spaces, can be useful to show that certain topological spaces are P-spaces.

Proposition 5.2. *Let X be a topological space. Then, X is a P-space if and only if the following conditions are satisfied:*

(i) *X is a Scott space;*
(ii) *$\mathsf{KOF}(X)$ is a base for X;*
(iii) *every up-directed subset of X (with respect to $\sqsubseteq$) has a join.*

Proof. First, we assume that X is a P-space and we prove that the three conditions above hold.

(i): Since X is sober, it is a T_0-space. Let U be an open set. Then U is an up-set of X and by Proposition 3.7, U is inaccessible by up-directed joins. Now let U be an up-set of X that is inaccessible by up-directed joins. Let $x \in U$. The set $D := \{a \in \mathsf{Fin}(X) : a \sqsubseteq x\}$ is up-directed and non-empty because $\mathsf{KOF}(X)$ is a base for X. Then, since X is sober, there exists $\bigsqcup^{\uparrow} D$. Let us see that $x = \bigsqcup^{\uparrow} D$. It is clear that $\bigsqcup^{\uparrow} D \sqsubseteq x$. To prove the reverse inequality, we use the fact that $\mathsf{KOF}(X)$ is a base. Let $\uparrow b \in \mathsf{KOF}(X)$ be such that $x \in \uparrow b$. So $b \sqsubseteq x$, and thus $b \in D$. Then $b \sqsubseteq \bigsqcup^{\uparrow} D$, which implies that $\bigsqcup^{\uparrow} D \in \uparrow b$. Then $x \sqsubseteq \bigsqcup^{\uparrow} D$. Thus, we have that $\bigsqcup^{\uparrow} D = x \in U$, and since U is inaccessible by up-directed joins, $D \cap U \neq \emptyset$. Hence, there is $a \in D$ such that $a \in U$. Consequently, $a \in \mathsf{Fin}(X)$ and $a \sqsubseteq x$, and hence $x \in \uparrow a$. So we obtain that $x \in \uparrow a \subseteq U$, which implies that U is an open set of X. Therefore, X is a Scott space.

(ii): By hypothesis, $\mathsf{KOF}(X)$ is a base for X.

(iii): By Proposition 3.7, every up-directed subset of X has a join.

Now we assume that X satisfies the three conditions of the proposition. We need only to prove that X is sober. Since X is a Scott space, X is T_0. Let $\mathcal{A}$ be a completely prime filter of the lattice of open sets. Let $D := \{a \in X : \uparrow a \in \mathcal{A}\}$. Since $\mathcal{A}$ is a filter, there exists $U \in \mathcal{A}$. So $U = \bigcup_{i \in I} \uparrow a_i$ for some family $\{a_i : i \in I\} \subseteq \mathsf{Fin}(X)$. As $\bigcup_{i \in I} \uparrow a_i \in \mathcal{A}$ and $\mathcal{A}$ is completely prime, there exists $i_0 \in I$ such that $\uparrow a_{i_0} \in \mathcal{A}$. So $a_{i_0} \in D$. Thus, D is non-empty.

Let us see that D is an up-directed subset of X. Let $a, b \in D$. Since $\mathcal{A}$ is a filter, $\uparrow a \cap \uparrow b \in \mathcal{A}$. As $\uparrow a \cap \uparrow b$ is open, $\uparrow a \cap \uparrow b = \bigcup_{j \in J} \uparrow c_j$ for some family $\{c_j : j \in J\} \subseteq \mathsf{Fin}(X)$. Hence, $\uparrow c_{j_0} \in \mathcal{A}$ for some $j_0 \in J$. Thus, $c_{j_0} \in D$, $a \sqsubseteq c_{j_0}$, and $b \sqsubseteq c_{j_0}$; hence, D is up-directed. By condition (iii), let $x := \bigsqcup^{\uparrow} D$. We show that $N^o(x) = \mathcal{A}$. Let $U \in N^o(x)$. So $x \in U$, and this implies that $\bigsqcup^{\uparrow} D \in U$. By condition (i), we have that $D \cap U \neq \emptyset$, and hence there exists some $a \in D \cap U$. Since U is an up-set of X, $\uparrow a \subseteq U$. As $a \in D$, $\uparrow a \in \mathcal{A}$, and then $U \in \mathcal{A}$. Hence, $N^o(x) \subseteq \mathcal{A}$. Conversely, let $U \in \mathcal{A}$. So $U = \bigcup_{i \in I} \uparrow a_i$ where $\uparrow a_i \in \mathsf{KOF}(X)$ for each $i \in I$. Since $\mathcal{A}$ is completely prime, there exists $i \in I$ such that $\uparrow a_i \in \mathcal{A}$. Thus, $a_i \in D$. This implies that $a_i \sqsubseteq x$. Then $x \in U$ because $a_i \in U$. Hence, $U \in N^o(x)$. Therefore, $\mathcal{A} \subseteq N^o(x)$. □

Theorem 5.3. *Let P be a poset. Then, X_P is a P-space.*

Proof. By definition, the space X_P is a Scott space. From Propositions 4.1 and 4.2, $\mathsf{KOF}(X_P)$ is a base for X_P. Lastly, since the specialization order $\sqsubseteq$ of

X_P is the inclusion order, it is clear that the joins of all up-directed subsets of X_P exist. Then the three conditions (i)–(iii) of Proposition 5.2 are satisfied, and therefore X_P is a P-space. □

Let X be a topological space. We denote the poset $\langle \mathsf{KOF}(X), \subseteq \rangle$ by P_X. Using the construction of the previous section, we obtain the space $X_{P_X} := \langle \mathsf{Fi}(P_X), \tau_{P_X} \rangle$.

Theorem 5.4. *Let X be a P-space. Then X is homeomorphic to X_{P_X}.*

Proof. We define the map $\theta\colon X \to X_{P_X}$ by $\theta(x) := \{U \in \mathsf{KOF}(X) : x \in U\}$ for each $x \in X$. We show that θ is a homeomorphism in several steps.

• θ is well defined. Let $x \in X$. It is clear that $\theta(x)$ is an up-set of $P_X = \langle \mathsf{KOF}(X), \subseteq \rangle$. Let $U_1, U_2 \in \theta(x)$. So $x \in U_1 \cap U_2$, and since $U_1 \cap U_2$ is open, there exists $U_3 \in \mathsf{KOF}(X)$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Then $U_3 \in \theta(x)$ and $U_3 \subseteq U_1, U_2$. Hence, $\theta(x)$ is a filter of the poset P_X.

• θ is injective. Let $x, y \in X$ and suppose that $x \not\sqsubseteq y$. Since $\mathsf{KOF}(X)$ is a base for X, there exists $U \in \mathsf{KOF}(X)$ such that $x \in U$ and $y \notin U$. Then $\theta(x) \not\subseteq \theta(y)$.

• θ is onto. Let $F \in X_{P_X} = \mathsf{Fi}(P_X)$. Let $D := \{a \in X : \uparrow a \in F\}$. As F is a filter of P_X, D is an up-directed subset of X. Then since X is a P-space, there exists $x := \bigsqcup^{\uparrow} D$. We want to show that $\theta(x) = F$. Let $\uparrow a \in F$. So $a \in D$, and then $x \in \uparrow a$. This implies that $\uparrow a \in \theta(x)$. Now let $\uparrow a \in \theta(x)$. By definition of θ and since $x = \bigsqcup^{\uparrow} D$, it follows that $\bigsqcup^{\uparrow} D \in \uparrow a$. As X is a P-space, the open subsets of X are inaccessible by up-directed joins, and so $D \cap \uparrow a \neq \emptyset$. Then there exists $d \in D \cap \uparrow a$. Since F is a filter, $\uparrow d \subseteq \uparrow a$ and $\uparrow d \in F$; it follows that $\uparrow a \in F$. Therefore, $F = \theta(x)$.

• θ is continuous. Let φ_U be a basic open set of the space X_{P_X}. Recall that for $U \in P_X = \mathsf{KOF}(X)$, we have $\varphi_U = \{F \in \mathsf{Fi}(P_X) : U \in F\}$. For every $x \in X$, we have

$$x \in \theta^{-1}[\varphi_U] \iff \theta(x) \in \varphi_U \iff U \in \theta(x) \iff x \in U.$$

Then $\theta^{-1}[\varphi_U] = U$ is an open set of X, and therefore θ is continuous.

• θ is an open map. Let $U \in \mathsf{KOF}(X)$. We show that $\theta[U] = \varphi_U$. Let $F \in \theta[U]$. So there is $x \in U$ such that $\theta(x) = F$. Since $x \in U$, $F \in \varphi_U$. Now let $F \in \varphi_U$. So $U \in F$. Since θ is onto, there exists $x \in X$ such that $\theta(x) = F$. As $U \in F = \theta(x)$, $x \in U$. Then $F \in \theta[U]$.

Therefore, from all these points, we can conclude that θ is a homeomorphism. □

Let us denote by $\mathbb{P}$ the category whose objects are posets and whose morphisms are the order-preserving maps between posets and such that the inverse image of a filter is a filter. That is, $j\colon P \to Q$ is a morphism of $\mathbb{P}$ if it is an order-preserving map and for all $G \in \mathsf{Fi}(Q)$, $j^{-1}(G) \in \mathsf{Fi}(P)$. A function $f\colon X \to Y$ from the P-space X to the P-space Y is called *F-continuous* if for all $U \in \mathsf{KOF}(Y)$, we have that $f^{-1}(U) \in \mathsf{KOF}(X)$. When this condition holds, we say that f^{-1} preserves compact open filters. By $\mathbb{TOP}(P)$, we denote the category of P-spaces and F-continuous functions. Note that every F-continuous map between P-spaces is continuous.

Now, we extend the representation theorem for posets to a duality between the categories $\mathbb{P}$ and $\mathbb{TOP}(P)$.

Theorem 5.5. *The categories $\mathbb{P}$ and $\mathbb{TOP}(P)$ are dually equivalent via the following functors:*

(1) $\Gamma\colon \mathbb{P} \to \mathbb{TOP}(P)$ *defined by*
  - $\Gamma(P) := X_P$, *for each poset* P;
  - *for every morphism* $j\colon P \to Q$ *of* $\mathbb{P}$, $\Gamma(j)\colon X_Q \to X_P$ *is given by* $\Gamma(j) := j^{-1}$.

(2) $\Delta\colon \mathbb{TOP}(P) \to \mathbb{P}$ *defined by*
  - $\Delta(X) := P_X$, *for each P-space* X;
  - *for every morphism* $f\colon X \to Y$ *of* $\mathbb{TOP}(P)$, $\Delta(f)\colon P_Y \to P_X$ *is given by* $\Delta(f) := f^{-1}$.

Proof. (1): Let $j\colon P \to Q$ be a morphism of $\mathbb{P}$. We need to show that $\Gamma(j) = j^{-1}\colon X_Q \to X_P$ is F-continuous. Let $U \in \mathsf{KOF}(X_P)$. By Proposition 4.4, there is $a \in P$ such that $U = \varphi_a$. Then we have

$$\begin{aligned} G \in \Gamma(j)^{-1}[\varphi_a] &\iff \Gamma(j)(G) \in \varphi_a \iff j^{-1}(G) \in \varphi_a \\ &\iff a \in j^{-1}(G) \iff j(a) \in G \iff G \in \varphi_{j(a)}. \end{aligned}$$

Hence, $\Gamma(j)^{-1}[\varphi_a] = \varphi_{j(a)} \in \mathsf{KOF}(X_Q)$.

(2): Let $f\colon X \to Y$ be a morphism of the category $\mathbb{TOP}(P)$. Since $\Delta(f) = f^{-1}$, it is clear that $\Delta(f)$ is an order-preserving map from P_Y to P_X. Now let $F \in \mathsf{Fi}(P_X)$. From Theorem 5.4, we know that $F = \theta(x)$ for some $x \in X$. Let $U \in \mathsf{KOF}(Y)$. Then

$$\begin{aligned} U \in \Delta(f)^{-1}[F] &\iff \Delta(f)(U) \in F = \theta(x) \\ &\iff f^{-1}(U) \in \theta(x) \iff f(x) \in U \iff U \in \theta(f(x)). \end{aligned}$$

Hence, we obtain that $\Delta(f)^{-1}[F] = \theta(f(x)) \in \mathsf{Fi}(P_Y)$. Thus, $\Delta(f)$ is a morphism of $\mathbb{P}$. To conclude the proof, we need to show that for every morphism $j\colon P \to Q$ of $\mathbb{P}$ and every morphism $f\colon X \to Y$ of $\mathbb{TOP}(P)$, the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\quad j \quad} & Q \\ \varphi \downarrow & & \downarrow \varphi \\ \mathsf{KOF}(X_P) & \xrightarrow[\Delta(\Gamma(j))]{} & \mathsf{KOF}(X_Q) \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \theta \downarrow & & \downarrow \theta \\ X_{P_X} & \xrightarrow[\Gamma(\Delta(f))]{} & X_{P_Y} \end{array}$$

Let $a \in P$ and $F \in \mathsf{Fi}(Q)$. Then we have

$$\begin{aligned} F \in (\Delta(\Gamma(j)) \circ \varphi)(a) &\iff F \in \Gamma(j)^{-1}[\varphi(a)] \iff \Gamma(j)(F) \in \varphi(a) \\ &\iff j^{-1}(F) \in \varphi(a) \iff a \in j^{-1}(F) \\ &\iff j(a) \in F \iff F \in \varphi(j(a)). \end{aligned}$$

Hence, $\Delta(\Gamma(j)) \circ \varphi = \varphi \circ j$.

Finally, let $x \in X$ and $U \in \mathsf{KOF}(Y)$. Then

$$\begin{aligned} U \in \Gamma(\Delta(f))(\theta(x)) &\iff U \in \Delta(f)^{-1}[\theta(x)] \iff \Delta(f)(U) \in \theta(x) \\ &\iff f^{-1}(U) \in \theta(x) \iff x \in f^{-1}(U) \\ &\iff f(x) \in U \iff U \in \theta(f(x)). \end{aligned}$$

Hence, $\Gamma(\Delta(f)) \circ \theta = \theta \circ f$. □

6. Canonical extension for posets

The canonical extension of a poset is introduced in [1, Definition 2.2]. We recall the definition here.

An *extension* of poset P is a pair $\langle Q, e\rangle$ where Q is a poset and $e\colon P \to Q$ is an order-embedding, i.e., for every $x, y \in P$, $x \leq y$ if and only if $e(x) \leq e(y)$. A *completion* of P is an extension $\langle Q, e\rangle$ of P where Q is a complete lattice.

Given an extension $\langle Q, e\rangle$ of P, an element of Q is called *closed* provided it is the infimum in Q of $e[F]$ for some filter F of P. Dually an element of Q is called *open* provided it is the supremum of $e[I]$ for some ideal I of Q.

An extension $\langle Q, e\rangle$ of P is *dense* provided each element of Q is both the supremum of all the closed elements below it and the infimum of all the open elements above it. An extension $\langle Q, e\rangle$ of P is *compact* provided that whenever D is a non-empty down-directed subset of P, U is a non-empty up-directed subset of P, and $\bigwedge_Q D \leq \bigvee_Q U$, then there are $x \in D$ and $y \in U$ with $x \leq y$. A *canonical extension* of P is any completion that is dense and compact. In [1] it is proved that if a poset has a canonical extension, then it is unique up to isomorphism, and that every poset has a canonical extension.

In this section, we use the duality between the categories $\mathbb{P}$ and $\mathbb{TOP}(P)$ of the previous section to show the existence of the canonical extension of a poset from a topological viewpoint. The proof we present is a topological alternative to the algebraic proof in [1], in a way similar to the proof of the existence of a canonical extension for lattices given in [8] that is a topological alternative to the purely algebraic proof supplied in [3].

Let X be a topological space. Recall that $\mathsf{OF}(X)$ denotes the family of all open filters of X. We take the closure system $\mathsf{Fsat}(X)$ on X generated by the family $\mathsf{OF}(X)$. That is, $\mathsf{Fsat}(X)$ is the collection of all subsets of X that are intersections of open filters. We denote the associated closure operator of $\mathsf{Fsat}(X)$ by $\mathsf{fsat}(\cdot)$. So for every $A \subseteq X$, $\mathsf{fsat}(A) = \bigcap\{F \in \mathsf{OF}(X) : A \subseteq F\}$. Then we have the complete lattice $\langle \mathsf{Fsat}(X), \bigcap, \bigvee\rangle$ where $\bigvee \mathcal{A} = \mathsf{fsat}(\bigcup \mathcal{A})$ for each $\mathcal{A} \subseteq \mathsf{Fsat}(X)$, and moreover, $\mathsf{KOF}(X) \subseteq \mathsf{OF}(X) \subseteq \mathsf{Fsat}(X)$. So it is clear that the lattice $\mathsf{Fsat}(X)$ is a completion of the poset $P_X = \langle \mathsf{KOF}(X), \subseteq \rangle$. The elements of $\mathsf{Fsat}(X)$ are called *F-saturated* sets.

Let P be a poset. We will prove that $\mathsf{Fsat}(X_P) = \langle \mathsf{Fsat}(X_P), \bigcap, \bigvee \rangle$ is the canonical extension of P with the embedding $\varphi\colon P \to \mathsf{Fsat}(X_P)$.

According to the terminology in [1], an element of $\mathsf{Fsat}(X_P)$ is a *closed element* if it is the infimum in $\mathsf{Fsat}(X_P)$ of $\varphi[F]$ for some filter F of P. And an element of $\mathsf{Fsat}(X_P)$ is an *open element* if it is the supremum in $\mathsf{Fsat}(X_P)$ of $\varphi[H]$ for some ideal H of P.

Lemma 6.1. *An element $U \in \mathsf{Fsat}(X_P)$ is a closed element if there is a filter F of P such that $U = {\uparrow}F$ in $\langle \mathsf{Fi}(P), \subseteq \rangle$. Similarly, U is an open element if there exists an ideal H of P such that $U = \{G \in \mathsf{Fi}(P) : G \cap H \neq \emptyset\}$.*

Proof. First note that if $F \in \mathsf{Fi}(P)$, then

$$ {\uparrow}F = \{G \in \mathsf{Fi}(P) : F \subseteq G\} = \bigcap \{\varphi(a) : a \in F\} = \bigcap \varphi[F]. $$

Thus, ${\uparrow}F \in \mathsf{Fsat}(X_P)$ and is closed. Now if $U \in \mathsf{Fsat}(X_P)$ is closed, then there exists $F \in \mathsf{Fi}(P)$ such that $U = \bigcap \varphi[F]$. Then, $U = {\uparrow}F$.

Let H be an ideal of P. Then $\{G \in \mathsf{Fi}(P) : G \cap H \neq \emptyset\} = \bigcup \varphi[H]$. Since H is an ideal, $\varphi[H]$ is up-directed. Thus, $\bigvee \varphi[H] = \mathsf{fsat}(\bigcup \varphi[H]) = \bigcup \varphi[H]$. Hence, $\{G \in \mathsf{Fi}(P) : G \cap H \neq \emptyset\} \in \mathsf{Fsat}(X_P)$ and is open. Now if $U \in \mathsf{Fsat}(X_P)$ is open, let H be an ideal of P such that we have $U = \bigvee \varphi[H]$. Then we have $U = \{G \in \mathsf{Fi}(P) : G \cap H \neq \emptyset\}$. □

Lemma 6.2. *If $\mathcal{F}$ is an open filter of X_P, then there exists an ideal H of P such that $\mathcal{F} = \bigvee \varphi[H]$.*

Proof. Let $\mathcal{F}$ be an open filter of X_P. Thus, it is an up-set that is down-directed, and since it is an open set, it is inaccessible by up-directed joins. Let $H := \{a \in P : {\uparrow}a \in \mathcal{F}\}$. We claim that H is an ideal of P. If $a \in H$ and $b \leq a \in P$, then ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \subseteq {\uparrow}b$. Hence, ${\uparrow}b \in \mathcal{F}$ and so $b \in H$. Suppose now that $a, b \in H$, so that ${\uparrow}a, {\uparrow}b \in \mathcal{F}$. There exists $F \in \mathcal{F}$ such that $F \subseteq {\uparrow}a, {\uparrow}b$. Note that since F is a filter of P, the set $\{{\uparrow}c : c \in F\}$ is up-directed and its join is F. Using that $\mathcal{F}$ is inaccessible by up-directed joins, there exists $c \in F$ such that ${\uparrow}c \in \mathcal{F}$. It follows that $a, b \leq c \in H$.

To conclude the proof, we show that $\mathcal{F} = \bigvee \varphi[H]$. First note that $\varphi[H]$ is up-directed because H is an ideal. Thus, $\bigvee \varphi[H] = \bigcup \varphi[H]$. Let $G \in \bigcup \varphi[H]$. So there exists $a \in H$ such that $a \in G$. Hence, since ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \subseteq G$, we have $G \in \mathcal{F}$. To prove the other inclusion, suppose that $G \in \mathcal{F}$. Since $G = \bigvee \{{\uparrow}c : c \in G\} \in \mathcal{F}$ and the set $\{{\uparrow}c : c \in G\}$ is up-directed, then there is $c_0 \in G$ such that ${\uparrow}c_0 \in \mathcal{F}$. Therefore, $c_0 \in H$ and $G \in \varphi(c_0)$, so that $G \in \bigcup \varphi[H]$. □

Proposition 6.3. *Let P be a poset. Then, the complete lattice $\mathsf{Fsat}(X_P)$ is the canonical extension of the poset P (with the embedding φ).*

Proof. *Density:* Let $U \in \mathsf{Fsat}(X_P)$. First note that U is an up-set of the poset $\langle \mathsf{Fi}(P), \subseteq \rangle$ because it is an intersection of open filters of X_P and these are up-sets. Thus, $U = \bigcup\{\uparrow F : F \in U\}$. Hence,

$$U = \mathsf{fsat}(U) = \bigvee\{V \in \mathsf{Fsat}(X_P) : V \text{ is a closed element and } V \subseteq U\}.$$

Now we prove that $U = \bigcap\{\bigvee \varphi[H] : H \text{ is an ideal of } P \text{ and } U \subseteq \bigvee \varphi[H]\}$, that is, that $U = \bigcap\{V \in \mathsf{Fsat}(X_P) : V \text{ is an open element and } U \subseteq V\}$. One inclusion is obvious; to prove the other inclusion, let $G \in \mathsf{Fi}(P)$ be such that $G \notin U$. We find an ideal H of P such that $U \subseteq \bigvee \varphi[H]$ and $G \notin \bigvee \varphi[H]$. Since $U \in \mathsf{Fsat}(X_P)$, there is a family $\{\mathcal{F}_i : i \in I\}$ of open filters of X_P such that $U = \bigcap_{i \in I} \mathcal{F}_i$. Hence, there exists $\mathcal{F}_i$ such that $G \notin \mathcal{F}_i$. We consider the set $H := \{a \in P : \uparrow a \in \mathcal{F}_i\}$. By Lemma 6.2, we have $\mathcal{F}_i = \bigvee \varphi[H]$. Thus, $U \subseteq \bigvee \varphi[H]$ and $G \notin \bigvee \varphi[H]$.

Compactness: Let D be a non-empty down-directed subset of P and E a non-empty up-directed subset of P such that $\bigwedge \varphi[D] \subseteq \bigvee \varphi[E]$. Then $\varphi[E]$ is up-directed, and therefore $\bigvee \varphi[E] = \bigcup \varphi[E]$. Hence, $\bigcap \varphi[D] \subseteq \bigcup \varphi[E]$. Let $F := \{d \in P : \exists a \in D,\ a \leq d\}$. This set is a filter of P and $F \in \bigcap \varphi[D]$. Thus, $F \in \bigcup \varphi[E]$. Then there exists $d \in E$ such that $d \in F$. Hence, there is $a \in D$ such that $a \leq d$. This proves the compactness condition. □

7. The dual space of P^∂

Given a P-space X, we consider the Scott topology of the poset $\langle \mathsf{OF}(X), \subseteq \rangle$. We refer to the resulting space simply by $\mathsf{OF}(X)$. For every $x \in X$, we define the set $\psi_x := \{F \in \mathsf{OF}(X) : x \in F\}$.

Proposition 7.1. *The family $\{\psi_x : x \in X\}$ is a base for the space $\mathsf{OF}(X)$.*

Proof. Let $x \in X$. First, we prove that ψ_x is a Scott open subset of $\mathsf{OF}(X)$. It is clear that ψ_x is an up-set of $\mathsf{OF}(X)$. Now, let $\{F_i : i \in I\}$ be an up-directed family of open filters of X and suppose that $\bigvee^{\uparrow} F_i \in \psi_x$. Since the family $\{F_i : i \in I\}$ is up-directed, $\bigvee^{\uparrow} F_i = \bigcup F_i$. So $x \in F_i$ for some $i \in I$. Then $\{F_i : i \in I\} \cap \psi_x \neq \emptyset$. Hence, ψ_x is a Scott open set of the space $\mathsf{OF}(X)$.

To prove that the family $\{\psi_x : x \in X\}$ is a base for $\mathsf{OF}(X)$, let U be a Scott open set of the space $\mathsf{OF}(X)$ and let $F \in U$. Since X is a P-space,

$$F = \bigcup\{\uparrow a : a \in F \cap \mathsf{Fin}(X)\} = \bigvee^{\uparrow}\{\uparrow a : a \in F \cap \mathsf{Fin}(X)\}.$$

As U is inaccessible by up-directed joins, there exists $a \in F \cap \mathsf{Fin}(X)$ such that $\uparrow a \in U$. Since U is an up-set, we have $F \in \psi_a \subseteq U$. □

In [9], Moshier and Jipsen consider for an HMS-space X, the topology on $\mathsf{OF}(X)$ generated by the family $\{\psi_x : x \in X\}$ and then they show that this family is a base. In our case, in the setting of posets, the proof that the family $\{\psi_x : x \in X\}$ is a base is completely different from the proof in [9]. Moreover, we are showing that this family is a base for the Scott topology on $\mathsf{OF}(X)$.

In the next proposition, we show a relation between a P-space X and the space $\mathsf{OF}(X)$. Consider the poset $\mathsf{Fin}(X) := \langle \mathsf{Fin}(X), \sqsubseteq \rangle$, which is a sub-poset of the space X with respect to the specialization order. So we can consider the dual P-space $X_{\mathsf{Fin}(X)} = \mathsf{Fi}(\mathsf{Fin}(X))$ of the poset $\mathsf{Fin}(X)$.

From the previous proposition, we know that the space $\mathsf{OF}(X)$ has the family $\{\psi_x : x \in X\}$ as a base, but since $\mathsf{KOF}(X)$ is a base for the space X, we can take a smaller family as a base for the space $\mathsf{OF}(X)$; this base will be $\{\psi_a : a \in \mathsf{Fin}(X)\}$. To show this, let U be an open set of the space $\mathsf{OF}(X)$ and let $F \in U$. So there is $x \in X$ such that $F \in \psi_x \subseteq U$. Since $x \in F$ and F is an open set of X, there exists $a \in \mathsf{Fin}(X)$ such that $x \in {\uparrow}a \subseteq F$. Hence, we obtain $F \in \psi_a \subseteq U$. We are ready to prove the following proposition.

Proposition 7.2. *Let X be a P-space. Then the spaces $X_{\mathsf{Fin}(X)}$ and $\mathsf{OF}(X)$ are homeomorphic.*

Proof. We define the map $\alpha\colon X_{\mathsf{Fin}(X)} \to \mathsf{OF}(X)$ by $\alpha(F) := \bigcup\{{\uparrow}a : a \in F\}$ for each $F \in X_{\mathsf{Fin}(X)}$. We show that α is a homeomorphism in several steps.

• α is well defined. Let $F \in X_{\mathsf{Fin}(X)}$. Since $F \subseteq \mathsf{Fin}(X)$, $\alpha(F)$ is an open subset of X and moreover, it is an up-set. Let $x, y \in \alpha(F)$. So there are $a, b \in F$ such that $x \in {\uparrow}a$ and $y \in {\uparrow}b$. Given that F is a filter of the poset $\mathsf{Fin}(X)$, there is $c \in F$ such that $c \sqsubseteq a, b$. Then, $c \sqsubseteq x, y$ and $c \in \alpha(F)$. Hence, $\alpha(F) \in \mathsf{OF}(X)$.

• α is injective. Let $F_1, F_2 \in X_{\mathsf{Fin}(X)}$ and assume that $\alpha(F_1) = \alpha(F_2)$. Since $F_1 \subseteq \alpha(F_1)$, we have $F_1 \subseteq \alpha(F_2)$. Let $x \in F_1$. So $x \in \alpha(F_2)$ and this implies that there exists $a \in F_2$ such that $x \in {\uparrow}a$. Then $x \in F_2$. Thus, $F_1 \subseteq F_2$. Similarly, we can show that $F_2 \subseteq F_1$. Hence, $F_1 = F_2$.

• α is onto. Let $G \in \mathsf{OF}(X)$. We take $F = G \cap \mathsf{Fin}(X)$. Let $a, b \in \mathsf{Fin}(X)$ and suppose that $a \sqsubseteq b$ and $a \in F$. Since G is a filter of X, $b \in G$, and then $b \in F$. Let $a, b \in F$. Since G is a filter, there is $c \in G$ such that $c \sqsubseteq a, b$. Given that G is an open set of X, there exists $c' \in \mathsf{Fin}(X)$ such that $c \in {\uparrow}c' \subseteq G$. So we have $c' \in G$ and $c' \sqsubseteq c$. Then $c' \in F$ and $c' \sqsubseteq a, b$. This implies that $F \in X_{\mathsf{Fin}(X)}$. Finally, we need to show that $\alpha(F) = G$. Let $x \in \alpha(F)$. So there is $a \in F$ such that $a \sqsubseteq x$. Then $a \in G$, and thus $x \in G$. Hence, $\alpha(F) \subseteq G$. Let $x \in G$. So there exists $a \in \mathsf{Fin}(X)$ such that $x \in {\uparrow}a \subseteq G$. Then $a \in G \cap \mathsf{Fin}(X) = F$, and consequently $x \in \alpha(F)$. Thus, $G \subseteq \alpha(F)$. Therefore, $\alpha(F) = G$.

• α is continuous. Let $a \in \mathsf{Fin}(X)$. We have that

$$F \in \alpha^{-1}[\psi_a] \iff \alpha(F) \in \psi_a \iff a \in \alpha(F) \iff F \in \varphi_a.$$

Then $\alpha^{-1}[\psi_a]$ is an open subset of the space $X_{\mathsf{Fin}(X)}$, and hence α is continuous. Notice that here φ_a is restricted to the poset $\langle \mathsf{Fin}(X), \sqsubseteq \rangle$, that is, we have $\varphi_a = \{F \in \mathsf{Fi}(\mathsf{Fin}(X)) : a \in F\}$.

• α is an open map. By the previous point, for $a \in \mathsf{Fin}(X)$, $\alpha^{-1}[\psi_a] = \varphi_a$. Since we know that α is a bijection, we obtain $\psi_a = \alpha[\alpha^{-1}[\psi_a]] = \alpha[\varphi_a]$. Hence, α is open. This completes the proof. □

It should be noted that by the previous proposition, for every P-space X, the space $\mathsf{OF}(X)$ is a P-space.

Corollary 7.3. *Let P be a poset. If X is the dual P-space of P, then the P-space $\mathsf{OF}(X)$ is the dual P-space of the poset P^∂.*

Proof. Let X be a P-space. It is clear that $\mathsf{KOF}(X)^\partial \cong \mathsf{Fin}(X)$. Then

$$\mathsf{Fi}(\mathsf{KOF}(X)^\partial) \cong \mathsf{Fi}(\mathsf{Fin}(X)) \cong \mathsf{OF}(X).$$

Therefore, the P-space $\mathsf{OF}(X)$ is the dual of the poset $\mathsf{KOF}(X)^\partial$. □

Proposition 7.4. *Let X be a P-space. Then $x \in \mathsf{Fin}(X)$ if and only if ψ_x is compact.*

Proof. Let $x \in \mathsf{Fin}(X)$. Notice that $\psi_x = \{F \in \mathsf{OF}(X) : {\uparrow}x \subseteq F\}$. Then since ${\uparrow}x \in \mathsf{OF}(X)$, ψ_x is the compact open filter of the space $\mathsf{OF}(X)$ generated by ${\uparrow}x$. Conversely, let $x \in X$ and assume that ψ_x is compact. So $\psi_x \in \mathsf{KOF}(\mathsf{OF}(X))$. Then there exists $G \in \mathsf{OF}(X)$ such that $\psi_x = \{F \in \mathsf{OF}(X) : G \subseteq F\}$. Since $G \in \psi_x$, $x \in G$. So we have ${\uparrow}x \subseteq G$. Now let $a \in G$ and suppose that $x \not\sqsubseteq a$. Thus, there is $U \in \mathsf{KOF}(X)$ such that $x \in U$ and $a \notin U$. Then we have $U \in \psi_x$, and hence $G \nsubseteq U$, which is a contradiction. Then $x \sqsubseteq a$. So we obtain that $G \subseteq {\uparrow}x$. Therefore, $G = {\uparrow}x$ and $x \in \mathsf{Fin}(X)$. □

Proposition 7.5. *If X is a P-space, then the map $\eta\colon X \to \mathsf{OF}(\mathsf{OF}(X))$ defined by $\eta(x) := \psi_x = \{F \in \mathsf{OF}(X) : x \in F\}$ for every $x \in X$, is a homeomorphism.*

8. Topological representation of quasi-monotone maps

Let $P_1, \dots, P_{n+1}$ be posets. The main aim of this section is to characterize topologically the maps $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ that in each coordinate either preserve or reverse the order. We will call such maps *quasi-monotone maps.* If P is a poset and $j\colon P^n \to P$ is a quasi-monotone map, then we say that j is an *n-ary quasi-monotone map.* In [1], a structure $\langle P, (j_i)_{i \in I}\rangle$, where P is a poset and every j_i is an n_i-ary quasi-monotone map on P, is called a *monotone poset expansion.*

For every quasi-monotone map $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$, there is a *monotonicity type* $\epsilon = \langle \epsilon_1, \dots, \epsilon_n\rangle$ associated with j where for every $i = 1, \dots, n$, $\epsilon_i = 1$ or $\epsilon_i = \partial$, depending on whether j preserves or reverses the order in the coordinate i. If we let $P_i^{\epsilon_i} = P_i$ or $P_i^{\epsilon_i} = P_i^\partial$, depending on whether $\epsilon_i = 1$ or $\epsilon_i = \partial$, then the map $j\colon P_1^{\epsilon_1} \times \cdots \times P_n^{\epsilon_n} \to P_{n+1}$ is order-preserving.

To represent topologically quasi-monotone maps $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ for arbitrary posets $P_1, \dots, P_{n+1}$, it is then enough to represent order-preserving maps in each coordinate. Indeed, if $\epsilon = \langle \epsilon_1, \dots, \epsilon_n\rangle$ is the monotonicity type associated with j, letting $X_k^{\epsilon_k}$ to be the dual space of $P_k^{\epsilon_k}$ for $1 \leq k \leq n$, we will take as a representation of $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ the representation $f\colon X_1^{\epsilon_1} \times \cdots \times X_n^{\epsilon_n} \to X_{n+1}$ of j considered as the order-preserving map $j\colon P_1^{\epsilon_1} \times \cdots \times P_n^{\epsilon_n} \to P_{n+1}$.

8.1. Product of P-spaces. Here we prove that the topological product $X_1 \times \cdots \times X_n$ of a finite number of P-spaces is a P-space. The facts in the following proposition are known and we omit their proofs.

Proposition 8.1. *Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces.*

(1) *If for every $i \in I$ the space X_i is a T_0-space, then $\prod_{i \in I} X_i$ is T_0.*
(2) *If for every $i \in I$ the space X_i is sober, then $\prod_{i \in I} X_i$ is a sober space.*

Next we characterize the compact open filters of finite products of P-spaces.

Proposition 8.2. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then $\mathsf{KOF}(X) = \{\uparrow x_1 \times \cdots \times \uparrow x_n : x_1 \in \mathsf{Fin}(X_1), \ldots, x_n \in \mathsf{Fin}(X_n)\}$.*

Proof. Let $x_1 \in \mathsf{Fin}(X_1), \ldots, x_n \in \mathsf{Fin}(X_n)$ and let $\overline{x} = \langle x_1, \ldots, x_n \rangle$. It is easy to see that $\uparrow \overline{x} = \uparrow x_1 \times \cdots \times \uparrow x_n$ and since this last set is an open set of X, $\overline{x} \in \mathsf{Fin}(X)$. Thus, $\uparrow x_1 \times \cdots \times \uparrow x_n \in \mathsf{KOF}(X)$. Let $\overline{y} = \langle y_1, \ldots, y_n \rangle \in \mathsf{Fin}(X)$. It is clear that $\uparrow \overline{y} = \uparrow y_1 \times \cdots \times \uparrow y_n$. Moreover, for every $i \in \{1, \ldots, n\}$, $\pi_i[\uparrow \overline{y}] = \uparrow y_i$; therefore, $\uparrow y_i$ is open and then $y_i \in \mathsf{Fin}(X_i)$. □

From the proof of the proposition, we have that for a finite number of P-spaces $X_1, \ldots, X_n$, if $X = X_1 \times \cdots \times X_n$ is their topological product space, then an element $\overline{x} = \langle x_1, \ldots, x_n \rangle$ of X is finite if and only if each x_i is finite in the space X_i.

Theorem 8.3. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then X is a P-space.*

Proof. For every $i \in \{1, \ldots, n\}$, $\mathsf{KOF}(X_i) = \{\uparrow x : x \in \mathsf{Fin}(X_i)\}$ is a base for X_i. Thus, from the previous proposition, it follows that $\mathsf{KOF}(X)$ is a base for X. Moreover, since the product of sober spaces is sober, X is sober. Hence, X is a P-space. □

The open filters of a finite product of P-spaces are characterized as follows:

Proposition 8.4. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then $\mathsf{OF}(X) = \{F_1 \times \cdots \times F_n : F_1 \in \mathsf{OF}(X_1), \ldots, F_n \in \mathsf{OF}(X_n)\}$.*

Proof. Let $F_1 \in \mathsf{OF}(X_1), \ldots, F_n \in \mathsf{OF}(X_n)$. Then $F_1 \times \cdots \times F_n$ is an open set of X and it is easy to see that it is a filter. Assume now that $F \in \mathsf{OF}(X)$. Let $F_i = \pi_i[F]$ for every $i \in \{1, \ldots, n\}$. Then F_i is an open filter of X_i. Moreover, using that F is a filter, it is easy to check that $F = F_1 \times \cdots \times F_n$. □

Now we move to some considerations on the P-space dual of the direct product of a finite number of posets. Let $P_1, \ldots, P_n$ be posets and consider their direct product $P = P_1 \times \cdots \times P_n$, whose order is given coordinatewise. Note that the filters of P are the sets of the form $F_1 \times \cdots \times F_n$ where F_i is a filter of P_i for every $i \in \{1, \ldots, n\}$.

Proposition 8.5. *The P-spaces X_P and $X_{P_1} \times \cdots \times X_{P_n}$ are homeomorphic.*

Proof. Define $f\colon X_{P_1} \times \cdots \times X_{P_n} \to X_P$ by $f(\langle F_1, \ldots, F_n \rangle) = F_1 \times \cdots \times F_n$ for all $\langle F_1, \ldots, F_n \rangle \in X_{P_1} \times \cdots \times X_{P_n}$. This map is clearly a bijection. The compact open filters of X_P are the sets of the form $\varphi_{\langle a_1,\ldots,a_n \rangle}$ with $\langle a_1, \ldots, a_n \rangle \in P$ and the compact open filters of $X_{P_1} \times \cdots \times X_{P_n}$ are sets of the form $\varphi_{a_1} \times \cdots \times \varphi_{a_n}$ with $\langle a_1, \ldots, a_n \rangle \in P$. Let $\langle a_1, \ldots, a_n \rangle \in P$. Then it is easy to check that

$$\begin{aligned} f^{-1}[\varphi_{\langle a_1,\ldots,a_n \rangle}] &= \{\langle F_1, \ldots, F_n \rangle : \langle a_1, \ldots, a_n \rangle \in F_1 \times \cdots \times F_n\} \\ &= \varphi_{a_1} \times \cdots \times \varphi_{a_n}, \end{aligned}$$

and hence a compact open filter of $X_{P_1} \times \cdots \times X_{P_n}$. Similarly, we have $f[\varphi_{a_1} \times \cdots \times \varphi_{a_n}] = \varphi_{\langle a_1,\ldots,a_n \rangle}$. Therefore, f is a continuous and open map. We conclude that f is a homeomorphism. □

8.2. Quasi-monotone maps. In [9], Moshier and Jipsen present a topological representation of n-ary quasioperators. From the definition of n-ary quasioperators, it clearly follows that they are n-ary quasi-monotone maps. We apply the ideas developed in [9] to the poset setting to obtain a topological representation of quasi-monotone maps as maps between P-spaces. Hence, the topological representation of n-ary quasi-monotone maps in the setting of posets that we develop in this section can be considered a generalization of the topological representation for n-ary quasioperators in the setting of lattices due to Moshier and Jipsen.

As we mentioned at the beginning of the section, to represent topologically quasi-monotone maps, it is enough to represent order-preserving maps. Let $P_1, \ldots, P_{n+1}$ be posets. Any map $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ that is order-preserving in each coordinate is an order-preserving map from the direct product $P_1 \times \cdots \times P_n$ of the posets $P_1, \ldots, P_n$ to the poset P_{n+1}. So, considering also Proposition 8.5, it will be enough to represent order-preserving maps between posets.

Let P, Q be posets and $j\colon P \to Q$ an order-preserving map. We define the map $f_j\colon X_P \to X_Q$ as follows:

$$f_j(F) = \{a \in Q : \exists b \in F,\ j(b) \le a\} = {\uparrow} j[F] \tag{8.1}$$

for every $F \in X_P$. Let us see that f is well defined in the sense that its range is included in X_Q. Clearly, $f_j(F)$ is an up-set. Let us see that it is down-directed. Let $a_1, a_2 \in f_j(F)$. Fix $b_1, b_2 \in F$ such that $j(b_1) \le a_1$ and $j(b_2) \le a_2$. Let $c \in F$ be such that $c \le b_1, b_2$. Then $j(c) \le j(b_1), j(b_2)$ and thus, $j(c) \le a_1, a_2$ and $j(c) \in f_j(F)$. Hence, $f_j(F)$ is down-directed.

Proposition 8.6. *The map f_j is continuous.*

Proof. Let U be a basic open subset of the space X_Q. We know that by Proposition 4.4, $U = \varphi_a$ for some $a \in Q$. Notice that by definition of f_j,

$$F \in f_j^{-1}[\varphi_a] \iff f_j(F) \in \varphi_a \iff a \in f_j(F) \iff \exists b \in F (j(b) \le a).$$

Let $F \in f_j^{-1}[\varphi_a]$. So there exists $b \in F$ such that $j(b) \leq a$. Clearly, φ_b is an open subset of the space X_P and $F \in \varphi_b$. Next, we show that $\varphi_b \subseteq f_j^{-1}[\varphi_a]$. Let $G \in \varphi_b$. So $b \in G$, and since $j(b) \leq a$, then $G \in f_j^{-1}[\varphi_a]$. Thus, $f_j^{-1}[\varphi_a]$ is an open subset of the space X_P, and therefore f_j is continuous. □

Let X be a P-space. We define the binary relation $\ll$ on X as follows: for every $x_0, x_1 \in X$, $x_0 \ll x_1$ if and only if for some $F \in \mathsf{OF}(X)$ with $x_1 \in F$, $x_0 \in G$ implies $F \subseteq G$ for all $G \in \mathsf{OF}(X)$.

A map $f\colon X \to Y$ between P-spaces is *strongly-continuous* if it is continuous and preserves the relation $\ll$, that is, $x_0 \ll x_1$ implies $f(x_0) \ll f(x_1)$.

The next two propositions are easy consequences of the definition.

Proposition 8.7. *Let X be a P-space and let $x, y \in X$. Then, $x \ll y$ if and only if for some $a \in \mathsf{Fin}(X)$ with $y \in {\uparrow}a$, $x \in {\uparrow}b$ implies ${\uparrow}a \subseteq {\uparrow}b$ for all $b \in \mathsf{Fin}(X)$.*

Proposition 8.8. *Let X be a P-space. Then for every $x \in X$, we have $x \ll x$ if and only if $x \in \mathsf{Fin}(X)$.*

The following proposition is a useful characterization of the relation $\ll$ in a product of a finite number of P-spaces.

Proposition 8.9. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$ be the space with the product topology. Let $\overline{x}, \overline{y} \in X$. Then,*

$$\overline{x} \ll \overline{y} \text{ if and only if } x_i \ll y_i \text{ for all } i = 1, \ldots, n.$$

Proof. First, we assume that $\overline{x} \ll \overline{y}$. So there exists $F \in \mathsf{OF}(X)$ such that $\overline{y} \in F$. Then for every $i = 1, \ldots, n$, there exists $F_i \in \mathsf{OF}(X_i)$ such that $F = F_1 \times \cdots \times F_n$. Let $i \in \{1, \ldots, n\}$. Then $y_i \in F_i$. Let now $G_i \in \mathsf{OF}(X_i)$ be such that $x_i \in G_i$. Fix $G_j \in \mathsf{OF}(X_j)$ such that $x_j \in G_j$ for every $j \in \{1, \ldots, n\}$ different from i. Then $\overline{x} \in G = G_1 \times \cdots \times G_n \in \mathsf{OF}(X)$. Therefore, $F \subseteq G$. This implies that $F_i \subseteq G_i$. Hence, $x_i \ll y_i$.

Conversely, we assume that $x_i \ll y_i$ for all $i = 1, \ldots, n$. So, for each $i = 1, \ldots, n$, there exists $a_i \in \mathsf{Fin}(X_i)$ such that $y_i \in {\uparrow}a_i$, and we have that $\forall b_i \in \mathsf{Fin}(X_i)\big(x_i \in {\uparrow}b_i \Longrightarrow {\uparrow}a_i \subseteq {\uparrow}b_i\big)$. We define $\overline{a} := \langle a_1, \ldots, a_n\rangle \in \mathsf{Fin}(X)$. Notice that $\overline{y} \in {\uparrow}\overline{a}$. Let $\overline{b} \in \mathsf{Fin}(X)$ be such that $\overline{x} \in {\uparrow}\overline{b}$. Then for every $i = 1, \ldots, n$, $b_i \in \mathsf{Fin}(X_i)$ and $x_i \in {\uparrow}b_i$. Thus, ${\uparrow}a_i \subseteq {\uparrow}b_i$ for all $i = 1, \ldots, n$, and consequently ${\uparrow}\overline{a} \subseteq {\uparrow}\overline{b}$. Therefore, $\overline{x} \ll \overline{y}$. □

Remark 8.10. Let P be a poset and consider the dual P-space X_P of P. Then the relation $\ll$ on X_P reduces to

$$F \ll G \iff \exists a \in P(F \subseteq {\uparrow}a \subseteq G).$$

Proposition 8.11. *The map $f_j\colon X_P \to X_Q$ is strongly-continuous.*

Proof. By Proposition 8.6, it only remains to prove that f_j preserves the relation $\ll$. Let $F, G \in X_P$ be such that $F \ll G$. By the above remark, there is $a \in P$ such that $F \subseteq {\uparrow}a \subseteq G$. We take $b := j(a) \in Q$. Then by definition of f_j and since the map j is order-preserving, $f_j(F) \subseteq {\uparrow} b \subseteq f_j(G)$. Hence, $f_j(F) \ll f_j(G)$. $\square$

To obtain the reverse construction, let X, Y be P-spaces and let $f\colon X \to Y$ be a strongly-continuous map. The map $j_f\colon \mathsf{KOF}(X) \to \mathsf{KOF}(Y)$ is defined by

$$j_f({\uparrow} x) = {\uparrow} f(x), \tag{8.2}$$

for every $x \in \mathsf{Fin}(X)$. Notice that if $x \in \mathsf{Fin}(X)$, $x \ll x$. Then, given that f preserves the relation $\ll$, $f(x) \ll f(x)$, which implies that $f(x) \in \mathsf{Fin}(Y)$. So we have that j_f is well defined.

Proposition 8.12. *The map j_f is order-preserving.*

Proof. Let $x_1, x_2 \in \mathsf{Fin}(X)$ be such that ${\uparrow} x_1 \subseteq {\uparrow} x_2$. Then $x_2 \sqsubseteq x_1$. Since f is a continuous map, f is order-preserving (with respect to the specialization order). Then $f(x_2) \sqsubseteq f(x_1)$. Thus, it follows that ${\uparrow} f(x_1) \subseteq {\uparrow} f(x_2)$. Hence, $j_f({\uparrow} x_1) \subseteq j_f({\uparrow} x_2)$. Therefore, j_f is order-preserving. $\square$

We are in a position to show that the function that sends order-preserving maps to strongly-continuous maps $j \mapsto f_j$ and the function that sends strongly-continuous maps to order-preserving maps $f \mapsto f_j$ are inverses of one another.

Let $j\colon P \to Q$ be an order-preserving map and consider $f_j\colon X_P \to X_Q$ defined as in (8.1). Then we have the map $j_{f_j}\colon \mathsf{KOF}(X_P) \to \mathsf{KOF}(X_Q)$ defined as in (8.2). We want to show that the maps j and j_{f_j} are, essentially, the same. Recall from Theorem 4.5 that $\varphi\colon P \to \mathsf{KOF}(X_P)$ is an order isomorphism. So we should prove that $j_{f_j}(\varphi_P(a)) = \varphi_Q(j(a))$ for all $a \in P$. That is, that the the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\;\;j\;\;} & Q \\ {\scriptstyle \varphi_P}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_Q} \\ \mathsf{KOF}(X_P) & \xrightarrow{\;j_{f_j}\;} & \mathsf{KOF}(X_Q) \end{array}$$

Let $a \in P$. First we observe, by definition of f_j, that $f_j(a) = {\uparrow} j(a)$, and moreover $j_{f_j}(\varphi_P(a)) = {\uparrow} f_j({\uparrow} a)$. Then for every $F \in \mathsf{Fi}(P)$, we have

$$F \in j_{f_j}(\varphi_P(a)) \iff f_j({\uparrow} a) \subseteq F \iff j(a) \in F \iff F \in \varphi_Q(j(a)).$$

Conversely, we now consider a strongly-continuous map $f\colon X \to Y$ from a P-spaces X to a P-space Y. So we have the map $j_f\colon P_X \to P_Y$ given by (8.2). Then we consider the strongly-continuous map $f_{j_f}\colon \mathsf{Fi}(P_X) \to \mathsf{Fi}(P_Y)$, and we prove that f and f_{j_f} are, essentially, the same maps. That is, we prove that $f_{j_f}(\theta_X(x)) = \theta_Y(f(x))$ for all $x \in X$, where $\theta_X\colon X \to \mathsf{Fi}(P_X)$ is the homeomorphism given by Theorem 5.4 and similarly for θ_Y. In other words, we prove that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ \mathsf{Fi}(P_X) & \xrightarrow{\quad f_{j_f} \quad} & \mathsf{Fi}(P_Y) \end{array}$$

Let $x \in X$ and let $\uparrow y \in \mathsf{KOF}(Y)$. First, we assume $\uparrow y \in f_{j_f}(\theta_X(x))$. So there exists $\uparrow z \in \theta_X(x)$ such that $j_f(\uparrow z) \subseteq \uparrow y$. Then $\uparrow f(z) \subseteq \uparrow y$, which implies that $f(z) \in \uparrow y$. Since $x \in \uparrow z$ and f is order-preserving, we have that $f(z) \sqsubseteq f(x)$. Thus, $f(x) \in \uparrow y$. Hence, $\uparrow y \in \theta_Y(f(x))$.

Now we assume that $\uparrow y \in \theta_Y(f(x))$. So $f(x) \in \uparrow y$. Then $x \in f^{-1}[\uparrow y]$, and so there exists $z \in \mathsf{Fin}(X)$ such that $x \in \uparrow z \subseteq f^{-1}[\uparrow y]$. Thus, $f[\uparrow z] \subseteq \uparrow y$. Then $\uparrow z \in \theta_X(x)$ and $f(z) \in \uparrow y$. Therefore, $\uparrow y \in f_{j_f}(\theta_X(x))$.

Let $P_1, \ldots, P_{n+1}$ be posets and let $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ be a map that is order-preserving in each coordinate. Let P be the direct product of $P_1, \ldots, P_n$. Note that the filters of P are the sets of the form $F_1 \times \cdots \times F_n$ where for every $i = 1, \ldots, n$, $F_i \in \mathsf{Fi}(P_i)$. So for every $F_1 \times \cdots \times F_n \in \mathsf{Fi}(P)$,

$$f_j(F_1 \times \cdots \times F_n) = \{a \in P_{n+1} : \exists b_1 \in F_1, \ldots, \exists b_n \in F_n,\ j(\langle b_1, \ldots, b_n\rangle) \leq a\}.$$

Thus we can obtain a map $\overline{f_j}\colon X_{P_1} \times \cdots \times X_{P_n} \to X_{P_{n+1}}$ defined by

$$\overline{f_j}(\langle F_1, \ldots, F_n\rangle) = \{a \in P_{n+1} : \exists b_1 \in F_1, \ldots, \exists b_n \in F_n,\ j(\langle b_1, \ldots, b_n\rangle) \leq a\}.$$

This map is strongly-continuous, thanks to the homeomorphism between X_P and the product space $X_{P_1} \times \cdots \times X_{P_n}$.

Now let $X_1, \ldots, X_{n+1}$ be P-spaces and let $f\colon X_1 \times \cdots \times X_n \to X_{n+1}$ be a strongly-continuous map. Let P be the poset of compact open filters of the product space $X_1 \times \cdots \times X_n$. This poset is isomorphic to the direct product $\mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n)$. Consequently, we have the order-preserving map $j_f\colon P \to \mathsf{KOF}(X_{n+1})$. Using the isomorphism, we obtain a map

$$\overline{j_f}\colon \mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n) \to \mathsf{KOF}(X_{n+1}),$$

which is order-preserving in each coordinate and is given by

$$\overline{j_f}(\langle \uparrow a_1, \ldots, \uparrow a_n\rangle) = \uparrow f(\langle a_1, \ldots, a_n\rangle)$$

for every $a_i \in \mathsf{Fin}(X_i)$ with $i = 1, \ldots, n$.

9. The extension of a strongly-continuous map between P-spaces to their lattices of F-saturated sets

Let X and Y be P-spaces. We show in this short section how to extend a strongly-continuous map f from X to Y to maps from $\mathsf{Fsat}(X)$ to $\mathsf{Fsat}(Y)$ in such a way that the image of $\uparrow x$, with $x \in X$, is $\uparrow f(x)$.

We will exploit the fact that according to the results in Section 6, for every P-space X, $\mathsf{Fsat}(X)$ is (up to isomorphism) the canonical extension of $\mathsf{KOF}(X)$, with the identity as the embedding map, and the theory developed in [1] of the extension of maps between posets to their canonical extensions.

Let $f\colon X \to Y$ be a strongly-continuous map; then $j_f\colon \mathsf{KOF}(X) \to \mathsf{KOF}(Y)$ is order-preserving. Thus, according to [1, Definition 3.2], the map j_f has two extensions $(j_f)^\sigma$ and $(j_f)^\pi$ from the canonical extension $\mathsf{Fsat}(X)$ of $\mathsf{KOF}(X)$ to the canonical extension $\mathsf{Fsat}(Y)$ of $\mathsf{KOF}(Y)$. We provide a description in our setting of the maps $(j_f)^\sigma$ and $(j_f)^\pi$.

First, let us characterize, for a given P-space X, the open and closed elements of $\mathsf{Fsat}(X)$ taken as the canonical extension of $\mathsf{KOF}(X)$. According to [1], a set $U \in \mathsf{Fsat}(X)$ is a closed element if there is a filter $\mathcal{F}$ of the poset $\langle \mathsf{KOF}(X), \subseteq \rangle$ such that $U = \bigcap \mathcal{F}$. And it is an open element if there is an ideal $\mathcal{I}$ of the poset $\langle \mathsf{KOF}(X), \subseteq \rangle$ such that $U = \bigvee \mathcal{I}$.

Note that $\mathcal{I}$ is an ideal of $\langle \mathsf{KOF}(X), \subseteq \rangle$ if and only if there is $F_{\mathcal{I}} \in \mathsf{OF}(X)$ such that $\mathcal{I} = \{\uparrow x : x \in F_{\mathcal{I}}\}$ and is in $\mathsf{Fsat}(X)$, $\bigvee \mathcal{I} = F_{\mathcal{I}}$. Thus, the open elements of $\mathsf{Fsat}(X)$ are the open filters of X. Now note that $\mathcal{F}$ is a filter of $\langle \mathsf{KOF}(X), \subseteq \rangle$ if and only if there is an ideal I of $\mathsf{Fin}(X)$ with $\mathcal{F} = \{\uparrow y : y \in I\}$.

Thus, $U \in \mathsf{Fsat}(X)$ is an open element if and only if there exists $F \in \mathsf{OF}(X)$ such that $U = \bigvee \{\uparrow x : x \in F\}$ and it is a closed element if and only if there exists an ideal I of $\mathsf{Fin}(X)$ such that $U = \bigcap \{\uparrow x : x \in I\}$.

Proposition 9.1. *Let X, Y be P-spaces and $f\colon X \to Y$ a strongly-continuous map. Then for every $U \in \mathsf{Fsat}(X)$,*

(1) $(j_f)^\pi(U) = \bigcap \Big\{ \bigvee \{\uparrow f(x) : x \in F \cap \mathsf{Fin}(X)\} : F \in \mathsf{OF}(X) \;\; U \subseteq F \Big\}$.

(2) $(j_f)^\sigma(U) = \bigvee \Big\{ \bigcap \{\uparrow f(x) : x \in I\} : I \in \mathsf{Id}(\mathsf{Fin}(X)), \bigcap \{\uparrow x : x \in I\} \subseteq U \Big\}$.

Proof. (1): By definition, if $U \in \mathsf{Fsat}(X)$, then $U = \bigcap \{F \in \mathsf{OF}(X) : U \subseteq F\}$. Thus, by definition of the map $(j_f)^\pi\colon \mathsf{Fsat}(X) \to \mathsf{Fsat}(Y)$,

$$(j_f)^\pi(U) = \bigcap \Big\{ \bigvee \{j_f(\uparrow x) : \uparrow x \in \mathcal{I}\} : \mathcal{I} \in \mathsf{Id}(\mathsf{KOF}(X)) \text{ and } U \subseteq \bigvee \mathcal{I} \Big\}.$$

Hence, $(j_f)^\pi(U) = \bigcap \Big\{ \bigvee \{\uparrow f(x) : x \in F \cap \mathsf{Fin}(X)\} : F \in \mathsf{OF}(X) \text{ and } U \subseteq F \Big\}$.
(2) Similarly, by the definition of the map $(j_f)^\sigma\colon \mathsf{Fsat}(X) \to \mathsf{Fsat}(Y)$,

$$(j_f)^\sigma(U) = \bigvee \Big\{ \bigcap \{j_f(\uparrow x) : \uparrow x \in \mathcal{F}\} : \mathcal{F} \in \mathsf{Fi}(\mathsf{KOF}(X)) \text{ and } \bigcap \mathcal{F} \subseteq U \Big\}.$$

Thus, $(j_f)^\sigma(U) =$

$$\bigvee \Big\{ \bigcap \{\uparrow f(x) : x \in I\} : I \in \mathsf{Id}(\mathsf{Fin}(X)) \text{ and } \bigcap \{\uparrow x : x \in I\} \subseteq U \Big\}. \qquad \square$$

10. Meet-semilattices and maps that preserve meet

In [8], Moshier and Jipsen develop a topological duality for meet-semilattices with top element of which our duality for posets is a generalization. But our duality also provides a duality for meet-semilattices in general. We proceed to expound this duality and show how it restricts to the duality of Moshier and Jipsen.

A *meet-semilattice* M is a poset such that the greatest lower bound exists for every pair of elements of M. Equivalently, a meet-semilattice can be defined as an algebra $\langle M, \wedge \rangle$ where M is a non-empty set and $\wedge$ is a binary operation that is idempotent, associative and commutative. We consider the category of meet-semilattices (as posets) and meet-preserving maps. It is not hard to check that this category is a full subcategory of $\mathbb{P}$.

We say that a topological space X is an *almost HMS-space*, an AHMS-space for short, if it satisfies the following conditions:

(1) X is sober;
(2) $\mathsf{KOF}(X)$ forms a base;
(3) $\mathsf{KOF}(X)$ is closed under finite non-empty intersections (that is, if we have $U, V \in \mathsf{KOF}(X)$, then $U \cap V \in \mathsf{KOF}(X)$).

This notion of almost HMS-space is essentially due to Moshier and Jipsen [8]. Since they work with meet-semilattices with a top element, they require in addition $\mathsf{KOF}(X)$ to be closed under intersections of arbitrary finite subsets of $\mathsf{KOF}(X)$ or, equivalently, that X has a least element with respect to specialization order. Moshier and Jipsen call their spaces *HMS-spaces* in honor of Hofmann, Mislove and Stralka.

It is clear that every almost HMS-space is a P-space. Thus, we may consider the full subcategory $\mathbb{AHMS}$ of $\mathbb{TOP}(P)$ with objects the almost HMS-spaces (and hence with morphisms the F-continuous functions between them). The full subcategory $\mathbb{HMS}$ of $\mathbb{AHMS}$ of the HMS-spaces is the category that Moshier and Jipsen prove in [8] to be dually equivalent to the category of meet-semilattices with top element and meet-preserving maps that also preserve the top element.

If we apply the duality for posets given in Theorem 5.5 to the full subcategory of meet-semilattices we obtain, taking into account Remark 4.6, that this category is dual to the category $\mathbb{AHMS}$ and if we apply that theorem to the category of meet-semilattices with top element, we obtain, again taking into account Remark 4.6, the duality given by Moshier and Jipsen between that category and the category of HMS-spaces.

Now, we restrict our attention to those maps $j \colon M_1 \times \cdots \times M_n \to M_{n+1}$, where $M_1, \ldots, M_{n+1}$ are meet-semilattices that are meet-preserving in each coordinate. We apply the topological representation presented in Subsection 8.2 to the map j. First, observe that if M is a meet-semilattice and F_1, F_2 are filters of M, then the filter

$$F_1 \vee F_2 := \{a \in M : \ b_1 \wedge b_2 \leq a \text{ for some } b_1 \in F_1 \text{ and } b_2 \in F_2\}$$

is the least upper bound of F_1 and F_2 in $\mathsf{Fi}(M)$ with respect to inclusion order. Hence, using the duality between meet-semilattices and almost HMS-spaces, we have that every almost HMS-space X is a join-semilattice with respect to specialization order.

Proposition 10.1. *Let $M_1, \dots, M_n, M_{n+1}$ be meet-semilattices. The maps $j\colon M_1 \times \cdots \times M_n \to M_{n+1}$ that preserve meets in each coordinate are topologically represented by the maps $f\colon X_{M_1} \times \cdots \times X_{M_n} \to X_{M_{n+1}}$ that are strongly-continuous and preserve joins in each coordinate (with respect to the specialization order).*

Proof. Let $M_1, \dots, M_{n+1}$ be meet-semilattices and $j\colon M_1 \times \cdots \times M_n \to M_{n+1}$ a map that preserves meets in each coordinate. It is clear that j is order-preserving. We thus define the map $\overline{f_j}\colon X_{M_1} \times \cdots \times X_{M_n} \to X_{M_{n+1}}$ (where X_{M_i} is the dual almost HMS-space of the meet-semilattice M_i) as in Subsection 8.2. It only remains to prove that $\overline{f_j}$ preserves joins in each coordinate. Let $H, G \in \mathsf{Fi}(M_1)$ and $F_i \in \mathsf{Fi}(M_i)$ for every $i = 2, \dots, n$. We need to prove that

$$\overline{f_j}(\langle H \vee G, F_2, \dots, F_n\rangle) = \overline{f_j}(\langle H, F_2, \dots, F_n\rangle) \vee \overline{f_j}(\langle G, F_2, \dots, F_n\rangle).$$

Let $a \in \overline{f_j}(\langle H \vee G, F_2, \dots, F_n\rangle)$. Therefore, $j(\langle a_1, b_2, \dots, b_n\rangle) \leq a$ for some $a_1 \in H \vee G$ and $b_2 \in F_2, \dots, b_n \in F_n$. Then there exist $h \in H$ and $g \in G$ such that $h \wedge g \leq a_1$. Thus, we have $j(\langle h \wedge g, b_2, \dots, b_n\rangle) \leq j(\langle a_1, b_2, \dots, b_n\rangle) \leq a$, and since j preserves meets in each coordinate, we have

$$j(\langle h, b_2, \dots, b_n\rangle) \wedge j(\langle g, b_2, \dots, b_n\rangle) \leq a.$$

Moreover, it is clear that we have both $j(\langle h, b_2, \dots, b_n\rangle) \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle)$ and $j(\langle g, b_2, \dots, b_n\rangle) \in \overline{f_j}(\langle G, F_2, \dots, F_n\rangle)$. Hence,

$$a \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle) \vee \overline{f_j}(\langle G, F_2, \dots, F_n\rangle).$$

On the other hand, if $a \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle) \vee \overline{f_j}(\langle G, F_2, \dots, F_n\rangle)$, then there exist $h \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle)$ and $g \in \overline{f_j}(\langle G, F_2, \dots, F_n\rangle)$ with $h \wedge g \leq a$. Thus, by definition of $\overline{f_j}$, we obtain $j(\langle h_1, b_2, \dots, b_n\rangle) \leq h$ for some $h_1 \in H$ and $b_i \in F_i$ for $i = 2, \dots, n$, and $j(\langle g_1, b'_2, \dots, b'_n\rangle) \leq g$ for some $g_1 \in G$ and $b'_i \in F_i$ for $i = 2, \dots, n$. Now, for each $2 \leq i \leq n$, we put $c_i := b_i \wedge b'_i$. We note that $c_i \in F_i$ for all $i = 2, \dots, n$. Then

$$j(\langle h_1, c_2, \dots, c_n\rangle) \leq h \text{ and } j(\langle g_1, c_2, \dots, c_n\rangle) \leq g.$$

So from the previous inequalities and since the map j preserves meet in each argument, we have $j(\langle h_1 \wedge g_1, c_2, \dots, c_n\rangle) \leq h \wedge g \leq a$. This implies

$$a \in \overline{f_j}(\langle H \vee G, F_2, \dots, F_n\rangle).$$

Now let $X_1, \dots, X_{n+1}$ be almost HMS-spaces and let $f\colon X_1 \times \cdots \times X_n \to X_{n+1}$ be a strongly-continuous map that preserves joins in each coordinate (with respect to the specialization order). We consider the map

$$\overline{j_f}\colon \mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n) \to \mathsf{KOF}(X_{n+1}).$$

Let $\uparrow a, \uparrow b \in \mathsf{KOF}(X_1)$, $\uparrow a_2 \in \mathsf{KOF}(X_2)$, $\dots$, $\uparrow a_n \in \mathsf{KOF}(X_n)$. Then,

$$\begin{aligned}\overline{f_j}(\langle \uparrow a \cap \uparrow b, \uparrow a_2, \dots, \uparrow a_n\rangle) &= \uparrow f(\langle a \sqcup b, a_2, \dots, a_n\rangle)\\ &= \uparrow (f(\langle a, a_2, \dots, a_n\rangle) \sqcup f(\langle b, a_2, \dots, a_n\rangle))\\ &= \uparrow f(\langle a, a_2, \dots, a_n\rangle) \cap \uparrow f(\langle b, a_2, \dots, a_n\rangle).\end{aligned}$$

The result follows similarly for the rest of the coordinates. It follows that $\overline{f_j}$ preserve meets in each coordinate. □

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Luciano J. González
Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, Avda. Uruguay 151, 6300 Santa Rosa, La Pampa, Argentina
e-mail: lucianogonzalez@exactas.unlpam.edu.ar

Ramon Jansana
Departament de Lògica, Història i Filosofia de la Ciència, Universitat de Barcelona, Montalegre 6, 08001 Barcelona, España
e-mail: jansana@ub.edu